

The density of the (α, d, β) -superprocess and singular solutions to a fractional non-linear PDE

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Abstract

We consider the density X_t of the critical (α, d, β) -superprocess with $\alpha \in (0, 2)$, $1 + \beta \in (1, 2]$ and $\beta < \frac{\alpha}{d}$. A recent result on partial differential equations [3] implies the following dichotomy when the process is conditioned on survival: if $x \in \mathbb{R}^d$ is fixed and $\beta \leq \beta^* = \frac{\alpha}{d + \alpha}$, then $X_t(x) > 0$ a.s.; otherwise, the probability that $X_t(x)$ is positive has power law decay. We strengthen this by proving that if $\beta < \beta^*$ and the density is continuous, then the density function is strictly positive almost surely.

We then consider the event that X_t charges a measure μ , that is, $\int X_t(x)\mu(dx)$ is positive. We give close to sharp conditions on μ under which X_t charges μ almost surely when conditioned on survival. Our characterization is based on the size of the support of μ , in the sense of Hausdorff measure and dimension. There is a critical dimension d_c such that all measures with support of dimension greater than d_c are charged almost surely. If μ has support of dimension less than d_c and satisfies a uniform density condition, we estimate the probability that X_t charges μ .

Our methods also give new results for the non-linear fractional partial differential equation

$$\partial_t u(t, x) = \Delta_\alpha u(t, x) - u(t, x)^{1+\beta} \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}^d,$$

where $\Delta_\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian. The initial trace of a positive solution consists of a Radon measure and a closed set \mathcal{S} , called the singular set, where the solution has non-integrable singularities as $t \rightarrow 0$. For $\beta < \frac{\alpha}{d}$, the existence of solutions with singular set \mathcal{S} is characterized via the critical dimension d_c . Solutions do not exist if the Hausdorff dimension of \mathcal{S} exceeds (and in some cases equals) d_c . If the dimension of \mathcal{S} is less than d_c and \mathcal{S} satisfies a technical condition, we prove that a solution with singular set \mathcal{S} exists.

1 Introduction, main results and discussion

The main object of study of this article is a measure-valued Markov process called the (α, d, β) -superprocess, also known as the super- α -stable process in \mathbb{R}^d with $(1 + \beta)$ -stable branching. We prove new properties of the density in the parameter regime in which the measure is absolutely continuous. The regularity (and irregularity) of the density has been studied in depth and is

^{*}The author was supported by NSERC and the Li Tze Fong Memorial Fellowship during the preparation of this work.

well understood [13, 14, 30]. Here, we consider positivity of the density. Globally, positivity is understood due to a classical theorem on the propagation of the superprocess' support; ours is then a local analysis and concerns positivity over Lebesgue-null sets. We approach the problem of determining where the density is positive by analysing the probability that the density charges certain measures, a concept we make rigorous. Our findings indicate that the density exhibits delicate and highly non-trivial behaviour at the local level.

Our results are partly motivated by recent work on the non-linear fractional partial differential equation (PDE) which is dual to the superprocess. A dichotomy concerning the existence and non-existence of very singular solutions to the dual equation was proved in [3]. The probabilistic interpretation of this dichotomy, in terms of the (α, d, β) -superprocess, is stated here for the first time (see Theorem 1.3). The result itself is quite interesting, and it also serves to motivate our other results. Throughout the work we emphasize the interplay of probabilistic and analytical aspects, and the proofs use techniques from both fields. Our probabilistic results have analytical counterparts which are new and of independent interest. These are stated in the last part of the introduction.

As our results do not have an obvious precedent in the superprocess literature, we give an expository, and therefore lengthy, introduction. We begin now by introducing the general framework and notation we will use throughout the work.

1.1 The (α, d, β) -superprocess

For $\alpha \in (0, 2)$, $1 + \beta \in (1, 2]$, $d \in \mathbb{N}$, the (α, d, β) -superprocess, or simply the (α, β) -superprocess when the dimension is fixed, is a strong Markov process taking values in $\mathcal{M}_F(\mathbb{R}^d)$, the space of finite measures on \mathbb{R}^d equipped with the topology of weak convergence. We will denote it by $X = (X_t : t \geq 0)$, so that $X_t \in \mathcal{M}_F(\mathbb{R}^d)$. The spatial Markov process associated to X is a symmetric α -stable process in \mathbb{R}^d and, for $\beta \in (0, 1)$, the branching mechanism is that of a continuous state branching process with $(1 + \beta)$ -stable branching. When $\beta = 1$ the superprocess is binary branching and the associated continuous state branching process is Feller's branching diffusion.

The paths of X belong to $\mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^d))$, the space of càdlàg paths in $\mathcal{M}_F(\mathbb{R}^d)$. The Markov transition kernel of X is specified by its Laplace functional, which is characterized via a dual relationship with a fractional non-linear evolution equation. For a general probability space realizing X with initial value $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$, we will write $(\Omega, \mathcal{F}, P_{X_0}^X)$, and denote the associated expectation by $E_{X_0}^X$. Let $\mathcal{B}_b^+(\mathbb{R}^d) = \mathcal{B}_b^+$ denote the space of bounded, measurable, non-negative functions on \mathbb{R}^d . Then for every $\phi \in \mathcal{B}_b^+$,

$$E_{X_0}^X(\exp(-X_t(\phi))) = \exp(-X_0(u_t^\phi)), \quad (1.1)$$

where $X_t(\phi) = \int \phi(x)X_t(dx)$, and $u_t^\phi(x)$ is the unique solution of the evolution equation

$$u_t(x) = S_t\phi(x) - \int_0^t S_{t-s}(u_s^{1+\beta})(x) ds \quad (1.2)$$

for $(t, x) \in Q := (0, \infty) \times \mathbb{R}^d$. $(S_t)_{t \geq 0}$ denotes the transition semigroup of the isotropic (symmetric) α -stable process in \mathbb{R}^d . (See Theorem 4.4.1 of [7] for the existence and uniqueness of solutions to (1.2) and the derivation of (1.1).) The generator of the α -stable process is the fractional Laplacian $\Delta_\alpha = -(-\Delta)^{\frac{\alpha}{2}}$. We will use the probabilistic convention, so that Δ_α corresponds to the α -stable

process; the actual “fraction” associated to this operator, and the parameter commonly used in the PDE literature, is then $\frac{\alpha}{2} \in (0, 1)$. In the singular integral formulation, Δ_α is defined as

$$\Delta_\alpha f(x) := \lim_{\epsilon \rightarrow 0^+} -a_{\alpha,d} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} \chi_\epsilon(|x - y|) dy$$

for a constant $a_{\alpha,d} > 0$, where

$$\chi_\epsilon(r) = \begin{cases} 1 & \text{if } r > \epsilon, \\ 0 & \text{if } r \in [0, \epsilon]. \end{cases}$$

The integral equation (1.2) corresponds to the fractional PDE

$$\partial_t u = \Delta_\alpha u - u^{1+\beta} \tag{1.3}$$

with initial data $u_0 = \phi$. When they exist, solutions of (1.3) and (1.2) generally coincide. In Section 2.3 we define weak solutions to (1.3) (see Definition 2.4) and make this correspondence rigorous in certain cases. In addition to studying solutions of (1.3) (or (1.2)) as a means of proving properties of X , we also prove novel results concerning the existence and non-existence of solutions to (1.3) with very singular initial conditions.

We are interested in the density of X_t for $t > 0$. It is a classical result of Fleischmann [12] that X_t is absolutely continuous if and only if $d < \frac{\alpha}{\beta}$. (When $d > \frac{\alpha}{\beta}$, X_t has a Borel support of dimension $\frac{\alpha}{\beta}$; see Theorem 7.3.1 of [6]. There is a very precise description of the support of X_t due to Perkins [32] when $\beta = 1$.) This work is concerned only with absolutely continuous case, and we restrict to it now.

Assumption. For the remainder of this work, we assume $\beta \in (0, 1]$, $\alpha \in (0, 2)$, and $\beta < \frac{\alpha}{d}$.

Under this assumption, X_t has a density for $t > 0$. We denote it $X_t(x)$, so that $X_t(dx) = X_t(x)dx$. A priori, a generic density is only defined up to Lebesgue-null sets. Much of our work concerns the behaviour of the density on such sets, so this is not sufficient. In particular, we need the object

$$\mu(X_t) := \int_{\mathbb{R}^d} X_t(x) \mu(dx) \tag{1.4}$$

to be well-defined for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. Let $B(x, r)$ denote the closed ball of radius $r > 0$ around $x \in \mathbb{R}^d$. We define

$$X_t^\epsilon(x) = \frac{X_t(B(x, \epsilon))}{|B(x, \epsilon)|}, \tag{1.5}$$

where $|A|$ denotes the Lebesgue measure of $A \subset \mathbb{R}^d$. By the Lebesgue Differentiation Theorem, $X_t^\epsilon(x)$ converges as $\epsilon \downarrow 0$ for Lebesgue-a.e. $x \in \mathbb{R}^d$, and consequently

$$X_t(x) := \liminf_{\epsilon \downarrow 0} X_t^\epsilon(x) \tag{1.6}$$

is a density for X_t . In fact, more holds.

Lemma 1.1. *For fixed $x \in \mathbb{R}^d$, $X_t^\epsilon(x) \rightarrow X_t(x)$ $P_{X_0}^X$ -a.s as $\epsilon \downarrow 0$. Moreover, for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, $X_t^\epsilon(x) \rightarrow X_t(x)$ for μ -a.e. x almost surely and $\mu(X_t^\epsilon) \rightarrow \mu(X_t)$ in $L^1(P_{X_0}^X)$ as $\epsilon \downarrow 0$.*

The above is in fact an abridged version of Lemma 2.1, which is proved in Section 2.2. Alongside this, as we discuss in Section 2.3, the evolution equation has a unique solution with initial condition given by a finite measure. Consequently, both sides of (1.1) are well-defined when ϕ is replaced with a finite measure. The measure case can then be summarized as follows: (see Lemma 2.9 for a precise statement) for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, there is a unique solution $u_t^\mu(x)$ to (1.3) on Q such that $u_t^\mu \rightarrow \mu$ weakly in the sense of measures as $t \downarrow 0$. The solution u_t^μ satisfies

$$E_{X_0}^X(\exp(-\mu(X_t))) = \exp(-X_0(u_t^\mu)).$$

In the above, $\mu(X_t)$ is defined by (1.4) with the density $X_t(x)$ from (1.6).

Another approach to specifying a canonical version of the density is via a Green's function representation for X_t , which is the method used by Fleischmann, Mytnik and Wachtel in [13] and other works. This version of the density is given by

$$X_t(x) = X_0 * p_t(x) + \int_{(0,t] \times \mathbb{R}^d} p_{t-s}(y-x) M(d(s,y)). \quad (1.7)$$

In the above, p_t is the transition density of the symmetric α -stable process (see Section 2.1) and the measure M is a compensated stable martingale measure associated to X . Although we do not show it here, the density above will agree with the version we use.

The canonical measure associated with the (α, β) -superprocess, which we denote by \mathbb{N}_0 , is a σ -finite measure supported on $\mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^d))$, the space of càdlàg $\mathcal{M}_F(\mathbb{R}^d)$ -valued paths. In the construction of superprocesses as scaling limits of discrete spatial branching models, one takes the number of individuals in the population to infinity while their masses are simultaneously scaled to 0. The canonical measure \mathbb{N}_0 describes the behaviour of a cluster started from a single ancestor at the origin given that the cluster survives for some positive time interval. Likewise, \mathbb{N}_x describes the superprocess descending from an ancestor located at $x \in \mathbb{R}^d$. \mathbb{N}_x can be obtained as the weak limit $\lim_{N \rightarrow \infty} NP_{N^{-1}\delta_x}^X(X \in \cdot)$. A version of this convergence result for branching particle systems is proved for $\beta = 1$ in [34]; see Theorem II.7.3 of that work.

In Section 2.5 we describe the relationship between canonical measure and the superprocess with law $P_{X_0}^X$. The following formula will be used quite frequently: for any $x \in \mathbb{R}^d$, for $t > 0$,

$$\mathbb{N}_x(X_t \neq 0) = U_t := \left(\frac{1}{\beta t}\right)^{\frac{1}{\beta}}. \quad (1.8)$$

(This is shown, for example, in (5.4.2) of [6].) We remark that U_t is the maximal solution on $(0, \infty)$ to the ODE $u' = -u^{1+\beta}$. By (1.8), when considering X_t under \mathbb{N}_x for some fixed $t > 0$, one is essentially working with a finite measure. The formula (1.8) is a consequence of the fact that the canonical measure also shares a close relationship with the dual evolution equation. If $u_t^\phi(x)$ is as in (1.2) for $\phi \in \mathcal{B}_b^+$, then

$$\mathbb{N}_x(1 - \exp(-(X_t, \phi))) = u_t^\phi(x). \quad (1.9)$$

As with $P_{X_0}^X$, the above relationship can be generalized to include measures when X_t has a density. If u_t^μ is the unique solution to (1.3) with initial condition $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, (see Lemma 2.9 for details) then

$$\mathbb{N}_x(1 - \exp(-\mu(X_t))) = u_t^\mu(x). \quad (1.10)$$

1.2 Propagation theorems

We now state two theorems related to the propagation of X_t . One is a well-known property of superprocesses associated to stable motions. The other is new and concerns the density at a fixed point; it is a consequence of a theorem about solutions to (1.3), which we also state. We begin with the former, classical result. It is originally due to Perkins, whose proof of the $\beta = 1$ case appeared in [33]; Evans and Perkins gave another proof in [11]. The proof for $\beta < 1$ appears in a more recent work of Li and Zhou [21]. We define $\text{supp}(\mu)$ to be the closed support of $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and will refer to it as the support of μ .

Theorem A: Perkins (1990), Li and Zhou (2008). *For $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and $t > 0$,*

$$P_{X_0}^X(\text{supp}(X_t) = \mathbb{R}^d \text{ or } \emptyset) = 1.$$

Similarly, $\mathbb{N}_0(\text{supp}(X_t) = \mathbb{R}^d \mid X_t \neq 0) = 1$.

The superprocesses we study are critical and therefore go extinct almost surely. That is, $P_{X_0}^X(X_t \neq 0 \text{ for all } t > 0) = 0$. (This is analogous to the almost sure extinction of critical branching processes, whose spatial analogues have scaling limits given by superprocesses.) Thus, the above statement can be understood as saying that, conditioned on non-extinction at time $t > 0$, $\text{supp}(X_t) = \mathbb{R}^d$ a.s.

Theorem A is sometimes called *instantaneous propagation*. Regardless of the choice of initial measure, X_t has mass “everywhere” in \mathbb{R}^d ; if $X_0 = \delta_x$, varying x over \mathbb{R}^d has no influence on the support of X_t , which is always equal to \mathbb{R}^d . The condition $\text{supp}(X_t) = \mathbb{R}^d$ is equivalent to: $X_t(U) > 0$ for every open set $U \subset \mathbb{R}^d$, where $X_t(U) = (X_t, 1_U)$ and 1_U is the indicator function of U . So for open U , $X_t \neq 0$ implies $X_t(U) > 0$ almost surely. For $\lambda > 0$, consider now $u^{\lambda 1_U}$, which is the solution to

$$\partial_t u = \Delta_\alpha u - u^{1+\beta}, \quad u_0 = \lambda 1_U.$$

By (1.9),

$$\mathbb{N}_x(1 - \exp(-\lambda X_t(U))) = u_t^{\lambda 1_U}(x).$$

Taking $\lambda \rightarrow \infty$ with standard limiting arguments, we observe that

$$\lim_{\lambda \rightarrow \infty} \mathbb{N}_x(1 - \exp(-\lambda X_t(U))) = \mathbb{N}_x(X_t(U) > 0).$$

By instantaneous propagation, the right hand side is equal to $\mathbb{N}_x(X_t \neq 0)$, which, by translation invariance, is equal for any choice of x . The implication is that

$$\lim_{\lambda \rightarrow \infty} u_t^{\lambda 1_U}(x) = \mathbb{N}_0(X_t \neq 0) \quad \text{for all } x \in \mathbb{R}^d.$$

Hence $u_t^{\infty 1_U} = \lim_{\lambda \rightarrow \infty} u_t^{\lambda 1_U}$ is constant in space for each $t > 0$. In particular, by (1.8), $u_t^{\infty 1_U}(x) = U_t$ for all $x \in \mathbb{R}^d$.

Remark 1.2. *We can similarly argue that $\mathbb{N}_0(X_t(\phi) > 0) = U_t$ for any measurable function $\phi \geq 0$ which is positive on a set of positive Lebesgue measure. In particular, $\lim_{\lambda \rightarrow \infty} u_t^{\lambda \phi} = U_t$ for any such function. These probabilistic results, which we believe are not widely known in the PDE community, imply the non-existence of positive solutions to (1.3) whose singular sets (see (1.20)) have positive Lebesgue measure.*

The next result concerns solutions to (1.3) and is due to Chen, Véron, and Wang [3]. To state it, we introduce some notation. Let u_t^λ denote the solution to (1.3) with initial data $\lambda\delta_0$, and define $u_t^\infty = \lim_{\lambda \rightarrow \infty} u_t^\lambda$. Let $\beta^*(\alpha) = \frac{\alpha}{d+\alpha}$. We will generally view d as fixed and therefore omit the dependence of $\beta^*(\alpha)$ on d .

Theorem B: Chen, Véron, Wang (2017), Chen, Véron (2019).

- (a) Let $\beta \leq \beta^*(\alpha)$. Then $u_t^\infty(x) = U_t$ for all $(t, x) \in Q$.
(b) Let $\beta^*(\alpha) < \beta < \frac{\alpha}{d}$. Then $u_t^\infty(x)$ satisfies

$$C_1 \frac{t^{-\frac{1}{\beta}}}{1 + |t^{-\frac{1}{\alpha}}x|^{d+\alpha}} \leq u_t^\infty(x) \leq C_2 \frac{t^{-\frac{1}{\beta}} \log(e + |t^{-\frac{1}{\alpha}}x|)}{1 + |t^{-\frac{1}{\alpha}}x|^{d+\alpha}}$$

for all $(t, x) \in Q$ for constants $0 < C_1 < C_2$.

We introduce some terminology: if a solution to (1.3), or a limit of solutions is equal to U_t , then we will call the solution (or limit) *flat*, because these solutions are constant for fixed $t > 0$. Otherwise it is *non-flat*. Thus u_t^∞ is flat when $\beta \leq \beta^*(\alpha)$ and non-flat when $\beta^*(\alpha) < \beta < \frac{\alpha}{d}$.

With the exception of the case that $\beta = \beta^*(\alpha)$, this result was proved in [3], while the $\beta = \beta^*(\alpha)$ case was proved by the first two authors of that paper in [4]. Rather surprisingly, depending on the parameters (α, d, β) , $u_t^\infty(x)$ is either flat or has almost the same asymptotic decay as the heat kernel associated to Δ_α .

Let us consider what this means for X_t . From (1.10) with $\mu = \lambda\delta_x$, we have

$$\mathbb{N}_0(1 - \exp(-\lambda X_t(x))) = u_t^\lambda(x),$$

where $X_t(x)$ is the density of X_t at x . Taking $\lambda \rightarrow \infty$ yields

$$\mathbb{N}_0(X_t(x) > 0) = u_t^\infty(x), \tag{1.11}$$

and Theorem B implies the following new result for X_t .

Theorem 1.3. *The following hold under \mathbb{N}_{x_0} and $P_{X_0}^X$ for $x_0 \in \mathbb{R}^d$ and $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$.*

- (a) Let $\beta \leq \beta^*(\alpha)$ and $t > 0$. Then for fixed $x \in \mathbb{R}^d$, $X_t(x) > 0$ almost surely on $\{X_t \neq 0\}$. In particular, $X_t(x) > 0$ for Lebesgue-a.e. $x \in \mathbb{R}^d$ almost surely on $\{X_t \neq 0\}$.
(b) Let $\beta^*(\alpha) < \beta < \frac{\alpha}{d}$. For $t > 0$, $\{x : X_t(x) > 0\}$ has finite Lebesgue measure almost surely. Moreover, we have

$$C_1 \frac{t^{-\frac{1}{\beta}}}{1 + |t^{-\frac{1}{\alpha}}x|^{d+\alpha}} \leq \mathbb{N}_0(X_t(x) > 0) \leq C_2 \frac{t^{-\frac{1}{\beta}} \log(e + |t^{-\frac{1}{\alpha}}x|)}{1 + |t^{-\frac{1}{\alpha}}x|^{d+\alpha}} \tag{1.12}$$

for all $(t, x) \in Q$ for constants $0 < C_1 < C_2$.

Theorem B(a) implies part (a) of the above as follows: by (1.8) and (1.11), when $\beta \leq \beta^*(\alpha)$ we have

$$\mathbb{N}_{x_0}(X_t(x) > 0) = \mathbb{N}_{x_0}(X_t \neq 0).$$

Since $\{X_t(x) > 0\} \subseteq \{X_t \neq 0\}$, we have $\mathbb{N}_{x_0}(X_t(x) = 0 | X_t \neq 0) = 0$, and Fubini's theorem implies that $X_t(x) > 0$ for Lebesgue-a.e. x almost surely under $\mathbb{N}_{x_0}(\cdot | X_t \neq 0)$. In part (b), the fact that $\{x : X_t(x) > 0\}$ has finite Lebesgue measure under \mathbb{N}_{x_0} follows from integrability of the

upper bound in (1.12) and Fubini's theorem. For $P_{X_0}^X$, the result then follows from the cluster representation for the superprocess (see Section 2.5).

In keeping with the terminology of instantaneous propagation for the behaviour from Theorem A, we propose to call *strong instantaneous propagation* the property described for $\beta \leq \beta^*(\alpha)$. Both results say, in some sense, that X_t has mass "everywhere." Instantaneous propagation describes this on the level of mass on open sets, whereas strong instantaneous propagation concerns the density at a fixed point.

A priori, taken as an immediate consequence of Theorem B, we lack a probabilistic intuition for Theorem 1.3, as Theorem B is proved using analytical methods. In Section 3 we give a probabilistic proof of part (a). The arguments there are prototypical of other arguments used later on to prove some of our other results. We state as an open problem the task of giving a probabilistic proof of Theorem 1.3(b).

1.3 Strict positivity

We now strengthen one of the conclusions of Theorem 1.3 when a continuous version of the density $X_t(\cdot)$ exists. To do so we recall a dichotomy due to Fleischmann, Mytnik and Wachtel [13].

Theorem C: Fleischmann, Mytnik, Wachtel (2010). *Fix $t > 0$ and consider X_t under $P_{X_0}^X$ with $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$.*

(a) *If $d = 1$ and $\alpha > 1 + \beta$, then there is a version of the density $X_t(\cdot)$ which is locally η -Hölder continuous for all $\eta < \frac{\alpha}{1+\beta} - 1$.*

(b) *If $d > 1$, or $d = 1$ and $\alpha \leq 1 + \beta$, then $\|1_U X_t(\cdot)\|_\infty = \infty$ for all open sets U almost surely on $\{X_t \neq 0\}$.*

We will refer to the parameter regime $d = 1, \alpha > 1 + \beta$ as the *continuous case*. In this case it is understood that we will always work with the continuous version of the density $x \rightarrow X_t(x)$, and it is easy to see that this version agrees with the version defined in (1.6). The case from Theorem C(b) is then the *discontinuous case*. For $d = 1$, in the parameter regime with $\alpha > 1 + \beta$ and $\beta < \beta^*(\alpha)$, the density enjoys both continuity and strong instantaneous propagation, and we are able to show the following strict positivity result.

Theorem 1.4. *Let $d = 1, \alpha > 1 + \beta$ and $\beta < \beta^*(\alpha) = \frac{\alpha}{1+\alpha}$. Then for $t > 0$,*

$$X_t(x) > 0 \text{ for all } x \in \mathbb{R} \text{ almost surely on } \{X_t \neq 0\}$$

under both $P_{X_0}^X$ and \mathbb{N}_{x_0} .

Remark 1.5. *Strong instantaneous propagation still holds when $\beta = \beta^*(\alpha)$, but the proof of the above does not, and this case is left open. Of course, $\{x : X_t(x) > 0\}$ has finite Lebesgue measure when $\beta > \beta^*(\alpha)$ by Theorem 1.3(b).*

This is an interesting property and is perhaps best understood in the context of similar results for non-negative solutions to stochastic PDE (SPDE). Consider first the stochastic heat equation

$$\partial_t Y_t(x) = \Delta Y_t(x) + Y_t(x)^\gamma \dot{W}(t, x) \tag{1.13}$$

on $(0, \infty) \times \mathbb{R}$, where $\dot{W}(t, x)$ is space-time white Gaussian noise and $\gamma > 0$. It was shown by Mueller in [25] that if $\gamma \geq 1$, then a non-negative solution $Y_t(x)$ satisfies $Y_t(x) > 0$ for all $(t, x) \in (0, \infty) \times \mathbb{R}$. On the other hand, Perkins and Mueller [26] have shown that when $\gamma < 1$, if

Y_0 has compact support then Y_t has compact support for all $t > 0$ a.s. The case $\gamma = \frac{1}{2}$ corresponds to the density of super-Brownian motion, in which case the result is originally due to Iscoe [17].

In the SPDE associated to the (α, β) -superprocess with $\beta < 1$ in dimension one, the diffusion term is replaced with fractional diffusion and the noise is stable rather than Gaussian. In particular, the density $X_t(x)$ of the (α, β) -superprocess solves the SPDE

$$\partial_t X_t(x) = \Delta_\alpha X_t(x) + X_{t-}(x)^{\frac{1}{1+\beta}} \dot{L}(t, x), \quad (1.14)$$

where $\dot{L}(t, x)$ is a spectrally positive white stable noise of index $1 + \beta$. The Green's function representation (1.7) established in [13] can be viewed as a mild form of solution but only applies at fixed times. Alternatively, weak solutions to the SPDE obtained by replacing Δ_α with Δ in (1.14) were studied in [27]. The key results of that work (Propositions 4.1 and 5.1) could be generalized to construct weak solutions to (1.14).

In this context, Theorem 1.4 is a (fixed time) strict positivity result for a fractional SPDE with stable noise, and it is the first such result that we are aware of. Furthermore, because $\{x : X_t(x) > 0\}$ has finite Lebesgue measure when $\beta > \beta^*(\alpha)$, it is apparent that the interplay of fractional diffusion and stable noise with non-Lipschitz coefficients lead to non-trivial behaviour which is not seen in SPDE with Gaussian noise like (1.13) (c.f. the results from [25] and [26] discussed above).

1.4 Charging results for general measures

In order to best understand the results that follow, it will be useful to view Theorem A and Theorem 1.3 from a unified perspective. Let m_U denote the Lebesgue measure restricted to $U \subset \mathbb{R}^d$. Theorem A states that if U is open (so m_U is non-zero), $\int X_t(x) m_U(dx) > 0$ a.s. on $\{X_t \neq 0\}$. Thus $X_t(\cdot)$ has mass everywhere in \mathbb{R}^d at a *macroscopic* level—the level of open sets. In terms of the PDE (1.3), the equivalent statement is that $\lim_{\lambda \rightarrow \infty} u_t^{\lambda m_U}(x)$ is equal to the flat solution U_t . The Lebesgue measure on an open set U is, locally speaking, the most spread out measure on \mathbb{R}^d . The least spread out measure, that is the most concentrated measure, is δ_x for some $x \in \mathbb{R}^d$. Theorem 1.3 states that $X_t(\cdot)$ integrated against the measure δ_x , i.e. $X_t(x)$, is positive a.s. on $\{X_t \neq 0\}$ if and only if $\beta \leq \frac{\alpha}{d+\alpha}$, again with an equivalent interpretation that $\lim_{\lambda \rightarrow \infty} u^{\lambda \delta_x} = U_t$. In this case, $X_t(\cdot)$ has mass everywhere at a *microscopic* level—at a single point. Taken together, these two results describe the almost sure behaviour of $X_t(\cdot)$ on both the most concentrated and least concentrated measures on \mathbb{R}^d : δ_x and m_U , respectively. Our results that follow interpolate between these extremal cases to describe the almost sure behaviour of $\mu(X_t)$ for general measures μ . We are therefore also able to answer some questions about flatness and non-flatness of solutions to (1.3) that are the limits as $\lambda \rightarrow \infty$ of solutions with initial condition $\lambda\mu$. Indeed, a general principle underlying this work can be summarized as follows:

$$\mu(X_t) > 0 \text{ almost surely on } \{X_t \neq 0\} \text{ if and only if } \lim_{\lambda \rightarrow \infty} u_t^{\lambda\mu} = U_t.$$

To formalize this, we recall from (1.10) that

$$\mathbb{N}_x(1 - \exp(-\lambda\mu(X_t))) = u_t^{\lambda\mu}(x).$$

As $\lambda \rightarrow \infty$ the left hand side increases to $\mathbb{N}_x(\mu(X_t) > 0) < \infty$.

Definition 1.6. Let $u_t^{\infty\mu}(x) = \lim_{\lambda \rightarrow \infty} u_t^{\lambda\mu}(x)$.

We therefore have

$$\mathbb{N}_x(\mu(X_t) > 0) = u_t^{\infty\mu}(x).$$

Since $\{\mu(X_t) > 0\} \subseteq \{X_t \neq 0\}$, (1.8) and the above imply that

$$\sup_x u_t^{\infty\mu}(x) \leq U_t < \infty. \quad (1.15)$$

In these terms, the principle above reads that $\mu(X_t) > 0$ a.s. on $\{X_t \neq 0\}$ if and only if $u_t^{\infty\mu} = U_t$.

We say that $X_t(\cdot)$ charges μ if $\mu(X_t) > 0$. Our results give a nearly sharp answer to the question: *which measures does $X_t(\cdot)$ charge almost surely?* The characterization of measures μ which X_t does and does not charge is, roughly speaking, in terms of the size of the support of μ . We characterize this in two ways: directly via Hausdorff measure, and by local mass distribution properties. Although the concept of the “dimension of a measure” is ill-defined, both of these correspond to a notion of dimension, and in particular are closely related to the Hausdorff dimension of the measure’s support.

We now introduce the critical parameter associated to measures with s -dimensional supports for $s \in [0, d]$.

Definition 1.7. For $s \in [0, d]$, let $\beta^*(\alpha, s) = \frac{\alpha}{(d-s)+\alpha}$.

We remark that $\beta^*(\alpha, 0) = \beta^*(\alpha)$, the critical parameter from Theorem 1.3. Furthermore, $\beta^*(\alpha, d) = 1$, which is the critical parameter implicit in Theorem A, since the (α, β) -superprocess has instantaneous propagation for all $\beta \leq 1$.

For a Borel set $\mathcal{S} \subset \mathbb{R}^d$, we write $\mathcal{H}^s(\mathcal{S})$ to denote the x^s -Hausdorff measure of \mathcal{S} and $\dim(\mathcal{S})$ to denote its Hausdorff dimension. In the following, $x_0 \in \mathbb{R}^d$ and $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ are arbitrary and X_0 is assumed to be non-zero.

Theorem 1.8. Let $s \in [0, d]$ and suppose $\beta \leq \beta^*(\alpha, s)$. If $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfies $\mathcal{H}^s(\text{supp}(\mu)) > 0$, then for $t > 0$, $\mu(X_t) > 0$ a.s. on $\{X_t \neq 0\}$ under $P_{X_0}^X$ and \mathbb{N}_{x_0} , and $u_t^{\infty\mu} = U_t$.

Recall that if $\dim(\text{supp}(\mu)) = s$, then $\mathcal{H}^{s'}(\text{supp}(\mu)) = +\infty$ for all $s' < s$. In particular, when $\beta < \beta^*(\alpha, s)$ the conclusions above hold when $\dim(\text{supp}(\mu)) \geq s$. The assertion in Theorem 1.8 that $u_t^{\infty\mu} = U_t$ complements Theorem G of [4], which proves a similar result when s is an integer and the set (in this setting $\text{supp}(\mu)$) is a line or hyperplane. Their result is stated in the language of initial traces, which we discuss later. Our critical parameter $\beta^*(\alpha, s)$ agrees with the critical parameter of their result.

We observe that $\beta^*(\alpha, \alpha) = \frac{\alpha}{d}$. Since $\beta^*(\alpha, s)$ is increasing in s , it follows that, if $\beta < \frac{\alpha}{d}$, then $\beta < \beta^*(\alpha, s)$ for all $s \geq \alpha$. The first condition is required for the existence of the density, and hence if $s \geq \alpha$, $\beta < \beta^*(\alpha, s)$ holds over the entire parameter set in which we are interested. In other words, whenever the density exists, almost surely it charges any $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ such that $\dim(\text{supp}(\mu)) \geq \alpha$. (This also complements an observation made in [4].)

In order to state the complimentary result when $\beta > \beta^*(\alpha, s)$, we introduce the following *mass distribution properties*. We say that a measure $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfies condition (F1) with parameter $s \in [0, d]$, or simply (F1)- s , if:

(F1)- s For some constant $\overline{C} > 0$, for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu(B(x, r)) \leq \overline{C}r^s.$$

Similarly, we say that $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfies property (F2) with parameter $s \in [0, d]$, or (F2)- s , if:

(F2)-s For some constant $\underline{C} > 0$, for all $x \in \text{supp}(\mu)$ and $r \in (0, 1]$,

$$\mu(B(x, r)) \geq \underline{C}r^s.$$

The relationship between measures satisfying these properties and the sets which support them is well-known. In short, if μ satisfies (F1)-s and $\mu(1) > 0$, then its support has dimension at least s ; and if μ satisfies (F2)-s, its support has dimension at most s . We now treat each case with more detail.

Let $\mathcal{S} \subset \mathbb{R}^d$ be a Borel set. A classical result due to Frostman (it is often called Frostman's Lemma) asserts that there exists a non-zero measure $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfying (F1)-s with $\text{supp}(\mu) \subseteq \mathcal{S}$ if and only if $\mathcal{H}^s(\mathcal{S}) > 0$ (which implies $\dim(\mathcal{S}) \geq s$). See Theorem 8.8 in [22] for a proof. Measures satisfying (F1) are much nicer to work with than general measures with the same support. In fact, we first prove Theorem 1.8 for (F1)-s measures. It is noteworthy that we can generalize from (F1)-s measures to general measures whose supports have positive \mathcal{H}^s -measure, because a measure with s -dimensional support does not necessarily satisfy (F1)-s' for any $s' \in (0, s]$. For a more detailed discussion of this matter, we refer the interested reader to Section 5.

In contrast with (F1), which implies that the mass of μ is spread out in a certain sense, (F2) tells us that the mass of μ is not *too* spread out in that same sense. In particular, if μ satisfies (F2)-s, then $\dim(\text{supp}(\mu)) \leq s$ (for example, see Section 8.7 of [16]).

We now state our result concerning the behaviour when $\beta > \beta^*(\alpha, s)$. For a closed set $\mathcal{S} \subset \mathbb{R}^d$, let $d(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} |x - y|$ denote the distance between $x \in \mathbb{R}^d$ and \mathcal{S} . We define $\text{diam}(\mathcal{S}) = \sup\{|x - y| : x, y \in \mathcal{S}\}$. Finally, let $\mathcal{M}_F(\mathcal{S})$ denote the space of finite measures μ with $\text{supp}(\mu) \subseteq \mathcal{S}$. The restriction $s < \alpha$ in the following is because this is required for $(\beta^*(\alpha, s), \frac{\alpha}{d})$ to be non-empty. Of course, we also implicitly require $\beta \leq 1$.

Theorem 1.9. *Suppose that $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfies (F2)-s for some $s \in [0, \alpha)$ and has compact support $\text{supp}(\mu) = \mathcal{S}$. Let $\beta^*(\alpha, s) < \beta < \frac{\alpha}{d}$.*

(a) *For $t > 0$, $P_{X_0}^X(\mu(X_t) = 0 | X_t \neq 0) > 0$ and $\mathbb{N}_{x_0}(\mu(X_t) = 0 | X_t \neq 0) > 0$ for all non-zero $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and all $x_0 \in \mathbb{R}^d$.*

(b) *$\mathbb{N}_x(\mu(X_t) > 0) = u_t^{\infty\mu}(x)$ satisfies the following: there are constants $C_{1.16} > c_{1.16} > 0$ such that for all $(t, x) \in Q$,*

$$\frac{c_{1.16}t^{-\frac{1}{\beta}}}{1 + |t^{-\frac{1}{\alpha}}d(x, \mathcal{S})|^{d+\alpha}} \leq \mathbb{N}_x(\mu(X_t) > 0) \leq C_{1.16} \left[t^{-\frac{1}{\beta} - \frac{s}{\alpha}} \vee t^{-\frac{1}{\beta}} \right] \frac{\log(e + t^{-\frac{1}{\alpha}}d(x, \mathcal{S}))}{1 + |t^{-\frac{1}{\alpha}}d(x, \mathcal{S})|^{d+\alpha}}. \quad (1.16)$$

In particular, $\mathbb{N}_x(\mu(X_t) > 0)$ vanishes uniformly on $\{x : d(x, \mathcal{S}) \geq \rho\}$ as $t \downarrow 0$, for all $\rho > 0$.

(c) *If, in addition, μ satisfies (F1)-s, then there is a constant $c_{1.17} > 0$ such that for all $x \in \mathbb{R}^d$ and $t \in (0, 1]$,*

$$\mathbb{N}_x(\mu(X_t) > 0) \geq c_{1.17} \left[\frac{t^{-\frac{1}{\beta} - \frac{s}{\alpha}}}{1 + |t^{-\frac{1}{\alpha}}[d(x, \mathcal{S}) + \text{diam}(\mathcal{S})]|^{d+\alpha}} \right]. \quad (1.17)$$

(d) *For any $\nu \in \mathcal{M}_F(\mathcal{S})$, the conclusions of part (a) hold when μ is replaced with ν . Furthermore, for constants $C_{1.18} > c_{1.18} > 0$ we have*

$$\frac{c_{1.18}t^{-\frac{1}{\beta}}}{1 + |t^{-\frac{1}{\alpha}}d(x, \text{supp}(\nu))|^{d+\alpha}} \leq \mathbb{N}_x(\nu(X_t) > 0) \leq C_{1.18} \left[t^{-\frac{1}{\beta} - \frac{s}{\alpha}} \vee t^{-\frac{1}{\beta}} \right] \frac{\log(e + t^{-\frac{1}{\alpha}}d(x, \mathcal{S}))}{1 + |t^{-\frac{1}{\alpha}}d(x, \mathcal{S})|^{d+\alpha}}. \quad (1.18)$$

The upper and lower bounds in (1.16) and (1.18) generalize the bounds for the point case (i.e. $\mu = \delta_0$) given in Theorem 1.3. In order to obtain matching asymptotics (in terms of the power

of t) as $t \downarrow 0$, we require additional regularity of μ , as in part (c). We also note that for $x \in \mathcal{S}$ and small t , instead of the upper bound in (1.16) and (1.18) one should use $\mathbb{N}_x(\mu(X_t) > 0) \leq U_t$, which gives an upper bound of order $t^{-\frac{1}{\beta}}$.

Theorems 1.8 and 1.9 give a sharp picture of the behaviour of $X_t(\cdot)$ when integrated against an s -dimensional measure. Together, they give the analogues of the two parts of Theorem 1.3 when we replace δ_0 with an arbitrary measure. Restrict to the event $\{X_t \neq 0\}$. Then we have that when μ is *at least* s -dimensional (i.e. $\mathcal{H}^s(\text{supp}(\mu)) > 0$) and $\beta \leq \beta^*(\alpha, s)$, $X_t(\cdot)$ charges μ a.s., and when μ is *at most* s -dimensional (by this we mean μ satisfies (F2)- s) and $\beta > \beta^*(\alpha, s)$, the probability that $X_t(\cdot)$ charges μ has spatial decay similar to an α -stable heat kernel. Theorem A and Theorem 1.3 are the special cases of $s = d$ and $s = 0$, respectively; our results cover $s \in [0, d]$. Furthermore, for $s \in [0, \alpha)$ there is a non-trivial transition as we vary β over the critical value $\beta^*(\alpha, s)$. For $s \in [\alpha, d]$ the density charges any s -dimensional set or measure almost surely for all $\beta < \frac{\alpha}{d}$.

In the flat case, our most general result requires no conditions on μ apart from a size condition on its support, i.e. that $\mathcal{H}^s(\text{supp}(\mu)) > 0$. This is achieved by first proving the result for (F1)- s measures and then using a comparison argument which allows us to bootstrap to the general case. For more details, see Section 5. In the non-flat case, our most general result is Theorem 1.9(d), which holds for all ν whose support is contained in that of an (F2)- s measure and is therefore at most s -dimensional. It is an attractive open problem to show that this can be extended to measures supported on general compact sets of dimension at most s . With the exception of the critical case $\beta = \beta^*(\alpha, s)$, this would provide a complete characterization of $\mathbb{N}_x(\mu(X_t) > 0)$ for compactly supported μ .

Remark 1.10. *For the special case of $\beta = 1$, which requires $d = 1$ and $\alpha \in (1, 2)$ in order for the density to exist, we observe that $\beta^*(\alpha, s) < 1$ for all $s \in [0, 1)$. By Theorem 1.9, this implies that for any $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ with $\dim(\text{supp}(\mu)) < 1$, the $(\alpha, 1)$ -superprocess in \mathbb{R}^1 fails to charge μ with positive probability on $\{X_t \neq 0\}$.*

To obtain matching upper and lower bounds on $\mathbb{N}_x(\mu(X_t) > 0)$ as $t \downarrow 0$ in the non-flat case requires μ to satisfy both (F1)- s and (F2)- s (see Theorem 1.9(c)). This is a strong condition which is sometimes called Ahlfors-David regularity. Still, the properties (F1)- s and (F2)- s are satisfied by many measures which are, in an appropriate sense, uniform over some s -dimensional set \mathcal{S} . First consider the case where s is an integer. If \mathcal{S} is a rectifiable curve in \mathbb{R}^d and μ is its length measure, then μ satisfies (F1)-1 and (F2)-1. If \mathcal{S} is a surface, or two-dimensional manifold, then its surface measure μ satisfies (F1)-2 and (F2)-2. (In view of the discussion following Theorem 5.1, this implies that for every $\beta < \frac{\alpha}{d}$, if μ is the surface or volume measure for a manifold of dimension at least two, then $\mu(X_t) > 0$ a.s. when conditioned on survival). For non-integer s , it is known that if \mathcal{S} is a self-similar Cantor set of dimension s , then its uniform measure μ satisfies (F1)- s and (F2)- s . (For example, see the discussion in Section 3 in [23] or Section 1.2 of [5].) Our results can be applied directly to all of the examples above.

Many random sets support measures satisfying (F1) and/or (F2). To illustrate our dichotomy for the (α, β) -superprocess we consider the range of an independent fractional Brownian motion in \mathbb{R}^d with Hurst parameter $H \in (0, 1)$. We denote this process by $(B_t)_{t \geq 0}$ and let $R(B) = \{B_t : t \in [0, 1]\}$. In order to attain both sides of the dichotomy we assume that $H > d^{-1} \vee \alpha^{-1}$, which requires $d \geq 2$ and $\alpha \in (1, 2)$. (As we discuss below, $\dim(R(B)) = H^{-1} \wedge d$. Our assumption guarantees that $\dim(R(B)) = H^{-1} < \alpha$, and we have $\beta^*(\alpha, H^{-1}) < \frac{\alpha}{d}$.) The paths of B_t are η -Hölder continuous for $\eta \in (0, H)$, (for example see Proposition 1.6 of [31]) from which it can be easily shown that the measure μ_B defined by $\mu_B(A) = \int_0^1 1_A(B_t) dt$ satisfies (F2)- s almost surely

(with a random constant $\underline{C}(\omega)$) for all $s < H^{-1}$. Hence if $\beta > \beta^*(\alpha, H^{-1})$, by Theorem 1.9, $\mu_B(X_t) = 0$ with positive probability on $\{X_t \neq 0\}$. On the other hand, $\dim(R(B)) = H^{-1} \wedge d$. This is a special case of Theorem 2.1 of [35]. Given our assumption on H , $\dim(R(B)) = H^{-1} < \alpha$. Hence by Theorem 1.8, (and because $\text{supp}(\mu_B) = R(B)$) if $\beta < \beta^*(\alpha, H^{-1})$ then $\mu_B(X_t) > 0$ a.s. on $\{X_t \neq 0\}$.

Finally, we specialize the flatness and non-flatness results to the continuous case, which we recall from Theorem C is when $d = 1$ and $\alpha > 1 + \beta$. It is an elementary consequence of continuity of $X_t(\cdot)$ that $\mu(X_t) > 0$ if and only if $X_t(x) > 0$ for some $x \in \text{supp}(\mu)$. Theorems 1.8 and 1.9 then imply the following. (Part (b) is mainly included for contrast.)

Theorem 1.11. *Let $d = 1$, $\alpha > 1 + \beta$ and $s \in [0, d]$, and fix $t > 0$.*

(a) *Suppose $\beta \in (\beta^*(\alpha, s), \frac{\alpha}{d})$ (which implies $s < \alpha$) and let $\mathcal{S} = \text{supp}(\mu)$ for a compactly supported measure $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfying (F2)-s. Then $X_t(x) = 0$ for all $x \in \mathcal{S}$ with positive probability on $\{X_t \neq 0\}$ under $P_{X_0}^X$ and \mathbb{N}_{x_0} . Moreover, (1.16) holds when $\mathbb{N}_x(\mu(X_t) > 0)$ is replaced with $\mathbb{N}_x(\{X_t(x) > 0 \text{ for some } x \in \mathcal{S}\})$, and if μ satisfies (F1)-s, then so does (1.17).*

(b) *Suppose $\beta \leq \beta^*(\alpha, s)$ and let $\mathcal{S} \subset \mathbb{R}^d$ be Borel with $\mathcal{H}^s(\mathcal{S}) > 0$. Then almost surely on $\{X_t \neq 0\}$ there is a point $x \in \mathcal{S}$ such that $X_t(x) > 0$.*

Thus far we have generally viewed α as fixed and β as a variable parameter, with the critical β (i.e. $\beta^*(\alpha, s)$) depending on s , the dimension of \mathcal{S} or $\text{supp}(\mu)$. We would also like to draw attention to another perspective. We define the *saturation dimension* associated to the parameters (α, β, d) .

Definition 1.12. *For fixed parameters (α, β, d) , the saturation dimension d_{sat} is*

$$d_{\text{sat}} = d_{\text{sat}}(\alpha, \beta, d) = d + \alpha - \frac{\alpha}{\beta}$$

if this quantity is non-negative. Otherwise, $d_{\text{sat}} = 0$.

Thus d_{sat} is simply the value of s for which $\beta = \beta^*(\alpha, s)$, and therefore it is the critical dimension of the set or measure as pertains to this problem. It is the maximum dimension of a set which the density can fail to charge with positive probability: if $\mathcal{H}^{d_{\text{sat}}}(\mathcal{S}) > 0$, implying $\dim(\mathcal{S}) \geq d_{\text{sat}}$, then the behaviour of X_t is trivial on \mathcal{S} the sense that $\mu(X_t) > 0$ a.s. on $\{X_t \neq 0\}$ for every $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ with $\text{supp}(\mu) = \mathcal{S}$. The fact that $d_{\text{sat}} \leq d$ corresponds exactly to instantaneous propagation (Theorem A). On the other hand, strong instantaneous propagation (i.e. the conclusions of Theorem 1.3(a)) corresponds to parameters for which $d_{\text{sat}} = 0$. A summary of some of the results above in these terms is as follows:

- *If $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ with $\mathcal{H}^{d_{\text{sat}}}(\text{supp}(\mu)) > 0$, then $X_t(\cdot)$ charges μ almost surely on $\{X_t \neq 0\}$.*
- *If $\mathcal{S} = \text{supp}(\mu)$ for some $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ which satisfies (F2)-s with $s < d_{\text{sat}}$ (implying $\dim(\mathcal{S}) < d_{\text{sat}}$), then for every $\nu \in \mathcal{M}_F(\mathcal{S})$, $\nu(X_t) = 0$ with positive probability on $\{X_t \neq 0\}$.*

1.5 The initial trace problem

We now turn our attention to the PDE (1.3) and state some new results concerning the initial trace theory for positive solutions to this equation. We emphasize that we only state results for $\beta < \frac{\alpha}{d}$, which in the PDE literature is called the *subcritical absorption* regime. Following [4], we define the initial trace. For $A \subseteq \mathbb{R}^d$, let $C_c(A)$ denote the space of continuous, compactly supported functions on A . A positive solution u to (1.3) has *initial trace* (\mathcal{S}, ν) , where $\mathcal{S} \subset \mathbb{R}^d$ is closed and ν is a Radon measure satisfying $\nu(\mathcal{S}) = 0$, if

- For all $\xi \in C_c(\mathcal{S}^c)$,

$$\lim_{t \rightarrow 0^+} \int \xi(x) u_t(x) dx = \int \xi d\nu. \quad (1.19)$$

- For every $z \in \mathcal{S}$ and $\rho > 0$,

$$\lim_{t \rightarrow 0^+} \int_{B(z, \rho)} u_t(x) dx = +\infty. \quad (1.20)$$

The set \mathcal{S} is called the *singular set* of u , and \mathcal{S}^c is called the *regular set*. When the singular set is empty, the initial trace reduces to a measure-valued initial condition. That is, a solution with initial trace (\emptyset, ν) is a solution to the initial value problem with $u_0 = \nu$, where the initial measure ν is attained in the sense of (1.19). The inclusion of non-empty singular sets allows for solutions with more general behaviour as $t \downarrow 0$, in particular by allowing non-integrable singularities to form.

Theorems A and B of [4] give general conditions under which a positive solution to (1.3) can be associated to an initial trace (\mathcal{S}, ν) . Our results concern the converse problem of determining if there exists a positive solution with a given initial trace.

The non-fractional analogue of (1.3) is

$$\partial_t u = \Delta u - u^p. \quad (1.21)$$

The initial trace theory for (1.21) is well understood. For an analytic approach which covers a wide range of parameters p (including $p = 1 + \beta$ for $\beta \in (0, 1]$), see Marcus and Véron [24]. Because of the dual relationship with super-Brownian motion, the problem is also amenable to probabilistic analysis. For the case $p = 2$, Le Gall [19] characterized the positive solutions to (1.21) using the Brownian snake.¹

The theory for the fractional equation (1.3) is more recent and is far from complete. Theorem B hints at this, as it shows that existence of a solution with initial trace $(\{0\}, 0)$ (sometimes called a *very singular solution*) depends on the values of the parameters. A one-point singular set is the smallest non-trivial singular set, and so a consequence of the theorem is that when $\beta \leq \beta^*(\alpha)$, the singular initial trace problem is trivial in the sense that if \mathcal{S} is non-empty, the solution equals or exceeds U_t (see also Theorem F of [4]). This issue does not arise for solutions to (1.21) with $p \in (1, 2]$ because the associated superprocess has compact support which is localized near its initial conditions, and hence the event that the support of the superprocess does not intersect a given closed set (or that the superprocess does not charge a measure supported on that set in an appropriate sense) always has positive probability.

Our results about flatness and non-flatness of $u_t^{\infty\mu}$ allow us to advance the theory of existence for solutions to (1.3) with a given initial trace. At this stage we are largely concerned with existence and we do not consider the regularity of solutions. For this reason we study weak solutions. A precise definition of a weak solution to (1.3) with initial trace (\mathcal{S}, ν) is given in Section 7. In that section we also restrict our attention to solutions $u(t, x)$ which are bounded above by U_t , i.e. $u(t, x) \leq U_t$. We define $\mathcal{U} = \mathcal{U}(Q)$ by

$$\mathcal{U} = \{u : Q \rightarrow [0, \infty) : u_t \leq U_t \ \forall t > 0\}.$$

By duality, in particular (1.8), (1.9) and (1.10), \mathcal{U} includes all solutions which admit a probabilistic representation in terms of the (α, β) -superprocess, which includes all solutions obtained as the

¹Probabilistic approaches using superprocesses have been very effective in studying the elliptic equation related to (1.21), $\Delta u = u^p$ with $p \in (1, 2]$. The $p = 2$ case has been studied by Le Gall in [18, 20], again using the Brownian snake. The work of Dynkin and Kuznetsov (see [9, 10]) gives results covering $p \in (1, 2]$, which is the entire range for which the superprocess approach applies.

limit of solutions with initial data in $\mathcal{M}_F(\mathbb{R}^d)$ or \mathcal{B}_b^+ . See also Theorem D of [4], which proves (for classical solutions) that solutions satisfying a mild integrability property are bounded by U_t and hence belong to \mathcal{U} . It remains unresolved if there exist positive solutions to (1.3) which do not belong to \mathcal{U} ; because they cannot be obtained as the limits of solutions with truncated function-valued initial conditions, it is unclear how one might construct one.

We observe that the flat solution U_t has singular set equal to \mathbb{R}^d . Hence a solution with singular set \mathbb{R}^d always exists and we exclude the case $\mathcal{S} = \mathbb{R}^d$ in the following result.

Theorem 1.13. (a) (Non-existence) *Let $\mathcal{S} \subset \mathbb{R}^d$ be closed with $\mathcal{H}^{d_{\text{sat}}}(\mathcal{S}) > 0$. If $u \in \mathcal{U}$ is a solution to (1.3) whose singular set contains \mathcal{S} , then $u_t = U_t$ (and u_t has singular set \mathbb{R}^d). Hence there is no solution to (1.3) in \mathcal{U} with initial trace (\mathcal{S}, ν) for any Radon measure ν .*

(b) (Existence) *Suppose that $\mathcal{S} \subset \mathbb{R}^d$ is compact and there exists $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfying (F2)-s with $s < d_{\text{sat}}$ such that $\mathcal{S} = \text{supp}(\mu)$. Then there exists a weak solution to (1.3) in \mathcal{U} with initial trace $(\mathcal{S}, 0)$. The solution is given by $\lim_{\lambda \rightarrow \infty} u_t^{\lambda\mu}(x) = \mathbb{N}_x(\mu(X_t) > 0)$ and satisfies (1.16).*

The problem of existence for solutions with initial trace $(\mathcal{S}, 0)$ is therefore characterized by the saturation dimension. This generalizes Theorem B(a), which can be viewed as the special case of the above when $d_{\text{sat}} = 0$, as well as Theorem F of [4]. When $d_{\text{sat}} = 0$ there is no transition: all non-empty singular sets fall into the non-existence regime.

We do not address the uniqueness of solutions in this work. It is reasonable to expect that $\beta = \frac{\alpha}{d}$ is the threshold for uniqueness to solutions with non-empty singular sets. That is, given there exists a solution with initial trace (\mathcal{S}, ν) , it is unique if $\beta < \frac{\alpha}{d}$, but uniqueness fails in general if $\beta \geq \frac{\alpha}{d}$. (This is known for the equation when $\alpha = 2$; see [24].) If this is indeed the case, it would imply that the solutions constructed in this work are the unique solutions associated to their traces.

Organization of the paper. The rest of the paper is organized as follows. Section 2 provides background information and preliminary results; the key subsections are Section 2.2, which covers the density of X_t and $\mu(X_t)$, and Section 2.3, which discusses solutions to (1.3) with measure-valued initial data and extends the dual relationship to include finite measures. In Section 3 we give a probabilistic proof of Theorem 1.3(a), and in Section 4 we prove Theorem 1.4. Sections 5 and 6 cover, respectively, flatness and non-flatness of $u_t^{\infty\mu}$ for general measures μ , in particular Theorems 1.8 and 1.9. Finally, we discuss solutions to (1.3) with non-empty singular sets and prove Theorem 1.13 in Section 7.

2 Preliminaries

2.1 Transition densities

We denote by $p_t(x)$ the fundamental solution (or heat kernel) to the fractional heat equation on \mathbb{R}^d . That is, $p_t(x)$ is the solution to

$$\partial_t u = \Delta_\alpha u, \quad u_0 = \delta_0$$

on $Q = (0, \infty) \times \mathbb{R}^d$. The semigroup $(S_t)_{t \geq 0}$ which we have already introduced is a convolution semigroup with kernel p_t , i.e.

$$S_t \phi(x) = \phi * p_t(x) = \int \phi(y) p_t(x - y) dy.$$

We note that for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, $S_t\mu(x)$ can be defined in the same way with no difficulty. The kernel p_t is radial and radially decreasing. In a slight abuse of notation, for $\rho > 0$ we will sometimes write $p_t(\rho)$ to mean $p_t(x)$, where $|x| = \rho$.

$(S_t)_{t \geq 0}$ is the transition semigroup of the (rotationally) symmetric α -stable process, so we have the following: if W is a symmetric α -stable process with law and expectation P_x^W and E_x^W , respectively, when started at x , then for an appropriate class of functions (e.g. bounded and measurable),

$$E_x^W(\phi(W_t)) = S_t\phi(x).$$

In particular, p_t is also the transition density of W , that is

$$P_x^W(W_t \in dy) = p_t(y - x)dy.$$

The symmetric α -stable process is self-similar, which is reflected by the scaling property for p_t :

$$p_t(x) = t^{-\frac{d}{\alpha}} p_1(t^{-\frac{1}{\alpha}}x).$$

The asymptotic behaviour of p_t is well understood (for example, see [1]). There are universal constants $0 < c_{2.1} < C_{2.1} < \infty$ such that

$$c_{2.1} \frac{t^{-\frac{d}{\alpha}}}{1 + |t^{-1/\alpha}x|^{d+\alpha}} \leq p_t(x) \leq C_{2.1} \frac{t^{-\frac{d}{\alpha}}}{1 + |t^{-1/\alpha}x|^{d+\alpha}}. \quad (2.1)$$

It is occasionally useful to write the above as

$$c_{2.1} \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \right) \leq p_t(x) \leq C_{2.1} \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \right), \quad (2.2)$$

where in order to do so one may have to adjust the constants. Without loss of generality we will fix $c_{2.1}$ and $C_{2.1}$ so that both bounds hold.

2.2 The density of the (α, β) -superprocess

In this section we give an overview of the density of X_t . The main purpose is to show that we can take a version of the density which is defined almost surely at any fixed point $x \in \mathbb{R}^d$ and that with this version the quantity $\mu(X_t)$ from (1.4) is well-defined almost surely. We also discuss the regularity properties of the density in the continuous regime.

Recall from (1.5) and (1.6) that for $x \in \mathbb{R}^d$ and $\epsilon > 0$,

$$X_t^\epsilon(x) = \frac{X_t(B(x, \epsilon))}{|B(x, \epsilon)|}$$

and

$$X_t(x) := \liminf_{\epsilon \downarrow 0} X_t^\epsilon(x).$$

Since X_t is absolutely continuous, $x \rightarrow X_t(x)$ is a density for it by the Lebesgue differentiation theorem (for example see Theorem 3.21 of [15]). Because we are interested in the behaviour of the density at fixed points and more generally on Lebesgue-null sets, we require more than the standard a.e.-convergence of $X_t^\epsilon(\cdot)$ to $X_t(\cdot)$. The desired conditions are shown to hold in the next lemma.

For $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\epsilon > 0$, let $\psi_\epsilon(x) := \epsilon^{-d}\psi(\epsilon^{-1}x)$.

Lemma 2.1. Let $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and $t > 0$ and consider X_t under $P_{X_0}^X$.

(a) For every $x \in \mathbb{R}^d$, $X_t^\epsilon(x) \rightarrow X_t(x)$ a.s. and in L^1 as $\epsilon \downarrow 0$.

(b) For every $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, $X_t^\epsilon(x) \rightarrow X_t(x)$ for μ -a.e. x almost surely and $\mu(X_t^\epsilon) \rightarrow \mu(X_t)$ in L^1 , where $\mu(X_t) = \int X_t(x)\mu(dx)$. Moreover, we have

$$E_{X_0}^X(\mu(X_t)) = \mu(S_t X_0). \quad (2.3)$$

(c) If $\psi : \mathbb{R}^d \rightarrow [0, \infty)$ satisfies $\int \psi = 1$ and $\sup_x |\psi(x)|(1 + |x|)^{d+\delta} < \infty$ for some $\delta > 0$, then the conclusions of part (a) and (b) hold when X_t^ϵ is replaced with $X_t * \psi_\epsilon$.

We prove this lemma at the end of the section. Part (c) is included because it is sometimes useful to approximate $X_t(x)$ using convolutions with general kernels. A particularly useful example is when $\psi = p_1$, in which case $\psi_{\frac{1}{\epsilon^\alpha}} = p_\epsilon$.

We now turn our attention to the Hölder regularity of the density in the continuous regime, i.e. $d = 1$ and $\alpha > 1 + \beta$ (c.f. Theorem C). The following was proved in [13].

Theorem D: Fleischmann, Mytnik and Wachtel (2010). Let $d = 1$ and $\alpha > 1 + \beta$. Let $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and $t > 0$. Under $P_{X_0}^X$, there is a continuous version $X_t(\cdot)$ of the density such that for every $\eta < \eta_c = \frac{\alpha}{1+\beta} - 1$, $X_t(\cdot)$ is locally Hölder of index η , i.e.

$$\sup_{x_1, x_2 \in K, x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^\eta} < \infty \quad \text{for all compact } K \subset \mathbb{R}.$$

Furthermore, the value η_c is optimal in that, for any $\eta \geq \eta_c$, with probability one, for any open $U \subseteq \mathbb{R}$,

$$\sup_{x_1, x_2 \in U, x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^\eta} = \infty \quad \text{whenever } X_t \neq 0.$$

As we have noted in the introduction, when $d = 1$ and $\alpha > 1 + \beta$, the density we define in (1.6) is the same as the continuous version. The statement in the above implies the following. Let $\eta < \frac{\alpha}{1+\beta} - 1$ and $X_0 \in \mathcal{M}_F(\mathbb{R})$. Then

$$P_{X_0}^X\text{-a.s.}, \text{ for all compact } K \subset \mathbb{R}, \text{ there exists } C(K, \eta, \omega) > 0 \text{ such that for all } x, y \in K, \\ |X_t(x) - X_t(y)| \leq C(K, \eta, \omega)|x - y|^\eta. \quad (2.4)$$

We conclude the section with the proof of Lemma 2.1. The proof uses a classical result concerning absolute continuity of the laws of superprocesses. As with Theorem A, the result originates in [11], where it was proved for binary branching superprocesses. The proof for $(1 + \beta)$ -stable branching superprocesses appears in [21]. We do not state the result in full generality, but refer the reader to Theorems 1.1 and 2.2, respectively, of [11] and [21]. For any $X_0, \hat{X}_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and $t > 0$, we have

$$\text{The laws of } X_t \text{ under } P_{X_0}^X \text{ and } P_{\hat{X}_0}^X \text{ are mutually absolutely continuous.} \quad (2.5)$$

We use the above when \hat{X}_0 is a translation of X_0 . In particular, for $x \in \mathbb{R}^d$ suppose that $\hat{X}_0 = X_0 + x$, where for a measure μ we define $\mu + x$ by $(\mu + x)(A) = \int 1_A(y - x)\mu(dy)$. Applying (2.5), the laws of X_t under $P_{X_0}^X$ and $P_{X_0+x}^X$ are mutually absolutely continuous. On the other hand, by

translation invariance of the superprocess, the law of X_t under $P_{X_0+x}^X$ is equal to the law of $X_t - x$ under $P_{X_0}^X$. Consequently, we have that for $x \in \mathbb{R}^d$ and $t > 0$,

$$\text{The laws of } X_t \text{ and } X_t + x \text{ are mutually absolutely continuous under } P_{X_0}^X. \quad (2.6)$$

We will also use the following moment bound.

Lemma 2.2. *Let $0 < \theta < \beta$ and $\phi \geq 0$. Then*

$$E_{X_0}^X (X_t(\phi)^{1+\theta}) \leq 1 + C \left[\int_0^t X_0(S_{t-s}((S_s\phi)^{1+\beta})) ds + X_0((S_t\phi)^{1+\beta}) \right]$$

for a constant $C = C(\alpha, d, \theta) > 0$.

The $\alpha = 2$ case of this result appears as Lemma 2.1 of [29]. The same argument used in [29] also works in the $\alpha \in (0, 2)$ when one replaces the heat semigroup with the fractional heat semigroup, so we omit the proof and continue with the proof of Lemma 2.1. The arguments used to bound moments below are also borrowed from [29].

Proof of Lemma 2.1. Fix $t > 0$ and $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$. Let $\psi : \mathbb{R}^d \rightarrow [0, \infty)$ satisfy $\int \psi = 1$ and $\sup_x |\psi(x)|(1 + |x|)^{d+\delta} < \infty$ for some $\delta > 0$ and recall that $\psi_\epsilon(x) = \epsilon^{-d}\psi(\epsilon^{-1}x)$. These conditions include the case where ψ is the normalized indicator function of the unit ball. A priori this is the case we are interested in, but we carry out the analysis for the general case as this is required for part (c).

Let $X_t^{\epsilon, \psi}(x) = X_t * \psi_\epsilon(x)$. Since X_t is absolutely continuous almost surely, we have

$$P_{X_0}^X\text{-almost surely, } X_t^{\epsilon, \psi}(x) \text{ converges for Lebesgue-a.e. } x \in \mathbb{R}^d \text{ as } \epsilon \downarrow 0. \quad (2.7)$$

If ψ is the normalized indicator function of the unit ball, the above is simply Lebesgue's differentiation theorem. Otherwise, the result follows from Theorem 8.15 of [15].

We now show that the convergence holds pointwise almost surely. Suppose there exists $x_0 \in \mathbb{R}^d$ such that $P_{X_0}^X(\{X_t^{\epsilon, \psi}(x_0) \text{ does not converge as } \epsilon \downarrow 0\}) > 0$. By (2.6) this implies that $P_{X_0}^X(\{X_t^{\epsilon, \psi}(x) \text{ does not converge as } \epsilon \downarrow 0\}) > 0$ for all $x \in \mathbb{R}^d$. However, by Fubini's theorem this implies that the set of points for which $X_t^{\epsilon, \psi}(x)$ does not converge has positive Lebesgue measure a.s., which contradicts (2.7). Hence we must have that

$$X_t^{\epsilon, \psi}(x) \text{ converges } P_{X_0}^X\text{-a.s. as } \epsilon \downarrow 0 \text{ for all } x \in \mathbb{R}^d. \quad (2.8)$$

Thus we have shown the almost sure convergence stated in part (a). Before proving the L^1 convergence and part (b), we show how absolute continuity also implies the a.s. convergence in part (c). When ψ is the normalized indicator function of a unit ball, we denote $X_t^{\epsilon, \psi}(x)$ by $X_t^\epsilon(x)$. Let ψ be any other function satisfying the conditions as above. If there exists $x_0 \in \mathbb{R}^d$ such that $P_{X_0}^X(\liminf_{\epsilon \downarrow 0} X_t^\epsilon(x_0) \neq \liminf_{\epsilon \downarrow 0} X_t^{\epsilon, \psi}(x_0)) > 0$, then (2.6) implies that the probability is positive when we replace x_0 with x for all $x \in \mathbb{R}^d$. By Fubini's theorem, $\liminf_{\epsilon \downarrow 0} X_t^\epsilon(x) \neq \liminf_{\epsilon \downarrow 0} X_t^{\epsilon, \psi}(x)$ on a set of positive Lebesgue measure almost surely. Since $\liminf_{\epsilon \downarrow 0} X_t^\epsilon(\cdot)$ and $\liminf_{\epsilon \downarrow 0} X_t^{\epsilon, \psi}(\cdot)$ are both densities for X_t , this is a contradiction. Hence $\liminf_{\epsilon \downarrow 0} X_t^\epsilon(x) = \liminf_{\epsilon \downarrow 0} X_t^{\epsilon, \psi}(x)$ a.s. for every $x \in \mathbb{R}^d$. The left hand side is simply equal to $X_t(x)$, and so for all ψ satisfying the stated conditions, (2.8) can be improved to

$$X_t^{\epsilon, \psi}(x) \rightarrow X_t(x) \text{ } P_{X_0}^X\text{-a.s. as } \epsilon \downarrow 0 \text{ for all } x \in \mathbb{R}^d,$$

which proves that the almost sure convergence in (a) holds for all ψ from (c). Furthermore, the above implies that the limit of $X_t^{\epsilon, \psi}(x)$ does not depend on the choice of ψ . Given this, we can work with general ψ and prove the remaining claims of (a) and (b) simultaneously with (c).

In order to establish that $X_t^{\epsilon, \psi}(x) \rightarrow X_t(x)$ in L^1 , we will show L^p -boundedness of the quantity $X_t^{\epsilon, \psi}(x)$ for $p = 1 + \theta$ with $\theta < \beta$. To do so we apply Lemma 2.2. Let $\psi_\epsilon^x(\cdot) = \psi_\epsilon(\cdot - x)$ and note that $X_t^{\epsilon, \psi}(x) = X_t(\psi_\epsilon^x)$. For $\theta < \beta$, we have

$$E_{X_0}^X((X_t^{\epsilon, \psi}(x))^{1+\theta}) \leq 1 + C \left[\int_0^t X_0(S_{t-s}((S_s \psi_\epsilon^x)^{1+\beta})) ds + X_0((S_t \psi_\epsilon^x)^{1+\beta}) \right]. \quad (2.9)$$

Since $\int \psi_\epsilon^x = 1$, we can view ψ_ϵ^x as the density of a random variable, which we will denote Y . Thus for $s > 0$,

$$\begin{aligned} (S_s \psi_\epsilon^x)(z)^{1+\beta} &= \left[\int p_s(y-z) \psi_\epsilon^x(y) dy \right]^{1+\beta} = E^Y(p_s(Y-z))^{1+\beta} \\ &\leq E^Y(p_s(Y-z)^{1+\beta}) \\ &= \int p_s(y-z)^{1+\beta} \psi_\epsilon^x(y) dy, \end{aligned} \quad (2.10)$$

We also have

$$p_s(y-z)^{1+\beta} = p_s(y-z) p_s(y-z)^\beta \leq C p_s(y-z) s^{-\frac{d\beta}{\alpha}}$$

by (2.2). Hence by (2.10) and the above,

$$\begin{aligned} X_0(S_{t-s}((S_s \psi_\epsilon^x)^{1+\beta})) &\leq C s^{-\frac{d\beta}{\alpha}} \iiint p_{t-s}(z-w) p_s(y-z) \psi_\epsilon^x(y) dy dz X_0(dw) \\ &= C s^{-\frac{d\beta}{\alpha}} \iint p_t(w-y) \psi_\epsilon^x(y) dy X_0(dw) \\ &\leq C X_0(1) s^{-\frac{d\beta}{\alpha}} t^{-\frac{d}{\alpha}}. \end{aligned} \quad (2.11)$$

Since $\beta < \frac{\alpha}{d}$, the power of s is integrable over $s \in [0, t]$, which implies that the first term in the square brackets in (2.9) is uniformly bounded in $\epsilon \in (0, 1]$. The second term can be handled the same way and is in fact easier to control. Thus $E_{X_0}^X((X_t^{\epsilon, \psi})^{1+\theta})$ is uniformly bounded for $\epsilon \in (0, 1]$, which implies that the family $\{X_t^{\epsilon, \psi}(x) : \epsilon \in (0, 1]\}$ is uniformly integrable. Since $X_t^{\epsilon, \psi}(x)$ converges almost surely to $X_t(x)$, this implies that the convergence is also in $L^1(P_{X_0}^X)$.

We now fix $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and show L^1 convergence of $\mu(X_t^{\epsilon, \psi})$ to $\mu(X_t)$. Note that the bound in (2.11) does not depend on x . Thus, rather than consider $L^{1+\theta}$ -boundedness of $X_t(\psi_\epsilon^x)$ for fixed x with respect to $P_{X_0}^X$, we can fix $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and consider $L^{1+\theta}$ -boundedness of $(\omega, x) \rightarrow X_t(\psi_\epsilon^x)(\omega)$ with respect to $P_{X_0}^X \otimes \mu$. Because the bounds are uniform in x and μ is finite, the argument requires no modification. Since $X_t(\psi_\epsilon^x) \rightarrow X_t(x)$ a.s. for every x , it follows that $X_t(\psi_\epsilon^x) \rightarrow X_t(x)$ in $L^1(P_{X_0}^X \otimes \mu)$ (as a function of (ω, x)). This implies that $\mu(X_t^{\epsilon, \psi}) \rightarrow \mu(X_t)$ in L^1 .

It remains to show the moment formula (2.3). First, let $\phi : \mathbb{R}^d \rightarrow [0, \infty)$ be bounded and measurable. The mean measure formula for superprocesses gives

$$E_{X_0}^X(X_t(\phi)) = X_0(S_t \phi). \quad (2.12)$$

Now fix $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. We have

$$\mu(X_t^{\epsilon, \psi}) = \iint \psi_\epsilon(x-y) X_t(dy) \mu(dx) = \iint \mu_\epsilon(x) X_t(dx) = X_t(\mu_\epsilon),$$

where $\mu_\epsilon = \mu * \hat{\psi}_\epsilon$ with $\hat{\psi}_\epsilon(\cdot) = \psi_\epsilon(-\cdot)$. Since μ_ϵ is a bounded and measurable function, by (2.12) the above implies

$$E_{X_0}^X(\mu(X_t^{\epsilon, \psi})) = X_0(S_t \mu_\epsilon).$$

Because $\mu(X_t^{\epsilon, \psi}) \rightarrow \mu(X_t)$ in $L^1(P_{X_0}^X)$, the left hand side converges to $E_{X_0}^X(\mu(X_t))$ as $\epsilon \downarrow 0$. We have $S_t \mu_\epsilon \leq \mu(1)t^{-\frac{d}{\alpha}}$ for all $\epsilon > 0$, so the right hand side converges to $X_0(S_t \mu)$ by Dominated Convergence. Since $X_0(S_t \mu) = \mu(S_t X_0)$, this proves (2.3) and the proof is complete. \square

2.3 The fractional PDE and $\mu(X_t)$

In this section we extend the duality between X_t and solutions to the evolution equation (1.2). Recall from (1.1) that in its basic form, the dual relationship states that $E_{X_0}^X(\exp(-X_t(\phi))) = \exp(-X_0(u_t^\phi))$, where for bounded and measurable $\phi \geq 0$, u_t^ϕ is the unique solution to (1.2) with initial data ϕ . The purpose of this section is to extend this relationship to allow ϕ to be replaced with a finite measure when $\beta < \frac{\alpha}{d}$. We also introduce weak solutions to (1.3) and some of their properties.

The integral equation (1.2) is a mild form of the PDE (1.3). We will work with weak solutions. Recall that $Q = (0, \infty) \times \mathbb{R}^d$. Let $C_c^{1,2}(Q)$ denote the space of compactly supported functions on Q which are once and twice continuously differentiable in time and space, respectively. For $T > 0$, let $Q_T = (0, T] \times \mathbb{R}^d$ and $\bar{Q}_T = [0, T] \times \mathbb{R}^d$, and let $C_b^{1,2}(\bar{Q}_T)$ denote the space of bounded functions on \bar{Q}_T with bounded, continuous derivatives up to order one in time and order two in space. For $p \geq 1$, we let $L_{loc}^p(Q)$ denote the space of functions ϕ such that $\int_K |\phi|^p < \infty$ for every compact $K \subset Q$.

Definition 2.3. *A function $u : Q \rightarrow \mathbb{R}$ is a weak solution to (1.3) if $(t, x) \rightarrow u(t, x)$ is continuous, $u \in L_{loc}^{1+\beta}(Q)$, and*

$$\int_Q (u(t, x)[- \partial_t \xi(t, x) - \Delta_\alpha \xi(t, x)]) + u(t, x)^{1+\beta} \xi(t, x) dx dt = 0 \quad (2.13)$$

for all $\xi \in C_c^{1,2}(Q)$.

For measure-valued initial data, the PDE problem of interest is

$$\begin{cases} \partial_t u = \Delta_\alpha u - u^{1+\beta} & \text{for } (t, x) \in Q, \\ u_0 = \mu \end{cases} \quad (2.14)$$

for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. In the following, recall that by convergence in $\mathcal{M}_F(\mathbb{R}^d)$ we mean weak convergence of measures.

Definition 2.4. *For $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, we say that $u : Q \rightarrow \mathbb{R}^+$ is a weak solution to (2.14) if it is a weak solution to (1.3), $u \in L^1(Q_T) \cap L^{1+\beta}(Q_T)$ for all $T > 0$, and $u_t \rightarrow \mu$ in $\mathcal{M}_F(\mathbb{R}^d)$ as $t \downarrow 0$.*

Proposition 2.5. *For $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, there exists a unique weak solution $u : Q \rightarrow [0, \infty)$ to (2.14). Moreover, for $T > 0$ we have*

$$\begin{aligned} & \int_{Q_T} (u(t, x)[- \partial_t \xi(t, x) - \Delta_\alpha \xi(t, x)]) + u(t, x)^{1+\beta} \xi(t, x) dx dt \\ &= \int_{\mathbb{R}^d} \xi(0, x) \mu(dx) - \int_{\mathbb{R}^d} u(T, x) \xi(T, x) dx \end{aligned} \quad (2.15)$$

for all $\xi \in C_b^{1,2}(\bar{Q}_T)$.

We will denote the unique solution to (2.14) by $u_t^\mu(x)$ or $u^\mu(t, x)$. The above is proved in Theorem 1.1 of [3]. The authors of that work use a slightly different definition which incorporates (2.15). However, a short argument which we omit shows that given a solution in the sense of Definition 2.4, (2.15) holds for all $\xi \in C_b^{1,2}(\bar{Q}_T)$. The definition of a solution used in [3] also does not require continuity, but it can be verified that $u_t^\mu(x)$ is jointly continuous, for example using its correspondence with the solution to the integral equation given in Lemma 2.7(b) below. (The proof of Lemma 2.7(b) requires only a small modification if one does not assume a priori that $u_t^\mu(x)$ is continuous.) The following stability result holds as a consequence of Theorem 1.1 of [3].

Lemma 2.6. (a) If $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and $(\mu_n)_{n \geq 1}$ is a sequence of measures such that $\mu_n \rightarrow \mu$ in $\mathcal{M}_F(\mathbb{R}^d)$, then $u_t^{\mu_n}(x) \rightarrow u_t^\mu(x)$ locally uniformly in Q .
(b) The map $\mu \rightarrow u_t^\mu$ is increasing.

Solutions of (2.14) are bounded above by solutions of the homogeneous fractional heat equation with the same initial data. Again by Theorem 1.1 of [3], we have

$$u_t^\mu(x) \leq S_t \mu(x) \text{ for } (t, x) \in Q, \quad (2.16)$$

where $(t, x) \rightarrow S_t \mu(x)$ is the unique solution to $\partial_t v = \Delta_\alpha v$ on Q with $v_0 = \mu$ (see Theorems 3.1 and 5.1 of [2]).

Lemma 2.7. Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. (a) The integral equation

$$u_t(x) = S_t \mu(x) - \int_0^t S_{t-s}(u_s^{1+\beta})(x) ds, \quad (t, x) \in Q \quad (2.17)$$

has a unique, non-negative, jointly continuous solution in $L^1(Q_T) \cap L^{1+\beta}(Q_T)$ for all $T > 0$.
(b) The weak solution to (2.14) with initial data μ and the solution to (2.17) are equal.

We will therefore use the notation $u_t^\mu(x)$ and $u^\mu(t, x)$ to refer to the unique solution to (2.14) and (2.17). With the exception of continuity, part (a) of the above is proved in Lemma A.2 of [12], and continuity can be shown by a direct calculation which we omit.

Proof of Lemma 2.7(b). Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and let $u(t, x) = u^\mu(t, x)$, the unique weak solution to (2.14). For $T > 0$, $x_0 \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we take $\xi(t, x) = p_{T+n-1-t}(x - x_0) \in C_b^{1,2}(Q)$ in (2.15). Since p_t solves $\partial_t p_t(x) = \Delta_\alpha p_t(x)$, we have

$$\begin{aligned} & \int_{Q_T} u(t, x)^{1+\beta} p_{T+n-1-t}(x - x_0) dx dt \\ &= \int_{\mathbb{R}^d} p_{T+n-1}(x - x_0) \mu(dx) - \int_{\mathbb{R}^d} u(T, x) p_{n-1}(x - x_0) dx. \end{aligned} \quad (2.18)$$

For all $n \geq 1$, $\|p_{T+n-1}\|_\infty \leq CT^{-\frac{d}{\alpha}}$ by (2.2), and so by bounded convergence the first term in the second line converges to $S_T \mu(x_0)$. The second term converges to $u(T, x_0)$ by continuity of u . Now consider the first line. Let $\epsilon > 0$. For $n \geq 1$ and $t \leq T - \epsilon$, $\|p_{T+n-1-t}\|_\infty \leq C\epsilon^{-\frac{d}{\alpha}}$. Since $u \in L^{1+\beta}(Q_T)$, we can apply Dominated Convergence to obtain that

$$\lim_{n \rightarrow \infty} \int_0^{T-\epsilon} \int_{\mathbb{R}^d} u(t, x)^{1+\beta} p_{T+n-1-t}(x - x_0) dx dt = \int_0^{T-\epsilon} \int_{\mathbb{R}^d} u(t, x)^{1+\beta} p_{T-t}(x - x_0) dx dt.$$

On the other hand, by (2.16) there is a constant $K > 0$ such that $u(t, x)^{1+\beta} \leq K$ for all $x \in \mathbb{R}^d$ and $t \in [T - \epsilon, T]$. Consequently, we have

$$\limsup_{\epsilon \downarrow 0} \sup_{n \geq 1} \left| \int_{T-\epsilon}^T \int_{\mathbb{R}^d} u(t, x)^{1+\beta} p_{T+n-1-t}(x - x_0) dx dt \right| = 0.$$

Combining everything above, we take $n \rightarrow \infty$ in (2.18) to conclude that for all $(T, x_0) \in Q$,

$$u(T, x_0) = S_T \mu(x_0) - \int_{Q_T} u(t, x)^{1+\beta} p_{T-t}(x - x_0) dx dt,$$

and hence u is equal to the solution of (2.17). \square

Remark 2.8. *Along the same lines as the above, it can be shown that if $u(t, x)$ is a bounded weak solution to (1.3) and $u(t, x) \rightarrow \phi(x)$ a.e. as $t \downarrow 0$ for $\phi \in \mathcal{B}_b^+$, then $u(t, x) = u^\phi(t, x)$, the solution to (1.2) with $u_0 = \phi$. This also implies that such a solution is unique and has the probabilistic representation $u(t, x) = \mathbb{N}_x(1 - \exp(-X_t(\phi)))$ by (1.9).*

We now extend the dual relationship with the (α, β) -superprocess to measures.

Lemma 2.9. *Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. Then for $x \in \mathbb{R}^d$ and $t > 0$,*

$$\mathbb{N}_x(1 - \exp(-\mu(X_t))) = u_t^\mu(x), \quad (2.19)$$

and for $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$,

$$E_{X_0}^X(\exp(-\mu(X_t))) = \exp(-X_0(u_t^\mu)). \quad (2.20)$$

Proof. We give the proof under $P_{X_0}^X$ and note that it follows by essentially the same argument for the canonical measure. (One can restrict to the event $\{X_t \neq 0\}$ because $1 - \exp(-\mu(X_t)) = 0$ on $\{X_t = 0\}$, which allows us to treat \mathbb{N}_x as a finite measure.) Fix $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and define $\mu_n = S_{n-1}\mu$. Then μ_n is smooth, bounded and positive, so by (1.1) we have

$$E_{X_0}^X(\exp(-X_t(\mu_n))) = \exp(-X_0(u_t^{\mu_n})). \quad (2.21)$$

Since $\mu_n \rightarrow \mu$ in $\mathcal{M}_F(\mathbb{R}^d)$ as $n \rightarrow \infty$, by Lemma 2.6(a) it follows that $u^{\mu_n} \rightarrow u^\mu$ locally uniformly. In particular, $u_t^{\mu_n} \rightarrow u_t^\mu$ pointwise. By (2.16) and (2.2), we have

$$|u_t^{\mu_n}(x)| \leq C\mu(1)t^{-\frac{d}{\alpha}}$$

for all $n \geq 1$ and $x \in \mathbb{R}^d$. Hence $X_0(u_t^{\mu_n}) \rightarrow X_0(u_t^\mu)$ by Dominated Convergence, and consequently

$$\lim_{n \rightarrow \infty} \exp(-X_0(u_t^{\mu_n})) = \exp(-X_0(u_t^\mu)). \quad (2.22)$$

Consider now the left hand side of (2.21). Expanding (X_t, μ_n) , we have

$$X_t(\mu_n) = X_t(S_{n-1}\mu) = \int S_{n-1}\mu(x)X_t(dx) = \int S_{n-1}X_t(x)\mu(dx).$$

We have used the symmetry of S_{n-1} . Note that $S_{n-1}X_t(x) = X_t * p_{n-1}(x)$ is an approximation of the density which satisfies the conditions of Lemma 2.1(c), and hence $X_t(\mu_n) \rightarrow \mu(X_t)$ in $L^1(P_{X_0}^X)$. In particular, this implies the convergence of the left hand side of (2.21) to $E_{X_0}^X(\exp(-\mu(X_t)))$. Combined with (2.22), this implies (2.20) and completes the proof. \square

Remark 2.10. Note that (2.19) immediately implies that $\mu \rightarrow u_t^\mu$ is increasing, which we have already stated as Lemma 2.6(b). We will refer to this monotonicity in initial conditions of solutions to (1.3) as the comparison principle. Given Remark 2.8, the comparison principle also holds for bounded weak solutions with initial data in \mathcal{B}_b^+ .

Solutions to (1.3) satisfy a useful scaling property. For $\lambda > 0$ and $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, one can verify directly using elementary methods and uniqueness of solutions to problem (2.14) the following formula:

$$u^{\lambda\mu}(t, x) = \lambda^{\alpha\gamma} u^{\mu(\cdot/\lambda^{\beta\gamma})}(\lambda^{\alpha\beta\gamma}t, \lambda^{\beta\gamma}x), \quad (2.23)$$

where $\gamma = \frac{1}{\alpha-\beta}$, and the measure $\mu(\cdot/\lambda^{\beta\gamma})$ is the dilation of μ defined by

$$\mu(\cdot/\lambda^{\beta\gamma})(A) = \mu(A/\lambda^{\beta\gamma}) = \int 1(\lambda^{\beta\gamma}x \in A)\mu(dx)$$

for measurable $A \subseteq \mathbb{R}^d$. This leads to a very useful expression when we scale out the time variable t to obtain an expression involving a solution at time 1; in particular, we have

$$u_t^\mu(x) = t^{-\frac{1}{\beta}} u^{t^{\frac{\alpha-\beta}{\alpha\beta}} \mu(\cdot/t^{-\frac{1}{\alpha}})}(1, t^{-\frac{1}{\alpha}}x). \quad (2.24)$$

It follows that $u_t^{\infty\mu}(x) = \lim_{\lambda \rightarrow \infty} u_t^{\lambda\mu}(x)$ satisfies

$$u_t^{\infty\mu}(x) = t^{-\frac{1}{\beta}} u^{\infty\mu(\cdot/t^{-\frac{1}{\alpha}})}(1, t^{-\frac{1}{\alpha}}x). \quad (2.25)$$

2.4 A Feynman-Kac formula

We now state a Feynman-Kac formula for some functions related to solutions of (1.2). First, for $\phi, \psi \in \mathcal{B}_b^+$, we formally define

$$z^{\phi, \psi}(t, x) = \left. \frac{\partial}{\partial \epsilon} u^{\phi + \epsilon\psi}(t, x) \right|_{\epsilon=0} = \lim_{\epsilon \downarrow 0} \frac{u^{\phi + \epsilon\psi}(t, x) - u^\phi(t, x)}{\epsilon},$$

where $u^\phi(t, x)$ is taken to be the solution to (1.2) with $u_0 = \phi$. By (1.9) and the Dominated Convergence Theorem, it follows that the derivative with respect to ϵ exists and

$$z^{\phi, \psi}(t, x) = \mathbb{N}_x(X_t(\psi) \exp(-X_t(\phi))).$$

Recall that E_x^W denotes the expectation associated to an α -stable process W_t with $W_0 = x$.

Lemma 2.11. (a) For $\phi, \psi \in \mathcal{B}_b^+$,

$$z^{\phi, \psi}(t, x) = E_x^W \left(\psi(W_t) \exp \left(-(1 + \beta) \int_0^t u_{t-s}^\phi(W_s)^\beta ds \right) \right).$$

(b) For $\phi \in \mathcal{B}_b^+$ and $\lambda > 0$, $\frac{\partial}{\partial \lambda} u_t^{\lambda\phi}(x)$ exists for all $(t, x) \in Q$ and

$$\frac{\partial}{\partial \lambda} u_t^{\lambda\phi}(x) = E_x^W \left(\phi(W_t) \exp \left(-(1 + \beta) \int_0^t u_{t-s}^{\lambda\phi}(W_s)^\beta ds \right) \right)$$

In particular, for $(t, x) \in Q$ and $\Lambda > 0$ we have

$$u_t^{\Lambda\phi}(x) = \int_0^\Lambda \frac{\partial}{\partial \lambda} u_t^{\lambda\phi}(x) d\lambda. \quad (2.26)$$

A proof of part (a) is implicit in the proof of Theorem 6.3.1 in [6], where the first is established by showing that $z^{\phi, \psi}(t, x)$ solves a certain linear evolution equation. To see that (b) follows from (a), it suffices to consider $z^{\lambda\phi, \phi}(t, x)$ for $\phi \in \mathcal{B}_b^+$ and $\lambda > 0$.

2.5 Cluster decompositions

The connection between the (α, β) -superprocess and its canonical measure is via a cluster representation in which the superprocess is given by a Poisson superposition of clusters whose intensity is canonical measure. To make this precise, for $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ we define

$$\mathbb{N}_{X_0}(\cdot) = \int \mathbb{N}_x(\cdot) X_0(dx).$$

Let $\Xi(\cdot)$ be a Poisson point process on $\mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^d))$ with intensity \mathbb{N}_{X_0} . Then the process

$$X_t = \begin{cases} \int \nu_t \Xi(d\nu) & \text{if } t > 0, \\ X_0 & \text{if } t = 0 \end{cases}$$

is an (α, β) -superprocess with law $P_{X_0}^X$. This is a consequence of Theorem 4.2.1 of [8]. For fixed $t > 0$ the above implies that

$$X_t \stackrel{\mathcal{D}}{=} \sum_{i=1}^N X_t^i, \quad (2.27)$$

where $\stackrel{\mathcal{D}}{=}$ indicates equality of distribution. In the above, N is a Poisson random variable with mean $X_0(1)(\beta t)^{-\frac{1}{\beta}}$ and the X_t^i are iid random measures with distribution $\mathbb{N}_{X_0}(X_t \in \cdot | X_t \neq 0)$. This representation gives us a convenient way to compare path properties under $P_{X_0}^X$ and \mathbb{N}_x . In particular, suppose we realize X_t under $P_{\delta_x}^X$ via (2.27). Since the probability that $N = 1$ is positive, we can condition on this event, and it follows that

$$\mathbb{N}_x(X_t \in \cdot | X_t \neq 0) = P_{\delta_x}^X(X_t \in \cdot | N = 1). \quad (2.28)$$

Consequently, for measurable $A \subset \mathcal{M}_F(\mathbb{R}^d)$ we have

$$\text{If } P_{\delta_x}^X(X_t \in A | X_t \neq 0) = 1, \text{ then } \mathbb{N}_x(X_t \in A | X_t \neq 0) = 1. \quad (2.29)$$

3 The density at a fixed point

In the introduction, we noted that Theorem 1.3 is equivalent to Theorem B, which is proved analytically. In this section we give a probabilistic proof of Theorem 1.3(a), and therefore of Theorem B(a). We define $u_t^\lambda(x)$ as the solution to (2.14) with initial measure $\lambda \delta_0$. Then by translation invariance of the equation (1.3) and (2.20), for $x \in \mathbb{R}^d$ we have

$$E_{X_0}^X(\exp(-\lambda X_t(x))) = \exp\left(-\int u_t^\lambda(y-x) X_0(dy)\right) \quad (3.1)$$

We define $u_t^\infty = \lim_{\lambda \rightarrow \infty} u_t^\lambda$ and observe that, by taking $\lambda \rightarrow \infty$ in (3.1),

$$P_{X_0}^X(X_t(x) = 0) = \exp\left(-\int u_t^\infty(y-x) X_0(dy)\right). \quad (3.2)$$

The main purpose of this section is to show the following.

Proposition 3.1. *If $\beta \leq \beta^*(\alpha) = \frac{\alpha}{d+\alpha}$, then for $t > 0$ and fixed $x \in \mathbb{R}^d$, $X_t(x) > 0$ a.s. on $\{X_t \neq 0\}$ under $P_{X_0}^X$ and \mathbb{N}_{x_0} . Hence $u_t^\infty = U_t$.*

In fact, this result is a consequence of the more general Theorem 5.1, which we prove in Section 5. However, we state and prove it separately for a few reasons. First, while Theorem 5.1 (and several other results) concern the behaviour of $\mu(X_t)$ for certain families of measures, the measure δ_x is of particular interest because it corresponds to the density at a fixed point. The other reason is that, while the method used to prove Proposition 3.1 and Theorem 5.1 is largely the same, the proof of the latter involves technicalities that do not arise in the former. We therefore opt to include the simpler proof in the case which is particularly interesting.

Recall that we will sometimes write $u^\lambda(t, x)$ to denote $u_t^\lambda(x)$. We note the particular form that the scaling relationship (2.24) takes for the family $u^\lambda(t, x)$. Since $\delta_0(\cdot/r) = \delta_0$, it follows that

$$u^\lambda(t, x) = t^{-\frac{1}{\beta}} u^{t^{\frac{\alpha-\beta}{\alpha\beta}} \lambda}(1, t^{-\frac{1}{\alpha}} x) \quad (3.3)$$

Consequently, $u^\infty(t, x)$ satisfies

$$u^\infty(t, x) = t^{-\frac{1}{\beta}} u^\infty(1, t^{-\frac{1}{\alpha}} x). \quad (3.4)$$

The following lemma gives a lower bound for u_1^λ (for $\lambda \geq 1$) in terms of the heat kernel p_1 of the symmetric α -stable process and holds for all $0 < \beta < \frac{\alpha}{d}$. The statement of this result for u_1^∞ appeared in [3], where it was called Lemma 5.3. Here we give a probabilistic proof.

Lemma 3.2. *There is a constant $c_{3.2} = c_{3.2}(\alpha, \beta, d) > 0$ such that for all $\lambda \geq 1$,*

$$u_1^\lambda(x) \geq c_{3.2} p_1(x).$$

In particular, $u_1^\infty(x) = \mathbb{N}_0(X_1(x) > 0) \geq c_{3.2} p_1(x)$.

Proof. For $\lambda > 0$ and $\epsilon > 0$, consider $u^{\lambda p_\epsilon}(t, x)$, the unique solution to (1.2) with $u_0 = \lambda p_\epsilon$. By Lemma 2.11(b), we have

$$\frac{\partial}{\partial \lambda} u^{\lambda p_\epsilon}(1, x) = E_x^W \left(p_\epsilon(W_1) \exp \left(-(1 + \beta) \int_0^1 u^{\lambda p_\epsilon}(1 - s, W_s)^\beta ds \right) \right).$$

By (2.16), $u_s^{\lambda p_\epsilon} \leq \lambda S_s p_\epsilon = \lambda p_{s+\epsilon}$, which implies that

$$\begin{aligned} \frac{\partial}{\partial \lambda} u^{\lambda p_\epsilon}(1, x) &\geq E_x^W \left(p_\epsilon(W_1) \exp \left(-(1 + \beta) \int_0^1 \lambda^\beta p_{1-s+\epsilon}(x + W_s)^\beta ds \right) \right) \\ &\geq E_x^W (p_\epsilon(W_1)) \exp \left(-\lambda^\beta (1 + \beta) \int_0^1 p_{1-s+\epsilon}(0)^\beta ds \right). \end{aligned}$$

In the second line we have used the fact that $p_{1-s+\epsilon}$ is radially decreasing, hence $p_{1-s+\epsilon}(W_s) \geq p_{1-s+\epsilon}(0)$, and removed the exponential from the expectation because it no longer depends on W . We note that $E_x^W (p_\epsilon(W_1)) = S_1 p_\epsilon(x) = p_{1+\epsilon}(x)$ from the semigroup property. Using (2.2) and changing variables in the integral, we then have

$$\frac{\partial}{\partial \lambda} u^{\lambda p_\epsilon}(1, x) \geq p_{1+\epsilon}(x) \exp \left(-\lambda^\beta (1 + \beta) \int_0^1 C_{2.1}^\beta (s + \epsilon)^{-\frac{d\beta}{\alpha}} ds \right)$$

Because $\beta < \frac{\alpha}{d}$, the integral remains bounded as $\epsilon \downarrow 0$. It follows that for a constant $C > 0$, for all $\epsilon \in (0, 1]$,

$$\frac{\partial}{\partial \lambda} u^{\lambda p_\epsilon}(1, x) \geq p_{1+\epsilon}(x) \exp \left(-C \lambda^\beta \right).$$

In particular, (2.26) and the above imply that

$$u^{p_\epsilon}(1, x) \geq c_0 p_{1+\epsilon}(x)$$

for a constant $c_0 > 0$. By Lemma 2.6(a), the left hand side converges to $u^1(1, x)$ as $\epsilon \downarrow 0$, and the right hand side converges to $p_1(x)$. Thus we have $u^1(1, x) \geq c_0 p_1(x)$. Since $\lambda \rightarrow u^\lambda(1, x)$ is increasing, this implies the result. \square

By (3.4) and Lemma 3.2, one obtains that

$$u_t^\infty(x) \geq c_{3.2} t^{-\frac{1}{\beta}} p_1(t^{-\frac{1}{\alpha}} x). \quad (3.5)$$

For fixed $x \neq 0$, by (2.2) we then have

$$\liminf_{t \downarrow 0} \frac{u_t^\infty(x)}{t^{-\frac{1}{\beta} + \frac{d+\alpha}{\alpha}}} > 0.$$

It is therefore immediate that $\lim_{t \downarrow 0} u_t^\infty(x) = \infty$ for all $x \in \mathbb{R}^d$ when $\beta < \beta^*(\alpha)$. It does not give the same conclusion when $\beta = \beta^*(\alpha)$, and in neither case is it immediate that u_t^∞ is flat.

Let $(\mathcal{F}_t)_{t \geq 0}$ denote the standard right-continuous filtration associated to $X = (X_t : t \geq 0)$. For a $P_{X_0}^X$ -integrable function $f(X)$, to denote its conditional expectation we will omit the sub- and superscripts and simply write $E(f(X)|\mathcal{F})$. The (one-dimensional) Markov property for X is then expressed as

$$E(f(X_{t+s}) | \mathcal{F}_s)(\omega) = E_{X_s(\omega)}^X(f(X_t)).$$

Because X is càdlàg, $P_{X_0}^X(X_t = X_{t+} \forall t > 0) = 1$. The following lemma gives almost surely left continuity at a fixed time.

Lemma 3.3. *Fix $t > 0$ and $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$. Almost surely under $P_{X_0}^X$ there is no discontinuity of $s \rightarrow X_s$ at time t , and hence $X_s \rightarrow X_t$ in $\mathcal{M}_F(\mathbb{R}^d)$ as $s \uparrow t$. Moreover, for any open or closed ball B , $\lim_{s \uparrow t} X_s(B) = X_t(B)$ almost surely.*

Proof. Fix $t > 0$. The claim is a consequence of Lemma 1.6 of [13]. Part (a) of that lemma states that the discontinuities of $t \rightarrow X_t$ are described by a jump measure $N(d(s, x, r))$, and the form of the compensator of N given in part (b) implies that a.s. there is no jump at time t . Hence $X_s \rightarrow X_t$ weakly as $s \uparrow t$. For an open or closed ball B , the fact that $X_s(B) \rightarrow X_t(B)$ as $s \uparrow t$ follows from weak convergence and the fact that $X_t(\partial B) = 0$ (by absolute continuity). \square

Proof of Proposition 3.1. Let $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$. Our method is to show that $P_{X_0}^X(X_t(x) = 0, X_t \neq 0) = 0$. This implies that $P_{X_0}^X(X_t(x) > 0, X_t \neq 0) = P_{X_0}^X(X_t \neq 0)$. At the end of the proof we discuss the case for canonical measure and show that $u_t^\infty = U_t$.

Recall that $B(x, r) = \{y \in \mathbb{R}^d : |y - x| \leq r\}$. From Theorem A, we have $\text{supp}(X_t) = \mathbb{R}^d$ a.s. on $\{X_t \neq 0\}$. In particular, $P_{X_0}^X$ -a.s. on $\{X_t \neq 0\}$ we have $X_t(B(x, 1)) > 0$. Thus it suffices to show that

$$P_{X_0}^X(\{X_t(x) = 0\} \cap A^\kappa) = 0 \quad (3.6)$$

for every $\kappa > 0$, where we define the event

$$A^\kappa = \{X_t(B(x, 1)) \geq \kappa\}.$$

Let $\kappa > 0$ and consider the event $A^{2\kappa}$. By Lemma 3.3, $\lim_{s \uparrow t} X_s(B(x, 1)) = X_t(B(x, 1)) \geq 2\kappa$ on $A^{2\kappa}$. We let $\delta_n = 2^{-n}$ and note that the previous statement implies that $X_{t-\delta_n}(B(x, 1)) \geq \kappa$ for n sufficiently large (depending on ω). That is, if we define E_n^κ by

$$E_n^\kappa = \{X_{t-\delta_n}(B(x, 1)) \geq \kappa\},$$

then

$$\text{for a.e. } \omega \in A^{2\kappa}, \exists N = N(\omega) \text{ such that } \omega \in \bigcap_{n=N(\omega)}^\infty E_n^\kappa. \quad (3.7)$$

In particular, this implies that

$$\lim_{n \rightarrow \infty} P_{X_0}^X(A^{2\kappa} \cap (E_n^\kappa)^c) = 0. \quad (3.8)$$

Consider the conditional probability $P(X_t(x) = 0 | \mathcal{F}_{t-\delta_n})$. Applying the Markov property, we obtain

$$\begin{aligned} P(X_t(x) = 0 | \mathcal{F}_{t-\delta_n}) &= P_{X_{t-\delta_n}}^X(X_{\delta_n}(x) = 0) \\ &= \exp\left(-\int u_{\delta_n}^\infty(y-x) X_{t-\delta_n}(dy)\right), \end{aligned}$$

where the second equality uses (3.2). We now bound above by ignoring all the mass of $X_{t-\delta_n}$ outside of $B(x, 1)$. This gives

$$\begin{aligned} P(X_t(x) = 0 | \mathcal{F}_{t-\delta_n}) &\leq \exp\left(-\int_{B(x,1)} u_{\delta_n}^\infty(y-x) X_{t-\delta_n}(dy)\right) \\ &\leq \exp\left(-c_{3.2} \int_{B(x,1)} \delta_n^{-\frac{1}{\beta}} p_1(\delta_n^{-\frac{1}{\alpha}}(y-x)) X_{t-\delta_n}(dy)\right), \end{aligned}$$

where the second inequality uses (3.5). Since p_1 is radially decreasing, the minimum value it can attain in the integral above is $p_1(2\delta_n^{-\frac{1}{\alpha}})$, and $p_1(2\delta_n^{-\frac{1}{\alpha}}) \geq c_1 \delta_n^{\frac{d+\alpha}{\alpha}}$ for some $c_1 > 0$ by (2.2). We thus obtain that for $c_2 = c_1 \cdot c_{3.2} > 0$,

$$P(X_t(x) = 0 | \mathcal{F}_{t-\delta_n}) \leq \exp\left(-c_2 \delta_n^{-\frac{1}{\beta} + \frac{d+\alpha}{\alpha}} X_{t-\delta_n}(B(x, 1))\right).$$

Now suppose that $\omega \in E_n^\kappa$. Then $X_{t-\delta_n}(B(x, 1)) \geq \kappa$, and hence

For $P_{X_0}^X$ -a.e. $\omega \in E_n^\kappa$, we have

$$P_{X_0}^X(X_t(x) = 0 | \mathcal{F}_{t-\delta_n})(\omega) \leq \exp\left(-c_2 \kappa \delta_n^{-\frac{1}{\beta} + \frac{d+\alpha}{\alpha}}\right). \quad (3.9)$$

First suppose that $\beta < \beta^*(\alpha) = \frac{\alpha}{d+\alpha}$. In this case, the exponent of δ_n in (3.9) is negative and so the right hand side of (3.9) converges to 0 as $n \rightarrow \infty$. We thus obtain that

$$\begin{aligned} &P_{X_0}^X(\{X_t(x) = 0\} \cap A_{2\kappa}) \\ &\leq E_{X_0}^X(P(\{X_t(x) = 0\} \cap E_n^\kappa | \mathcal{F}_{t-\delta_n})) \\ &\quad + P_{X_0}^X(A^{2\kappa} \cap (E_n^\kappa)^c) \\ &\leq P_{X_0}^X(E_n^\kappa) \exp\left(-c_2 \kappa \delta_n^{-\frac{1}{\beta} + \frac{d+\alpha}{\alpha}}\right) \\ &\quad + P_{X_0}^X(A^{2\kappa} \cap (E_n^\kappa)^c) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

where we have used (3.8) and (3.9). We have therefore shown that

$$P_{X_0}^X(\{X_t(x) = 0\} \cap A_{2\kappa}) = 0.$$

This suffices to prove the result, so the proof is complete for $\beta < \frac{\alpha}{d+\alpha}$.

Now suppose that $\beta = \frac{\alpha}{d+\alpha}$. Here we use a martingale argument. The process $P(X_t(x) > 0 | \mathcal{F}_{t-\delta_n})$ is a bounded martingale with respect to the increasing sequence of σ -algebras $\{\mathcal{F}_{t-\delta_n}\}_{n=1}^\infty$. By the martingale convergence theorem it follows that

$$\lim_{n \rightarrow \infty} P(X_t(x) > 0 | \mathcal{F}_{t-\delta_n})(\omega) = P(X_t(x) > 0 | \mathcal{F}_{t-})(\omega) \quad (3.10)$$

for $P_{X_0}^X$ -a.e. ω , where $\mathcal{F}_{t-\delta_n} \uparrow \mathcal{F}_{t-} := \sigma(X_s : 0 \leq s < t)$. We now show that the right hand side of the above is equal to $1(X_t(x) > 0)(\omega)$ almost surely. By (1.6) we have

$$X_t(x) = \liminf_{\epsilon \downarrow 0} \frac{X_t(B(x, \epsilon))}{|B(x, \epsilon)|} = \liminf_{\epsilon \downarrow 0} \frac{X_{t-}(B(x, \epsilon))}{|B(x, \epsilon)|} \quad \text{a.s.},$$

where $X_{t-}(B(x, \epsilon)) = \lim_{s \uparrow t} X_s(B(x, \epsilon))$ exists and equals $X_t(B(x, \epsilon))$ a.s. for all $\epsilon > 0$ by Lemma 3.3. In a slight abuse of notation, let us denote by $X_{t-}(x)$ the quantity on the right hand side of the above. Then $X_{t-}(x) = X_t(x)$ almost surely and $X_{t-}(x)$ is \mathcal{F}_{t-} -measurable. We therefore have, for $P_{X_0}^X$ -a.e. ω ,

$$\begin{aligned} P(X_t(x) > 0 | \mathcal{F}_{t-})(\omega) &= P(X_{t-}(x) > 0 | \mathcal{F}_{t-})(\omega) \\ &= 1(X_{t-}(x) > 0)(\omega) \\ &= 1(X_t(x) > 0)(\omega). \end{aligned}$$

Hence (3.10) implies that

$$\lim_{n \rightarrow \infty} P(X_t(x) > 0 | \mathcal{F}_{t-\delta_n})(\omega) = 1(X_t(x) > 0)(\omega) \quad P_{X_0}^X\text{-a.s.} \quad (3.11)$$

Since $\beta = \frac{\alpha}{d+\alpha}$, the exponent of δ_n in (3.9) is zero, and hence the right hand side of the inequality in (3.9) is bounded above by $\exp(-c_2\kappa) =: c(\kappa) < 1$. From (3.7), (3.9) and (3.11), we can fix ω outside a null set such that (i) E_n^κ occurs for sufficiently large n on $A_{2\kappa}$, (ii) $P(X_t(x) > 0 | \mathcal{F}_{t-\delta_n})(\omega) \rightarrow 1(X_t(x) > 0)(\omega)$ as $n \rightarrow \infty$, and (iii) for all $n \geq 1$, if $\omega \in E_n^\kappa$ we have

$$P(X_t(x) = 0 | \mathcal{F}_{t-\delta_n})(\omega) \leq c(\kappa) < 1.$$

This third condition is equivalent to

$$P(X_t(x) > 0 | \mathcal{F}_{t-\delta_n})(\omega) \geq 1 - c(\kappa) > 0.$$

From (i) and (iii), $P(X_t(x) > 0 | \mathcal{F}_{t-\delta_n})(\omega) \geq 1 - c(\kappa)$ for n sufficiently large. From (ii) we conclude that $1(X_t(x) > 0)(\omega) \geq 1 - c(\kappa) > 0$, and hence $X_t(x) > 0$. We have therefore shown that $P_{X_0}^X(\{X_t(x) = 0\} \cap A_{2\kappa}) = 0$, that is (3.6) holds, for all $\kappa > 0$, which proves the result. Hence the proof is complete for the case $\beta = \frac{\alpha}{d+\alpha}$, and we are done.

Having shown that $P_{X_0}^X(X_t(x) > 0 | X_t \neq 0) = 1$, the result under \mathbb{N}_{x_0} then follows by (2.29). In particular, we obtain that $\mathbb{N}_0(X_t(x) > 0) = \mathbb{N}_0(X_t \neq 0)$, and hence that $u_t^\infty(x) = U_t$. \square

4 Strict positivity of the density

In this section we prove Theorem 1.4, which states that the density is strictly positive under certain conditions in the continuous case. In particular, in dimension one ($d = 1$) with $\alpha > 1 + \beta$ (continuity) and $\beta < \beta^*(\alpha)$ (strong instantaneous propagation), we have

$$X_t(x) > 0 \text{ for all } x \in \mathbb{R} \text{ almost surely on } \{X_t \neq 0\}$$

at a fixed time $t > 0$. The proof of the result hinges in part on the following result, which gives an exponential rate of decay for the left tail of the density conditional on non-negligible nearby mass. Its proof shares many ideas with the proof of Proposition 3.1.

The following holds for general dimensions $d \in \mathbb{N}$. We denote $B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$.

Lemma 4.1. *Let $\beta < \frac{\alpha}{d+\alpha}$. Let $R \geq 1$ and $t \in (0, 1]$. There is a constant $c_{4.1} > 0$ which depends only on (α, β, d) such that for any $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$,*

$$P_{X_0}^X \left(X_t(x) \leq t^{\frac{\alpha-\beta}{\alpha\beta}} \right) \leq e \cdot \exp \left(-c_{4.1} \frac{X_0(B_R)}{R^{d+\alpha}} t^{-q} \right)$$

for all $x \in B_R$, where $q = \frac{1}{\beta} - \frac{d+\alpha}{\alpha} > 0$.

Proof. Fix $t \in (0, 1]$, $R \geq 1$ and $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$. Let $\lambda = t^{-\frac{\alpha-\beta}{\alpha\beta}}$. One can verify directly that

$$1(X_t(x) \leq \lambda^{-1}) \leq \exp(1 - \lambda X_t(x)).$$

Using the above and applying (3.1), we have

$$\begin{aligned} P_{X_0}^X (X_t(x) \leq \lambda^{-1}) &\leq e \cdot E_{X_0}^X (\exp(-\lambda X_t(x))) \\ &= e \cdot \exp \left(- \int u^\lambda(t, y-x) X_0(dy) \right) \\ &\leq e \cdot \exp \left(- \int_{B_R} u^\lambda(t, y-x) X_0(dy) \right). \end{aligned} \quad (4.1)$$

In the last line we simply disregard the mass of μ outside of B_R . Next we obtain a lower bound on the integrand in the above. From (3.3), we have

$$\begin{aligned} u^\lambda(t, y-x) &= t^{-\frac{1}{\beta}} u^{\lambda t^{\frac{\alpha-\beta}{\alpha\beta}}} (1, t^{-\frac{1}{\alpha}}(y-x)) \\ &= t^{-\frac{1}{\beta}} u^1(1, t^{-\frac{1}{\alpha}}(y-x)) \\ &\geq c_{3.2} t^{-\frac{1}{\beta}} p_1(t^{-\frac{1}{\alpha}}(y-x)). \end{aligned}$$

The second line uses the fact that $\lambda t^{\frac{\alpha-\beta}{\alpha\beta}} = 1$ and the third follows from Lemma 3.2. Finally, for all $x, y \in B_R$, we use (2.2) and the above to obtain that

$$u^\lambda(t, y-x) \geq c \frac{t^{-\frac{1}{\beta} + \frac{d+\alpha}{\alpha}}}{R^{d+\alpha}} \quad \forall x, y \in B_R$$

for a constant $c > 0$ depending only on (α, β, d) . Using the above in (4.1), we obtain that

$$P_{X_0}^X (X_t(x) \leq \lambda^{-1}) \leq e \cdot \exp \left(-c X_0(B_R) \frac{t^{-\frac{1}{\beta} + \frac{d+\alpha}{\alpha}}}{R^{d+\alpha}} \right).$$

Since $\lambda = t^{-\frac{\alpha-\beta}{\alpha\beta}}$, this completes the proof. \square

Besides the above, the other main ingredient in the proof of Theorem 1.4 is the Hölder continuity of $X_t(x)$, which we discussed in Section 2.2. In particular we will use (2.4). As can be seen from the proof, the actual index of Hölder continuity is irrelevant. Any positive index works.

Proof of Theorem 1.4. Let $d = 1$ and $\alpha > 1 + \beta$. Let $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$. By scaling it is sufficient to consider the time $t = 1$. We will show that $X_1(x) > 0$ for all $x \in [-R, R]$ $P_{X_0}^X$ -a.s. for every $R \geq 1$, and hence that $X_1(x) > 0$ for all $x \in \mathbb{R}$.

Fix $R \in \mathbb{N}$. As in the proof of Proposition 3.1, we will use instantaneous propagation. In particular, by Theorem A we have

$$P_{X_0}^X(X_1([-R, R]) > 0 \mid X_1 \neq 0) = 1.$$

Let $E^{R, \kappa} = \{X_1([-R, R]) \geq \kappa\}$. By the above, it suffices to show that

$$X_1(x) > 0 \text{ for all } x \in [-R, R] \text{ a.s. on } E^{R, \kappa} \quad (4.2)$$

for all $\kappa > 0$. We fix $\kappa > 0$ and consider the event $E^{R, 2\kappa}$. For a sequence $\{\delta_n\}_{n \in \mathbb{N}} = \{2^{-\gamma n}\}_{n \in \mathbb{N}}$, with $\gamma > 0$ to be specified later, we define events $B_n^{R, \kappa}$ by

$$B_n^{R, \kappa} = \{X_{1-\delta_n}([-R, R]) \geq \kappa\}.$$

By Lemma 3.3, $\lim_{s \uparrow 1} X_s([-R, R]) = X_1([-R, R]) \geq 2\kappa$ a.s. on $E^{R, 2\kappa}$, and hence for a.e. $\omega \in E^{R, 2\kappa}$ there is $s_0(\omega) < 1$ such that $X_s([-R, R]) \geq \kappa$ for all $s \in [s_0(\omega), 1]$. Hence $B_n^{R, \kappa}$ occurs for sufficiently large n , that is,

$$E^{R, 2\kappa} \subseteq \{B_n^{R, \kappa} \text{ eventually}\}, \quad (4.3)$$

where

$$\{B_n^{R, \kappa} \text{ eventually}\} = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_n^{R, \kappa}.$$

For $n \in \mathbb{N}$, let Λ_n denote the set of dyadic lattice points at scale 2^{-n} , i.e. $\Lambda_n = 2^{-n}\mathbb{Z}$. Recall that $R \in \mathbb{N}$. We then let $\Lambda_n^R = [-R, R] \cap \Lambda_n$, that is

$$\Lambda_n = \{-R + k2^{-n} : k = 0, 1, \dots, 2R2^n\}.$$

Next, we define

$$F_n = F_n(R) = \{X_1(x) > \delta_n^{\frac{\alpha-\beta}{\alpha\beta}} \forall x \in \Lambda_n^R\}. \quad (4.4)$$

The first step of our proof is to show that

$$P_{X_0}^X(E^{R, 2\kappa} \cap \{F_n^c \text{ i.o.}\}) = 0, \quad (4.5)$$

where *i.o.* is short for *infinitely often*, meaning

$$\{F_n^c \text{ i.o.}\} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} F_n^c.$$

We now show that (4.5) holds. By (4.3), we have

$$P_{X_0}^X(E^{R, 2\kappa} \cap \{F_n^c \text{ i.o.}\}) \leq P_{X_0}^X(\{B_n^{R, \kappa} \text{ eventually}\} \cap \{F_n^c \text{ i.o.}\})$$

Suppose that ω is in the event on the right hand side of the above; then (i) there is $N(\omega)$ such that $\omega \in B_n^{R,\kappa}$ for all $n \geq N(\omega)$, and (ii) for any $m \in \mathbb{N}$, there is $n > m$ so that $\omega \in F_n^c$. Together, (i) and (ii) imply that for any $m \geq N(\omega)$, there is $n > m$ such that $\omega \in B_n^{R,\kappa} \cap F_n^c$. That is, the above event is a sub-event of $\{B_n^{R,\kappa} \cap F_n^c \text{ i.o.}\}$. Hence

$$\begin{aligned}
P_{X_0}^X(E^{R,2\kappa} \cap \{F_n^c \text{ i.o.}\}) &\leq P_{X_0}^X(\{B_n^{R,\kappa} \cap F_n^c \text{ i.o.}\}) \\
&= P_{X_0}^X\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_n^{R,\kappa} \cap F_n^c\right) \\
&= \lim_{N \rightarrow \infty} P_{X_0}^X\left(\bigcup_{n=N}^{\infty} B_n^{R,\kappa} \cap F_n^c\right) \\
&\leq \lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} P_{X_0}^X(B_n^{R,\kappa} \cap F_n^c). \tag{4.6}
\end{aligned}$$

We bound the probabilities arising in the final term using Lemma 4.1. We condition on $\mathcal{F}_{1-\delta_n}$ and note that $B_n^{R,\kappa} \in \mathcal{F}_{1-\delta_n}$. By the Markov property, we have

$$\begin{aligned}
P(B_n^{R,\kappa} \cap F_n^c | \mathcal{F}_{1-\delta_n}) &= 1(B_n^{R,\kappa}) P_{X_{1-\delta_n}}^X(F_n^c) \\
&\leq \sum_{x \in \Lambda_n^R} 1(B_n^{R,\kappa}) P_{X_{1-\delta_n}}^X\left(X(\delta_n, x) \leq \delta_n^{\frac{\alpha-\beta}{\alpha\beta}}\right).
\end{aligned}$$

By the definition of $B_n^{R,\kappa}$, in the above we need only to compute the probability for such $X_{1-\delta_n}$ as satisfy $X_{1-\delta_n}([-R, R]) \geq \kappa$, in which case we can apply Lemma 4.1 for each $x \in \Lambda_n^R$. From this we obtain (recall that $|\Lambda_n^R| = 2R2^n$)

$$P(B_n^{R,\kappa} \cap F_n^c | \mathcal{F}_{1-\delta_n}) \leq C_1(R) 2^n \exp\left(-c_{4.1} \frac{\kappa}{R^{1+\alpha}} \delta_n^{-q}\right),$$

where $C_1(R) = 2Re$. Recall that we have chosen $\delta_n = 2^{-\gamma n}$, and so, substituting the above into (4.6), we obtain

$$\begin{aligned}
P_{X_0}^X(E^{R,2\kappa} \cap \{F_n^c \text{ i.o.}\}) &\leq \lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} C_1(R) 2^n \exp\left(-c_{4.1} \frac{\kappa}{R^{1+\alpha}} 2^{\gamma q n}\right) \\
&= 0.
\end{aligned}$$

Thus we have shown that (4.5) holds, implying that $\{F_n \text{ eventually}\}$ occurs a.s. on $E^{R,2\kappa}$. Recalling the definition of F_n from (4.4) and that $\delta_n = 2^{-\gamma n}$, it therefore holds that

$$\begin{aligned}
\text{For a.e. } \omega \in E^{R,2\kappa}, \text{ there is } N(\omega) \in \mathbb{N} \text{ such that for all } n \geq N(\omega), \\
X_1(x) > 2^{-\gamma \frac{\alpha-\beta}{\alpha\beta} n} \text{ for each } x \in \Lambda_n^R. \tag{4.7}
\end{aligned}$$

Next we use the Hölder continuity of $X_t(\cdot)$. Let $0 < \eta < \frac{\alpha}{1+\beta} - 1$. By (2.4) with $K = [-R, R]$, for a random constant $C_2(R) = C_2(R, \eta, \omega) > 0$,

$$|X_1(x_1) - X_1(x_2)| \leq C_2(R) |x_1 - x_2|^\eta \text{ for all } x_1, x_2 \in [-R, R]. \tag{4.8}$$

Having chosen η , we can now choose a corresponding value of γ . Let $\gamma = \frac{\alpha\beta}{\alpha-\beta} \frac{\eta}{2}$. Then by (4.7), for n sufficiently large,

$$X_1(x) > 2^{-\frac{\eta}{2} n} \text{ for each } x \in \Lambda_n^R. \tag{4.9}$$

Let $y \in [-R, R]$. We define $[y]_n = \min\{x \in \Lambda_n^R : x \geq y\}$. That is, if $y \notin \Lambda_n^R$, then $[y]_n$ the point in Λ_n^R nearest to y on the right; if $y \in \Lambda_n^R$, then $[y]_n = y$. Note that $|y - [y]_n| < 2^{-n}$ for all $y \in [-R, R]$ by the definition of Λ_n^R . Hence by (4.8),

$$\sup_{y \in [-R, R]} |X_1(y) - X_1([y]_n)| \leq C_2(R) 2^{-\eta n}. \quad (4.10)$$

By the triangle inequality, for $y \in [-R, R]$,

$$X_1(y) \geq X_1([y]_n) - |X_1(y) - X_1([y]_n)|.$$

Note that $\{[y]_n : y \in [-R, R]\} = \Lambda_n^R$. Taking the infimum of the above over $[-R, R]$ and applying (4.10), we obtain that

$$\begin{aligned} \inf_{y \in [-R, R]} X_1(y) &\geq \left(\inf_{y \in [-R, R]} X_1([y]_n) \right) - \sup_{y \in [-R, R]} |X_1(y) - X_1([y]_n)| \\ &\geq \inf_{x \in \Lambda_n^R} X_1(x) - C_2(R) 2^{-\eta n}. \end{aligned}$$

By (4.9), for all sufficiently large n we therefore have

$$\begin{aligned} \inf_{y \in [-R, R]} X_1(y) &\geq 2^{-\frac{\eta}{2}n} - C_2(R) 2^{-\eta n} \\ &= 2^{-\frac{\eta}{2}n} (1 - C_2(R) 2^{-\frac{\eta}{2}n}). \end{aligned}$$

By taking n to be large enough in comparison to $C_2(R)$, the right hand side is positive. This proves that the density is strictly positive on $[-R, R]$ a.s. on $E^{R, 2\kappa}$. Hence (4.2) holds and the proof is complete under $P_{X_0}^X$. The result for \mathbb{N}_{x_0} then follows from (2.29). \square

5 Proof of Theorem 1.8

The goal of this section is to prove Theorem 1.8. We first prove the result for measures which satisfy the mass distribution property (F1)-s, which we now recall: $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfies (F1)-s if

(F1)-s For some constant \overline{C} , for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu(B(x, r)) \leq \overline{C} r^s.$$

We recall that $\beta \leq \beta^*(\alpha, s) = \frac{\alpha}{(d-s)+\alpha}$. In the following results, X_0 is any non-zero measure in $\mathcal{M}_F(\mathbb{R}^d)$ and x_0 is an arbitrary point in \mathbb{R}^d .

Theorem 5.1. *Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ be non-zero and satisfy (F1)-s for some $s \in [0, d]$. Then if $\beta \leq \beta^*(\alpha, s)$, for $t > 0$ we have $\mu(X_t) > 0$ a.s. on $\{X_t \neq 0\}$ under \mathbb{N}_{x_0} and $P_{X_0}^X$, and $u_t^{\infty\mu} = U_t$.*

Most of the work in this section goes into proving the above. We then establish Theorem 1.8 using two other results which we state now. Recall that $\mathcal{M}_F(\mathcal{S})$ is the space of finite measures supported on the set \mathcal{S} and that $\mathcal{H}^s(\mathcal{S})$ is the x^s -Hausdorff measure of \mathcal{S} . For a proof of Frostman's Lemma, see Theorem 8.8 in [22].

Theorem 5.2. *Let $\mu, \nu \in \mathcal{M}_F(\mathbb{R}^d)$ such that $\text{supp}(\mu) \subseteq \text{supp}(\nu)$. Then $u_t^{\infty\mu} = U_t$ implies that $u_t^{\infty\nu} = U_t$. Equivalently, if $\mu(X_t) > 0$ a.s. on $\{X_t \neq 0\}$, the same is true for $\nu(X_t)$.*

Frostman's Lemma. *Suppose that $\mathcal{S} \subset \mathbb{R}^d$ is Borel. Then $\mathcal{H}^s(\mathcal{S}) > 0$ if and only if $\exists \mu \in \mathcal{M}_F(\mathcal{S})$ satisfying (F1)-s.*

Proof of Theorem 1.8. Suppose $\nu \in \mathcal{M}_F(\mathbb{R}^d)$ and $\mathcal{S} = \text{supp}(\nu)$. If $\mathcal{H}^s(\mathcal{S}) > 0$, then by Frostman's Lemma there exists an (F1)-s measure μ supported on \mathcal{S} , and hence $\text{supp}(\mu) \subseteq \text{supp}(\nu)$. By Theorem 5.1, if $\beta \leq \beta^*(\alpha, s)$ then $u_t^{\infty\mu} = U_t$, and the result follows from Theorem 5.2 \square

Theorem 5.2 is proved in Section 7 as a corollary of Theorem 7.3. It can be viewed as a specialized version of the comparison principle for (1.3), and the methods used to prove it are based on the standard comparison principle.

As we alluded to in the introduction, Theorem 5.2 and its implications are somewhat surprising, because if $\mu, \nu \in \mathcal{M}_F(\mathbb{R}^d)$ and $\text{supp}(\mu) \subseteq \text{supp}(\nu)$, ν does not necessarily inherit any mass distribution properties from μ . This is best seen by observing that for any closed set \mathcal{S} , we can construct a purely atomic finite measure ν with $\text{supp}(\nu) = \mathcal{S}$. Hence for a given $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfying (F1)-s for some $s \in (0, d]$, we can always find a purely atomic measure ν with the same support. Moreover, conditional on $\{X_t \neq 0\}$, for such a pair of measures we have $\mu(X_t) > 0$ a.s. if and only if $\nu(X_t) > 0$ a.s., despite the fact that purely atomic measures do not satisfy (F1)-s' for any $s' > 0$.

We briefly consider the case of purely atomic measures in greater detail. Let $(a_i)_{i \geq 1}$ be a summable sequence of positive weights and let $(x_i)_{i \geq 1} \subset \mathbb{R}^d$ be a countable point set. We define $\nu \in \mathcal{M}_F(\mathbb{R}^d)$ by

$$\nu = \sum_{i=1}^{\infty} a_i \delta_{x_i}.$$

It follows from (1.4) that

$$\{\nu(X_t) > 0\} = \{X_t(x_i) > 0 \text{ for some } i \geq 1\}.$$

Let \mathcal{S} be the closure of the point set. Then the values of β for which this event occurs a.s. when X_t is conditioned on survival is determined by the Hausdorff dimension of \mathcal{S} . In particular, the probability that the density is positive on at least one point in a countable set of points is highly dependent on the dimension of the set's closure.

We now continue with the proof of Theorem 5.1. Without loss of generality, we can assume that $\text{supp}(\mu)$ is bounded. Indeed, if (F1)-s holds for μ , then it also holds for the restriction of μ to a bounded set. Furthermore, if μ' denotes this restriction, then $u_t^{\lambda\mu'} \leq u_t^{\lambda\mu}$ for $\lambda > 0$ by the comparison principle, and hence it suffices to show that $u_t^{\infty\mu'} = U_t$. We will further assume that $\text{supp}(\mu) \subseteq B_1 = \{x \in \mathbb{R}^d : |x| \leq 1\}$. This is allowable because, by translation invariance of (1.3), $u_t^{\infty\mu}$ is flat if and only if $u_t^{\infty\mu_z}$ is flat, where μ_z is the translate of μ by $z \in \mathbb{R}^d$. We can therefore translate μ so that it has positive mass in B_1 , then discard the mass outside B_1 by the previous argument.

We set the following standing assumption: for the remainder of this section, let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfy (F1)-s for some $s \in [0, d]$ and $\mu(B_1^c) = 0$. Without loss of generality we will suppose that $\mu(B_1) = \mu(\mathbb{R}^d) = 1$.

The proof of Theorem 5.1 uses a similar argument to the proof of Proposition 3.1. Recall that the bound $u^\infty(1, x) \geq cp_1(x)$ from Lemma 3.2 played a critical role in that result. We use a similar bound here, which however is adapted to $u_t^{\infty\mu}$ for μ satisfying (F1)-s. By monotonicity of $\lambda \rightarrow u_t^{\lambda\mu}$, we have the trivial bound that $u_t^{\infty\mu} \geq u_t^{\lambda\mu}$ for all $\lambda > 0$. As we will also be using scaling properties of these solutions, which involve rescaling the initial measure (see (2.23)), the critical scale turns out to be $u^{r^s\mu(\cdot/r)}(1, x)$, where we recall that for $r > 0$, $\mu(\cdot/r)$ is the measure given by

$\mu(A/r) = \int 1_A(rx) d\mu(x)$. If μ has support \mathcal{S} , then the support of $\mu(\cdot/r)$ is $r\mathcal{S} = \{rx : x \in \mathcal{S}\}$. The next result, which is analogous to Lemma 3.2, gives a lower bound for $u^{r^s \mu(\cdot/r)}(1, x)$.

Lemma 5.3. *Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfy (F1)-s for some $s \in [0, d]$. Then there is a constant $c_{5.3} = c_{5.3}(\mu, \alpha, \beta, d) > 0$ such that for all $r \geq 1$,*

$$u^{r^s \mu(\cdot/r)}(1, x) \geq c_{5.3} r^s S_1(\mu(\cdot/r))(x).$$

The proof of Lemma 5.3 requires the following boundedness result.

Lemma 5.4. *Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfy (F1)-s for $s \in [0, d]$ and fix $\alpha \in (0, 2)$. Then there is a constant $C_{5.4} = C_{5.4}(\mu, \alpha, d) > 0$ such that for all $t > 0$,*

$$\sup_{r \geq 1} \sup_{y \in \mathbb{R}^d} S_t(r^s \mu(\cdot/r))(y) \leq C_{5.4} t^{-\frac{(d-s)}{\alpha}}.$$

Proof. Fix $t, r > 0$. For $y \in \mathbb{R}^d$, we have

$$\begin{aligned} S_t(r^s \mu(\cdot/r))(y) &= r^s \int p_t(\tilde{z} - y) d\mu(\tilde{z}/r) \\ &= r^s \int p_t(rz - y) d\mu(z) \\ &= r^s \int_0^\infty \mu(\{z : p_t(rz - y) \geq k\}) dk, \end{aligned}$$

where the last line uses Fubini's theorem. By (2.2), it follows that

$$\{z : p_t(rz - y) \geq k\} \subseteq \{z : C_{2.1} \frac{t}{|rz - y|^{d+\alpha}} \geq k\} \cap \{C_{2.1} t^{-\frac{d}{\alpha}} \geq k\}.$$

Note that the first set on the right hand side is equal to $B(y/r, c(k^{-1}t)^{\frac{1}{d+\alpha}} r^{-1})$ with $c = C_{2.1}^{\frac{1}{d+\alpha}}$. Using this and the fact that μ satisfies (F1)-s, we have

$$\begin{aligned} S_t(r^s \mu(\cdot/r))(y) &\leq r^s \int_0^{C_{2.1} t^{-\frac{d}{\alpha}}} \mu\left(B(y/r, c(k^{-1}t)^{\frac{1}{d+\alpha}} r^{-1})\right) dk \\ &\leq \bar{C} r^s r^{-s} t^{\frac{s}{d+\alpha}} \int_0^{C_{2.1} t^{-\frac{d}{\alpha}}} k^{-\frac{s}{d+\alpha}} dk \\ &\leq C t^{-\frac{(d-s)}{\alpha}}, \end{aligned}$$

where we recall that $s \leq d$ and so C depends only on \bar{C} , d and α . □

Proof of Lemma 5.3. As in the proof of Lemma 3.2, we will use the Feynman-Kac formula from Section 2.4. We cannot apply Lemma 2.11 directly to $\frac{\partial}{\partial \lambda} u^{\lambda r^s \mu(\cdot/r)}$ because $r^s \mu(\cdot/r)$ is not a function, so for $\epsilon > 0$ we define

$$\psi_{\epsilon, r} = S_\epsilon(r^s \mu(\cdot/r)) = r^s (\mu(\cdot/r) * p_\epsilon). \quad (5.1)$$

Then $\psi_{\epsilon, r}$ is a smooth, bounded, non-negative function, so by Lemma 2.11(b), $w^{\lambda \psi_{\epsilon, r}}(t, x) = \frac{\partial}{\partial \lambda} u^{\lambda \psi_{\epsilon, r}}(t, x)$ exists and

$$w^{\lambda \psi_{\epsilon, r}}(t, x) = E_x^W \left(\psi_{\epsilon, r}(W_t) \exp \left(-(1 + \beta) \int_0^t u^{\lambda \psi_{\epsilon, r}}(t - \tau, W_\tau)^\beta d\tau \right) \right). \quad (5.2)$$

By (2.16), we have

$$u_\tau^{\lambda\psi_{\epsilon,r}} \leq S_\tau(\lambda\psi_{\epsilon,r}) = \lambda S_\tau(\psi_{\epsilon,r})$$

for $\tau > 0$. Hence by (5.2) with $t = 1$ we have

$$\begin{aligned} w^{\lambda\psi_{\epsilon,r}}(1, x) &\geq E_x^W \left(\psi_{\epsilon,r}(W_1) \exp \left(-(1 + \beta)\lambda^\beta \int_0^1 [S_{1-\tau}(\psi_{\epsilon,r})(W_\tau)]^\beta d\tau \right) \right) \\ &= E_x^W \left(\psi_{\epsilon,r}(W_1) \exp \left(-(1 + \beta)\lambda^\beta \int_0^1 [S_{1-\tau+\epsilon}(r^s\mu(\cdot/r))(W_\tau)]^\beta d\tau \right) \right), \end{aligned}$$

where to obtain the final expression we have used (5.1) and the semigroup property. By Lemma 5.4,

$$[S_{1-\tau+\epsilon}(r^s\mu(\cdot/r))(W_\tau)]^\beta \leq C_{5.4}^\beta (1 - \tau + \epsilon)^{-\frac{(d-s)\beta}{\alpha}},$$

and in particular there is a constant $C > 0$ such that for $0 < \epsilon \leq 1$,

$$[S_{1-\tau+\epsilon}(r^s\mu(\cdot/r))(W_\tau)]^\beta \leq C(1 - \tau + \epsilon)^{-\frac{d\beta}{\alpha}}.$$

Using this bound and changing variables, we obtain that

$$w^{\lambda\psi_{\epsilon,r}}(1, x) \geq E_x^W(\psi_{\epsilon,r}(W_1)) \exp \left(-C(1 + \beta)\lambda^\beta \int_0^1 (u + \epsilon)^{-\frac{d\beta}{\alpha}} du \right).$$

Since $\beta < \frac{\alpha}{d}$ the integral remains bounded as $\epsilon \downarrow 0$. Hence for a new constant $C > 0$, we obtain that for all $0 < \epsilon \leq 1$,

$$w^{\lambda\psi_{\epsilon,r}}(1, x) \geq E_x^W(\psi_{\epsilon,r}(W_1)) \exp(-C\lambda^\beta). \quad (5.3)$$

We now integrate over λ to obtain a lower bound for $u^{\psi_{\epsilon,r}}(1, x)$. Since $w^{\lambda\psi_{\epsilon,r}}(1, x) = \frac{\partial}{\partial \lambda} u^{\lambda\psi_{\epsilon,r}}(1, x)$, by (2.26) and (5.3) we have

$$\begin{aligned} u^{\psi_{\epsilon,r}}(1, x) &= \int_0^1 w^{\lambda\psi_{\epsilon,r}}(1, x) d\lambda \\ &\geq E_x^W(\psi_{\epsilon,r}(W_1)) \int_0^1 \exp(-C\lambda^\beta) d\lambda \\ &\geq c_0 E_x^W(\psi_{\epsilon,r}(W_1)) \end{aligned} \quad (5.4)$$

for a constant $c_0 > 0$. It remains to show that the left and right hand sides of the above converge to the desired quantities when $\epsilon \downarrow 0$. It follows from Lemma 2.6 that

$$\lim_{\epsilon \rightarrow 0} u^{\psi_{\epsilon,r}}(1, x) = u^{r^s\mu(\cdot/r)}(1, x). \quad (5.5)$$

Turning to the right hand side of (5.4), we first observe that $E_x^W(\psi_{\epsilon,r}(W_1)) = S_1\psi_{\epsilon,r}(x)$. Thus we may use the Dominated Convergence Theorem to see that

$$\lim_{\epsilon \rightarrow 0} E_x^W(\psi_{\epsilon,r}(W_1)) = S_1(r^s\mu(\cdot/r))(x). \quad (5.6)$$

Letting $\epsilon \downarrow 0$ in (5.4), from (5.5) and (5.6) we obtain that

$$u^{r^s\mu(\cdot/r)}(1, x) \geq c_0 r^s S_1(\mu(\cdot/r))(x)$$

for all $x \in \mathbb{R}^d$, which completes the proof. \square

We now have all the tools we need to prove Theorem 5.1.

Proof of Theorem 5.1. Fix $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$. Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfy (F1)-s as well as our assumptions that $\mu(1) = 1$ and $\mu(B_1^c) = 0$. By Lemma 2.9, in particular using (2.20) with $u_t^{\lambda\mu}$ and taking $\lambda \rightarrow \infty$, we obtain

$$P_{X_0}^X(\mu(X_t) = 0) = \exp(-X_0(u_t^{\infty\mu})). \quad (5.7)$$

Let $\delta_n = 2^{-n}$. Assume that n is large enough so that $\delta_n < t$, and consider the conditional probability $P(\mu(X_t) = 0 | \mathcal{F}_{t-\delta_n})$. Applying the Markov property and using (5.7), we obtain that

$$\begin{aligned} P(\mu(X_t) = 0 | \mathcal{F}_{t-\delta_n}) &= P_{X_{t-\delta_n}}^X(\mu(X_{\delta_n}) = 0) \\ &= \exp\left(-\int u^{\infty\mu}(\delta_n, x) dX_{t-\delta_n}(x)\right) \\ &\leq \exp\left(-\int_{B_2} u^{\infty\mu}(\delta_n, x) dX_{t-\delta_n}(x)\right), \end{aligned} \quad (5.8)$$

where $B_2 = \{x \in \mathbb{R}^d : |x| \leq 2\}$. Using monotonicity of $\lambda \rightarrow u^{\lambda\mu}(\delta_n, x)$ and the scaling relationship (2.24), we have

$$\begin{aligned} u^{\infty\mu}(\delta_n, x) &\geq u^{\delta_n^{-\frac{s}{\alpha} - \frac{\alpha-\beta}{\alpha\beta}}}(\delta_n, x) \\ &= \delta_n^{-\frac{1}{\beta}} u^{\delta_n^{-\frac{s}{\alpha}} \mu(\cdot/\delta_n^{-\frac{1}{\alpha}})}(1, \delta_n^{-\frac{1}{\alpha}} x) \\ &\geq c_{5.3} \delta_n^{-\frac{1}{\beta} - \frac{s}{\alpha}} S_1(\mu(\cdot/\delta_n^{-\frac{1}{\alpha}}))(\delta_n^{-\frac{1}{\alpha}} x). \end{aligned} \quad (5.9)$$

The final inequality follows from Lemma 5.3. We expand the semigroup term in the above as a convolution with p_1 . After a change of variables, we have

$$\begin{aligned} S_1(\mu(\cdot/\delta_n^{-\frac{1}{\alpha}}))(\delta_n^{-\frac{1}{\alpha}} x) &= \int p_1(\delta_n^{-\frac{1}{\alpha}}(x-y)) d\mu(y) \\ &\geq \mu(1) p_1(\delta_n^{-\frac{1}{\alpha}} d(x, \mathcal{S})), \end{aligned} \quad (5.10)$$

where $\mathcal{S} = \text{supp}(\mu)$ and $d(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} |x-y|$, and we recall that for $\rho > 0$, $p_1(\rho)$ denotes $p_1(|z|)$ with $|z| = \rho$. Because μ is supported on B_1 , for any $x \in B_2$ we have $d(x, \mathcal{S}) \leq 3$. In particular, using this in (5.10) and substituting it into (5.9), we obtain

$$u^{\infty\mu}(\delta_n, x) \geq c_{5.3} \delta_n^{-\frac{1}{\beta} - \frac{s}{\alpha}} p_1(3\delta_n^{-\frac{1}{\alpha}}) \quad \text{for all } x \in B_2,$$

where $\mu(1)$ does not appear because it equals one. Using (2.2) to bound p_1 below, we conclude that for a constant $c_1 > 0$,

$$\begin{aligned} u^{\infty\mu}(\delta_n, x) &\geq c_1 \delta_n^{-\frac{1}{\beta} - \frac{s}{\alpha}} (\delta_n^{-\frac{1}{\alpha}})^{-(d+\alpha)} \\ &= c_1 \delta_n^{-q} \quad \text{for all } x \in B_2, \end{aligned} \quad (5.11)$$

where $q := \frac{1}{\beta} - \frac{d-s+\alpha}{\alpha}$. Using (5.11) in (5.8), we obtain the following:

$$P(\mu(X_t) = 0 | \mathcal{F}_{t-\delta_n}) \leq \exp(-c_1 X_{t-\delta_n}(B_2) \delta_n^{-q}).$$

From this point, the proof is identical to that of Proposition 3.1. By instantaneous propagation, $X_t(B_2) > 0$ almost surely on $\{X_t \neq 0\}$. One considers the event $A_{2\kappa} = \{X_t(B_2) \geq 2\kappa\}$ for $\kappa > 0$

and notes that $X_{t-\delta_n}(B_2) \geq \kappa$ eventually a.s. on $A_{2\kappa}$. This leads to a statement analogous to (3.9). One then finishes the proof in the same way: by direct computation when $\beta < \beta^*(\alpha, s)$ and using martingale convergence when $\beta = \beta^*(\alpha, s)$. This completes the proof that $P_{X_0}^X(\mu(X_t) = 0 | X_t > 0)$. The result under \mathbb{N}_x follows from (2.29), which implies that

$$\mathbb{N}_x(\mu(X_t) = 0 | X_t \neq 0) = 0.$$

In particular, we have $\mathbb{N}_x(\mu(X_t) > 0) = \mathbb{N}_x(X_t \neq 0) = U_t$. Since $u_t^{\infty\mu}(x) = \mathbb{N}_x(\mu(X_t) > 0)$, this proves the last claim. \square

6 Proof of Theorem 1.9

This section is concerned with establishing conditions under which $\mathbb{N}_x(\mu(X_t) > 0) = u_t^{\infty\mu}(x)$ is non-flat (and hence $\mu(X_t) = 0$ with positive probability on $\{X_t \neq 0\}$) and quantifying its asymptotic behaviour under these conditions. The main result we prove is Theorem 1.9. The proofs are analytic and we pose it as an open problem to prove the same results using probabilistic arguments.

We will show that $u_t^{\infty\mu} = \mathbb{N}_x(\mu(X_t) > 0)$ is non-flat when $\beta > \beta^*(\alpha, s) = \frac{\alpha}{d-s+\alpha}$, where $s \in [0, d]$ and μ has compact support and satisfies (F2)-s, which we recall is the condition that

(F2)-s For some constant $\underline{C} > 0$, for all $x \in \text{supp}(\mu)$ and $r \in (0, 1]$,

$$\mu(B(x, r)) \geq \underline{C}r^s.$$

The method we use is to show the existence of certain *barrier functions* for the equation (1.3). A function $h : Q \rightarrow \mathbb{R}^+$ is a barrier function for μ if it is a super-solution to (1.3) on Q that explodes on $\text{supp}(\mu)$ with order $t^{-\frac{1}{\beta}}$ and vanishes on $\text{supp}(\mu)^c$ as $t \downarrow 0$. Our method is based on, and adapted from, a similar argument in [3].

First, we define $W : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$W(r) = \frac{\log(e + r^2)}{1 + r^{d+\alpha}}. \quad (6.1)$$

We also introduce $V : \mathbb{R}^d \rightarrow \mathbb{R}^+$, given by

$$V(x) = W(|x|) = \frac{\log(e + |x|^2)}{1 + |x|^{d+\alpha}}. \quad (6.2)$$

For $(t, x) \in Q$, we then define $w_t(x)$ by

$$w_t(x) = t^{-\frac{1}{\beta}}(1 + t^{-\frac{s}{\alpha}})V(t^{-\frac{1}{\alpha}}x). \quad (6.3)$$

Finally, for $k > 0$ and $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, let $h_k(t, x)$ be given by

$$h_k(t, x) = k(w_t * \mu)(x). \quad (6.4)$$

Note that $w_t \in C^{1,2}(Q)$, the space of functions which are once continuously differentiable in time and twice continuously differentiable in space. Consequently, we also have that $h_k \in C^{1,2}(Q)$. Recall that $\beta^*(\alpha, s) = \frac{\alpha}{(d-s)+\alpha}$. In what follows, we restrict to $s \in [0, \alpha)$, since this is required to have $\beta^*(\alpha, s) < \frac{\alpha}{d}$.

For closed $\mathcal{S} \subset \mathbb{R}^d$, recall that $\mathcal{M}_F(\mathcal{S})$ is the space of measures $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ with $\text{supp}(\mu) \subseteq \mathcal{S}$, and that $d(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} |x - y|$.

Proposition 6.1. *Suppose that $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfies (F2)-s for some $s \in [0, \alpha)$ and has compact support $\mathcal{S} \subset \mathbb{R}^d$. Let $\beta^*(\alpha, s) < \beta < \frac{\alpha}{d}$.*

(a) *There exists $\Lambda_0 > 0$ such that if $k \geq \Lambda_0$, h_k is a (strong) supersolution to (1.3) on Q , in the sense that for all $(t, x) \in Q$,*

$$(\partial_t - \Delta_\alpha)h_k(t, x) + h_k(t, x)^{1+\beta} \geq 0.$$

(b) *For $x \in \mathcal{S}$ and $t > 0$,*

$$h_k(t, x) \geq c_{6.5}kt^{-\frac{1}{\beta}}, \quad (6.5)$$

where $c_{6.5} = \underline{C} \cdot c_0$ and $c_0 > 0$ depends only on (α, d) . For all $(t, x) \in Q$,

$$h_k(t, x) \leq k\mu(1)[t^{-\frac{1}{\beta}-\frac{s}{\alpha}} \vee t^{-\frac{1}{\beta}}]W(t^{-\frac{1}{\alpha}}d(x, \mathcal{S})). \quad (6.6)$$

In particular, $\lim_{t \rightarrow 0} h_k(t, x) = \infty$ for $x \in \mathcal{S}$, and $h_k(t, \cdot)$ vanishes uniformly on $\{x : d(x, \mathcal{S}) \geq \rho\}$ as $t \downarrow 0$ for all $\rho > 0$.

(c) *For $(t, x) \in Q$ we have*

$$h_k(t, x) \geq kc_{6.7}t^{-\frac{1}{\beta}}W(t^{-\frac{1}{\alpha}}d(x, \mathcal{S})), \quad (6.7)$$

where $c_{6.7} = \underline{C} \cdot c_1$ and $c_1 > 0$ depends only on (α, d) .

(d) *For any $\nu \in \mathcal{M}_F(\mathcal{S})$, if $k \geq \Lambda_0$ then $h_k(t, x) \geq u^\nu(t, x)$ on Q .*

This proposition is the main result underlying Theorem 1.9 (as well as Theorem 1.13(b)). Before proving it, we comment on the technique. By and large, our method is adapted from the argument used by Chen, Veron and Wang in [3] to prove the result we called Theorem B(b) in the introduction. Our barrier function is modelled after theirs and we make use of some of their intermediate results. Define \tilde{w}_t by

$$\tilde{w}_t(x) = t^{-\frac{1}{\beta}}W(t^{-\frac{1}{\alpha}}|x|).$$

In [3], it is shown that $k\tilde{w}_t(x)$ is a supersolution to (1.3) for sufficiently large k . This barrier function is what the authors use to prove that $\lim_{\lambda \rightarrow \infty} u^{\lambda\delta_0}(t, x)$ is non-flat when $\beta > \frac{\alpha}{d+\alpha}$. Part of their proof was a detailed analysis of $-\Delta_\alpha V$. In equation (5.11) of [3], the following bound is established: there is a constant $c_1 > 0$ such that for $x \in \mathbb{R}^d$ with $|x| \geq 2$,

$$-\Delta_\alpha V(x) \geq -\frac{c_1}{1 + |x|^{d+\alpha}}. \quad (6.8)$$

This bound is critical to their argument and it is equally critical in ours which follows. As can be seen from (6.4), the function $h_k(t, x)$ is essentially \tilde{w}_t spread out over $\mathcal{S} = \text{supp}(\mu)$ via a convolution with μ , with an additional power of t to locally normalize mass of μ when $t \leq 1$. By spreading out \tilde{w}_t over \mathcal{S} , we construct a supersolution which is singular on \mathcal{S} as $t \downarrow 0$.

We make a few observations about the functions we have introduced. The function $W(z)$ is not globally decreasing for positive z , and correspondingly V and w_t are not globally radially decreasing. However, for any $d \geq 1$ and $\alpha \in (0, 2)$, W attains its maximum value at some $r_0 \in [0, 1)$ and $W(r)$ is decreasing for $r \geq r_0$. Furthermore, one can verify that $\min_{r \in [0, 1]} W(r) = W(1)$, and so for all $d \geq 1$ and $\alpha \in (0, 2)$, W is weakly decreasing in the sense that

$$\min_{r' \in [0, r]} W(r') = W(r) \quad \text{for all } r \geq 1. \quad (6.9)$$

V inherits this as a form of weak radial decreasing, i.e.

$$\min_{x \in B(0,R)} V(x) = W(R) \quad \text{for all } R \geq 1. \quad (6.10)$$

Finally, one can show that there is a constant $c_{6.11} > 0$ such that

$$\text{For all } R \geq 0, \quad \sup_{|x| \geq R} V(x) \leq c_{6.11} W(R). \quad (6.11)$$

Proof of Proposition 6.1(a). First let us consider the time derivative of $w_t(x)$. Expanding directly using (6.2) and (6.3), for $z = t^{-\frac{1}{\alpha}}|x|$ we have

$$\begin{aligned} \partial_t(w_t(x)) &= -\frac{1}{\beta} t^{-\frac{1}{\beta}-1} W(t^{-\frac{1}{\alpha}}|x|) + \left(-\frac{1}{\beta} - \frac{s}{\alpha}\right) t^{-\frac{1}{\beta}-\frac{s}{\alpha}-1} W(t^{-\frac{1}{\alpha}}|x|) \\ &\quad - \frac{1}{\alpha} t^{-\frac{1}{\beta}-1} (1 + t^{-\frac{s}{\alpha}}) (t^{\frac{1}{\alpha}}|x|) W'(t^{-\frac{1}{\alpha}}|x|) \\ &= t^{-\frac{1}{\beta}-\frac{s}{\alpha}-1} \left[\left(-\frac{1}{\beta} - \frac{s}{\alpha}\right) W(z) - \frac{1}{\alpha} z W'(z) \right] \\ &\quad + t^{-\frac{1}{\beta}-1} \left[-\frac{1}{\beta} W(z) - \frac{1}{\alpha} z W'(z) \right]. \end{aligned} \quad (6.12)$$

Computing W' directly, we obtain that

$$W'(z) = \frac{2z}{e+z^2} \frac{1}{1+z^{d+\alpha}} - (d+\alpha) \frac{\log(e+z^2) z^{d+\alpha-1}}{(1+z^{d+\alpha})^2}$$

From (6.12), it follows that

$$\begin{aligned} \partial_t(w_t(x)) &= t^{-\frac{1}{\beta}-\frac{s}{\alpha}-1} W(z) \left[\left(-\frac{1}{\beta} - \frac{s}{\alpha}\right) - \frac{1}{\alpha} \frac{2z^2(e+z^2)^{-1}}{\log(e+z^2)} + \frac{d+\alpha}{\alpha} \frac{z^{d+\alpha}}{1+z^{d+\alpha}} \right] \\ &\quad + t^{-\frac{1}{\beta}-1} W(z) \left[-\frac{1}{\beta} - \frac{1}{\alpha} \frac{2z^2(e+z^2)^{-1}}{\log(e+z^2)} + \frac{d+\alpha}{\alpha} \frac{z^{d+\alpha}}{1+z^{d+\alpha}} \right] \\ &= t^{-\frac{1}{\beta}-\frac{s}{\alpha}-1} W(z) f_1(z) + t^{-\frac{1}{\beta}-1} W(z) f_2(z). \end{aligned} \quad (6.13)$$

Consider $f_1(z)$, the square bracketed quantity in the first line. The second term in $f_1(z)$ vanishes as $z \rightarrow \infty$, and the third converges to $\frac{d+\alpha}{\alpha}$. The assumption $\beta > \frac{\alpha}{d-s+\alpha}$ is equivalent to $\frac{1}{\beta} + \frac{s}{\alpha} < \frac{d+\alpha}{\alpha}$; it follows that $f_1(z)$ is positive for sufficiently large z . Moreover, there are constants $R_0 > 0$ and $c_2 > 0$ such that for $z \geq R_0$, $f_1(z) \geq c_2$. Since $f_2(z) > f_1(z)$ for all $z \geq 0$, we also have $f_2(z) \geq c_2$ for $z \geq R_0$, and from (6.13) we have the following: for all (t, x) satisfying $t^{-\frac{1}{\alpha}}|x| \geq R_0$,

$$\partial_t(w_t(x)) \geq c_2 t^{-\frac{1}{\beta}-1} (1 + t^{-\frac{s}{\alpha}}) W(t^{-\frac{1}{\alpha}}|x|). \quad (6.14)$$

Next we consider $\Delta_\alpha w_t$. By the scaling of the α -stable process, if $g^\lambda(x) = g(\lambda x)$, then $\Delta_\alpha g^\lambda(x) = \lambda^\alpha (\Delta_\alpha g)(\lambda x)$. Using this and (6.3), it follows that

$$\Delta_\alpha w_t(x) = t^{-\frac{1}{\beta}-1} (1 + t^{-\frac{s}{\alpha}}) (\Delta_\alpha V)(t^{-\frac{1}{\alpha}}x). \quad (6.15)$$

Using the above and (6.8), for all (t, x) such that $t^{-\frac{1}{\alpha}}|x| \geq 2$,

$$-\Delta_\alpha w_t(x) \geq -c_1 t^{-\frac{1}{\beta}-1} (1 + t^{-\frac{s}{\alpha}}) \frac{1}{1 + |t^{-\frac{1}{\alpha}}x|^{d+\alpha}}. \quad (6.16)$$

This bound allows a direct comparison with $\partial_t(w_t(x))$. In particular, by (6.14) and (6.16), (and recalling the definition of W from (6.1)) we have, for $|t^{-\frac{1}{\alpha}}x| \geq 2 \vee R_0$,

$$(\partial_t - \Delta_\alpha)w_t(x) \geq \frac{t^{-\frac{1}{\beta}-1}(1+t^{-\frac{s}{\alpha}})}{1+|t^{-\frac{1}{\alpha}}x|^{d+\alpha}} \left[c_2 \log(e+|t^{-\frac{1}{\alpha}}x|^2) - c_1 \right].$$

It follows that for some $R_1 \geq R_0 \vee 2$,

$$(\partial_t - \Delta_\alpha)w_t(x) \geq 0 \quad \text{for all } (t, x) \text{ satisfying } |t^{-\frac{1}{\alpha}}x| \geq R_1. \quad (6.17)$$

Now consider $h_k(t, x)$. We can take differentiation under the integral in (6.4) to obtain

$$(\partial_t - \Delta_\alpha)h_k(t, x) = k \int (\partial_t - \Delta_\alpha)w_t(x-y)d\mu(y). \quad (6.18)$$

Recall that $d(x, \mathcal{S})$ denotes the distance from $x \in \mathbb{R}^d$ to the set \mathcal{S} . If $t^{-\frac{1}{\alpha}}d(x, \mathcal{S}) \geq R_1$, from (6.17) the integrand in (6.18) is positive for all $y \in \mathcal{S}$, i.e. for all $y \in \text{supp}(\mu)$, and hence

$$(\partial_t - \Delta_\alpha)h_k(t, x) \geq 0 \quad \text{for all } (t, x) \text{ satisfying } t^{-\frac{1}{\alpha}}d(x, \mathcal{S}) \geq R_1. \quad (6.19)$$

The condition on (t, x) that $t^{-\frac{1}{\alpha}}d(x, \mathcal{S}) \geq R_1$ will be important, so we introduce

$$Q^{\geq R_1} = \{(t, x) \in Q : t^{-\frac{1}{\alpha}}d(x, \mathcal{S}) \geq R_1\}.$$

The statement (6.19) then reads that

$$(\partial_t - \Delta_\alpha)h_k(t, x) \geq 0 \quad \text{for all } (t, x) \in Q^{\geq R_1}. \quad (6.20)$$

We also introduce

$$Q^{< R_1} = \{(t, x) \in Q : t^{-\frac{1}{\alpha}}d(x, \mathcal{S}) < R_1\},$$

and now consider the behaviour of $(\partial_t - \Delta_\alpha)h_k$ on $Q^{< R_1}$. We can apply (6.17) to the integrand in (6.18) to obtain that

$$(\partial_t - \Delta_\alpha)h_k(t, x) \geq k \int_{B(0, t^{\frac{1}{\alpha}}R_1)} (\partial_t - \Delta_\alpha)w_t(y)d\mu(y-x). \quad (6.21)$$

By (6.12) and (6.15), for $u = t^{-\frac{1}{\alpha}}x$ we have

$$\begin{aligned} (\partial_t - \Delta_\alpha)w_t(y) &= t^{-\frac{1}{\beta}-\frac{s}{\alpha}-1} \left[\left(-\frac{1}{\beta} - \frac{s}{\alpha} \right) W(|u|) - \frac{1}{\alpha}|u|W'(|u|) - \Delta_\alpha V(u) \right], \\ &\quad + t^{-\frac{1}{\beta}-1} \left[-\frac{1}{\beta}W(|u|) - \frac{1}{\alpha}|u|W'(|u|) - \Delta_\alpha V(u) \right]. \end{aligned}$$

Since all the terms in above are continuous functions of u and $|u| \leq R_1$ on $\{y : |y| \leq t^{\frac{1}{\alpha}}R_1\}$, the square-bracketed terms above are bounded on this set. Thus

$$\sup_{y:|y|\leq t^{\frac{1}{\alpha}}R_1} |(\partial_t - \Delta_\alpha)w_t(y)| \leq c_3 t^{-\frac{1}{\beta}-1} [t^{-\frac{s}{\alpha}} \vee 1]$$

for some $0 < c_3 < \infty$. Using this in (6.21) we obtain that

$$(\partial_t - \Delta_\alpha)h_k(t, x) \geq -kc_3 t^{-\frac{1}{\beta}-1} \left[\frac{\mu(B(x, t^{\frac{1}{\alpha}} R_1))}{t^{\frac{s}{\alpha}} \wedge 1} \right] \quad (6.22)$$

for all $(t, x) \in Q$. We now must show that the non-linear term in (1.3), given by $h_k(t, x)^{1+\beta}$, is sufficiently large on $Q^{<R_1}$ so that $h_k(t, x)$ is a super-solution to (1.3) even in the case of the worst-case bound given in (6.22). From (6.3) and (6.4), we have

$$\begin{aligned} \frac{1}{k^{1+\beta}} h_k(t, x)^{1+\beta} &= \left[\int w_t(x-y) d\mu(y) \right]^{1+\beta} \\ &= \left[t^{-\frac{1}{\beta}} (1 + t^{-\frac{s}{\alpha}}) \int V(t^{-\frac{1}{\alpha}}(x-y)) d\mu(y) \right]^{1+\beta} \\ &\geq \left[(t^{-\frac{1}{\beta}-\frac{s}{\alpha}} \vee t^{-\frac{1}{\beta}}) \int_{B(x, t^{\frac{1}{\alpha}}(R_1+1))} V(t^{-\frac{1}{\alpha}}(x-y)) d\mu(y) \right]^{1+\beta} \end{aligned}$$

where the final inequality holds because $V \geq 0$. Restricted to $B(x, t^{\frac{1}{\alpha}}(R_1+1))$, $V(t^{-\frac{1}{\alpha}}(x-y))$ is bounded below by $W(R_1+1)$ by (6.10). This implies that

$$\frac{1}{k^{1+\beta}} h_k(t, x)^{1+\beta} \geq c_4 t^{-\frac{1}{\beta}-1} \left[\frac{\mu(B(x, t^{\frac{1}{\alpha}}(R_1+1)))}{t^{\frac{s}{\alpha}} \wedge 1} \right]^{1+\beta} \quad (6.23)$$

on $Q^{<R_1}$, where $c_4 = W(R_1+1)^{1+\beta}$. Combining (6.22) and (6.23), we obtain that for $x \in Q_T^{<R_1}$,

$$\begin{aligned} &(\partial_t - \Delta_\alpha)h_k(t, x) + h_k(t, x)^{1+\beta} \\ &\geq kt^{-\frac{1}{\beta}-1} \left(c_4 k^\beta \left[\frac{\mu(B(x, t^{\frac{1}{\alpha}}(R_1+1)))}{t^{\frac{s}{\alpha}} \wedge 1} \right]^{1+\beta} - c_3 \left[\frac{\mu(B(x, t^{\frac{1}{\alpha}} R_1))}{t^{\frac{s}{\alpha}} \wedge 1} \right] \right) \\ &\geq kt^{-\frac{1}{\beta}-1} \left[\frac{\mu(B(x, t^{\frac{1}{\alpha}} R_1))}{t^{\frac{s}{\alpha}} \wedge 1} \right] \left(c_4 k^\beta \left[\frac{\mu(B(x, t^{\frac{1}{\alpha}}(R_1+1)))}{t^{\frac{s}{\alpha}} \wedge 1} \right]^\beta - c_3 \right). \quad (6.24) \end{aligned}$$

Since $(t, x) \in Q^{<R_1}$, there must be a point $y_0 \in \mathcal{S}$ such that $B(y_0, t^{\frac{1}{\alpha}}) \subset B(x, t^{\frac{1}{\alpha}}(R_1+1))$. In particular, since μ satisfies (F2)-s we have

$$\mu(B(x, t^{\frac{1}{\alpha}}(R_1+1))) \geq \mu(B(y_0, t^{\frac{1}{\alpha}})) \geq \underline{C} [t^{\frac{s}{\alpha}} \wedge 1].$$

The minimum above appears since for $y_0 \in \mathcal{S}$ and $t \geq 1$, $\mu(B(y_0, t^{\frac{1}{\alpha}})) \geq \mu(B(y_0, 1)) \geq \underline{C}$. Using this in (6.24), we obtain

$$(\partial_t - \Delta_\alpha)h_k(t, x) - h_k(t, x)^{1+\beta} \geq kt^{-\frac{1}{\beta}-1} \left[\frac{\mu(B(x, t^{\frac{1}{\alpha}} R_1))}{t^{\frac{s}{\alpha}} \wedge 1} \right] (c'_4 k^\beta - c_3)$$

for all $(t, x) \in Q^{<R_1}$, where $c'_4 = \underline{C}^\beta c_4$. For sufficiently large k the above is non-negative, and hence $h_k(t, x)$ is a super-solution to (1.3) on $Q^{<R_1}$. On the other hand, because $h_k(t, x)^{1+\beta} \geq 0$, (6.20) implies that $h_k(t, x)$ is a super-solution on $Q^{\geq R_1}$. Thus we have shown that for sufficiently large k , $h_k(t, x)$ is a super-solution to (1.3) on $Q^{<R_1} \cup Q^{\geq R_1} = Q$. \square

Proof of Proposition 6.1(b)-(d). We first show part (c). For fixed x , since \mathcal{S} is closed, there is a point $y_0 \in \mathcal{S}$ such that $|x - y_0| = d(x, \mathcal{S})$. Hence from (6.4),

$$\begin{aligned} h_k(t, x) &\geq k \int_{B(y_0, t^{\frac{1}{\alpha}})} w_t(x - y) d\mu(y) \\ &= kt^{-\frac{1}{\beta}} (1 + t^{-\frac{s}{\alpha}}) \int_{B(y_0, t^{\frac{1}{\alpha}})} W(t^{-\frac{1}{\alpha}} |x - y|) d\mu(y) \\ &\geq k [t^{-\frac{1}{\beta}} \vee t^{-\frac{1}{\beta} - \frac{s}{\alpha}}] W(t^{-\frac{1}{\alpha}} d(x, \mathcal{S}) + 1) \mu(B(y_0, t^{\frac{1}{\alpha}})), \end{aligned}$$

where the last line has used the triangle inequality and (6.9). Since μ satisfies (F2)-s, $\mu(B(y_0, t^{\frac{1}{\alpha}})) \geq \underline{C} t^{\frac{s}{\alpha}}$ for $t \leq 1$ and $\mu(B(y_0, t^{\frac{1}{\alpha}})) \geq \underline{C}$ for $t > 1$. This implies that above is bounded below by $k \underline{C} t^{-\frac{1}{\beta}} W(t^{-\frac{1}{\alpha}} d(x, \mathcal{S}) + 1)$. The result then follows from (6.11).

Next we prove part (b). The claim for $x \in \mathcal{S}$, i.e. (6.5), follows from part (c). Now let $x \in \mathbb{R}^d$. Applying (6.11), we obtain that for every $y \in \mathcal{S}$,

$$w_t(x - y) \leq c_{6.11} [t^{-\frac{1}{\beta} - \frac{s}{\alpha}} \vee t^{-\frac{1}{\beta}}] W(t^{-\frac{1}{\alpha}} d(x, \mathcal{S})).$$

One then uses this bound directly in the convolution defining $h_k(t, x)$, i.e. (6.4), to obtain (6.6). Using formula (6.1) for W and the fact that $\frac{1}{\beta} + \frac{s}{\alpha} < \frac{d+\alpha}{\alpha}$, the uniform convergence as $t \downarrow 0$ follows.

We now prove (d). Suppose that $\nu \in \mathcal{M}_F(\mathcal{S})$. (This includes the case $\nu = \mu$.) In order to show that $u^\nu(t, x) \leq h_k(t, x)$, we will need to consider a sequence of solutions corresponding to a sequence of approximations of ν . Let $Z > 0$ be the normalizing constant such that

$$Z^{-1} \int V(x) dx = 1.$$

Let $\tilde{V}(\cdot) = Z^{-1} V(\cdot)$ and define a sequence of C^2 mollifiers by $\phi_n(x) = n^d \tilde{V}(nx)$ for $n \geq 1$. Then it is immediate that $\nu * \phi_n \rightarrow \nu$ in the weak sense of measures as $n \rightarrow \infty$. Let $u_n(t, x) = u^{\nu * \phi_n}(t, x)$. By Lemma 2.6(a),

$$\lim_{n \rightarrow \infty} u_n(t, x) = u^\nu(t, x) \tag{6.25}$$

for all $(t, x) \in Q$. For $x \in \mathbb{R}^d$,

$$\begin{aligned} \nu * \phi_n(x) &= \frac{1}{Z} n^d \int W(n|y - x|) d\nu(y) \\ &\leq \frac{1}{Z} \nu(1) n^d \sup_{y \in \mathcal{S}} W(n|x - y|) \end{aligned}$$

Hence from (6.11) we obtain that

$$\nu * \phi_n(x) \leq \frac{c_{6.11} \nu(1)}{Z} [n^d W(nd(x, \mathcal{S}))]. \tag{6.26}$$

Let $t_n = n^{-\alpha}$. By (6.7), we have

$$\begin{aligned} h_k(t_n, x) &\geq c_{6.7} k n^{\frac{\alpha}{\beta}} W(nd(x, \mathcal{S})) \\ &= c_{6.7} k n^{\frac{\alpha}{\beta} - d} [n^d W(nd(x, \mathcal{S}))] \end{aligned} \tag{6.27}$$

By part (a), if $k \geq \Lambda_0$, then $(t, x) \rightarrow h_k(t_n + t, x)$ is a super-solution to (1.3) on Q with initial value $h_k(t_n, \cdot)$. Since $\frac{\alpha}{\beta} > d$, it follows from (6.26) and (6.27) that, for sufficiently large n ,

$$\nu * \phi_n(x) \leq h_k(t_n, x).$$

Since u_n has initial data $\nu * \phi_n$ and $(t, x) \rightarrow h_k(t_n + t, x)$ is a super-solution, the above and the comparison principle imply that

$$u_n(t, x) \leq h_k(t_n + t, x)$$

for all $(t, x) \in Q$. Taking $n \rightarrow \infty$ on both sides and using (6.25), we obtain that

$$u^\nu(t, x) \leq h_k(t, x)$$

for all $(t, x) \in Q$. □

The final ingredient in the proof of Theorem 1.9 is the following lower bound on $u_t^{\infty\mu}(x)$. We recall that $u_t^\infty(x) = \lim_{\lambda \rightarrow \infty} u_t^{\lambda\delta_0}(x)$.

Lemma 6.2. *Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. Then for every $z \in \text{supp}(\mu)$,*

$$u_t^{\infty\mu}(x) \geq u_t^\infty(x - z) \geq c_{6.2} t^{-\frac{1}{\beta}} p_1(t^{-\frac{1}{\alpha}}(x - z))$$

for all $(t, x) \in Q$, where $c_{6.2} > 0$ depends only on (α, β, d) .

This lemma follows from Lemma 7.2 and Proposition 7.4 of the next section. However, we note that it can be proven directly along very similar lines to Proposition 7.4 without introducing the notion of weak solutions.

Proof of Theorem 1.9. Fix $s \in [0, \alpha)$ and let μ satisfy (F2)- s and have compact support \mathcal{S} . Let $\beta^*(\alpha, s) < \beta < \frac{\alpha}{d}$. We begin with part (b). We need to show that the upper bound in (1.16) holds, i.e. for some constant $C > 0$,

$$u_t^{\infty\mu}(x) \leq C[t^{-\frac{1}{\beta} - \frac{s}{\alpha}} \vee t^{-\frac{1}{\beta}}]W(t^{-\frac{1}{\alpha}}d(x, \mathcal{S}))$$

on Q . Let $\Lambda_0 > 0$ be as in Proposition 6.1(a). Then by Proposition 6.1(d), $u_t^{\lambda\mu}(x) \leq h_{\Lambda_0}(t, x)$ for every $\lambda > 0$, and hence $u_t^{\infty\mu}(x) = \lim_{\lambda \rightarrow \infty} u_t^{\lambda\mu}(x) \leq h_{\Lambda_0}(t, x)$. The above bound then follows from Proposition 6.1(b) with $C = \mu(1)\Lambda_0$, which proves the upper bound in (1.16). The lower bound in (1.16) follows from Lemma 6.2.

Next we prove part (a). From the upper bound in (1.16), it is clear that for $t > 0$, $u_t^{\infty\mu}(x) \rightarrow 0$ as $d(x, \mathcal{S}) \rightarrow \infty$. We can thus take $x_0 \in \mathbb{R}^d$ so that $u_t^{\infty\mu}(x_0) < U_t$, which is equivalent to $\mathbb{N}_{x_0}(\mu(X_t) = 0 | X_t \neq 0) > 0$. By (2.28), we have $P_{\delta_{x_0}}^X(\mu(X_t) = 0 | X_t \neq 0) > 0$. Appealing to (2.5), it follows that

$$P_{X_0}^X(\mu(X_t) = 0 | X_t \neq 0) > 0$$

for every $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$. In particular, we can take $X_0 = \delta_x$ and use (2.28) again to see that $\mathbb{N}_x(\mu(X_t) = 0 | X_t \neq 0) > 0$ for every $x \in \mathbb{R}^d$. This completes the proof of (a).

To see that part (d) holds, recall that Proposition 6.1(d) applies to any $\nu \in \mathcal{M}_F(\mathcal{S})$. The upper bound in (1.18) then follows by the same argument used to prove the upper bound in (1.16) above. The argument used to prove (a) can then be used to prove that the claims from (a) hold when ν is replaced with μ . Finally, as in the proof of part (b), the lower bound in (1.18) follows from Lemma 6.2.

It remains to show part (c). Recall from (2.25) that

$$u_t^{\infty\mu}(x) = t^{-\frac{1}{\beta}} u_1^{\infty\mu}(\cdot/t^{-\frac{1}{\alpha}})(t^{-\frac{1}{\alpha}}x).$$

If μ satisfies (F1)-s, for $t \leq 1$ we apply Lemma 5.3 to the right hand side of the above to obtain

$$\begin{aligned} u_t^{\infty\mu}(x) &\geq c_{5.3} t^{-\frac{1}{\beta} - \frac{s}{\alpha}} \int p_1(y - t^{-\frac{1}{\alpha}}x) d\mu(y/t^{-\frac{1}{\alpha}}) \\ &= c_{5.3} t^{-\frac{1}{\beta} - \frac{s}{\alpha}} \int p_1(t^{-\frac{1}{\alpha}}(y - x)) d\mu(y). \end{aligned}$$

For every $y \in \mathcal{S}$, $|y - x| \leq d(x, \mathcal{S}) + \text{diam}(\mathcal{S})$, so from the above, for $x \in \mathbb{R}^d$ and $t \leq 1$ we have

$$u_t^{\infty\mu}(x) \geq c_{5.3} \mu(1) t^{-\frac{1}{\beta} - \frac{s}{\alpha}} p_1(t^{-\frac{1}{\alpha}}(d(x, \mathcal{S}) + \text{diam}(\mathcal{S}))).$$

The lower bound on p_1 from (2.1) then implies (1.17). \square

7 The initial trace problem

We now apply the results of Sections 5 and 6 to the initial trace theory for (1.3). Recall that the initial trace of a solution $u_t(x)$ to (1.3) was defined in (1.19) and (1.20). We restate the definition here for convenience. A pair (\mathcal{S}, ν) with $\mathcal{S} \subset \mathbb{R}^d$ a closed set and ν a Radon measure with $\nu(\mathcal{S}) = 0$ is the *initial trace* of $u_t(x)$ if:

- For all $\xi \in C_c(\mathcal{S}^c)$.

$$\lim_{t \rightarrow 0^+} \int \xi(x) u_t(x) dx = \int \xi d\nu. \quad (7.1)$$

- For every $z \in \mathcal{S}$ and $\rho > 0$,

$$\lim_{t \rightarrow 0^+} \int_{B(z, \rho)} u_t(x) dx = +\infty. \quad (7.2)$$

Our contribution is to the problem of determining when a solution with a given initial trace exists. We consider weak solutions. First, recall that we have defined a weak solution to (1.3) in Definition 2.3. Although we only consider the problem for initial traces in which the regular component (i.e. the Radon measure) is null, our definition applies for general initial traces.

Definition 7.1. For closed $\mathcal{S} \subset \mathbb{R}^d$ and a Radon measure ν satisfying $\nu(\mathcal{S}) = 0$, we say that $u : Q \rightarrow [0, \infty)$ is a weak solution to the initial trace problem with initial trace (\mathcal{S}, ν) if:

- u is a weak solution to (1.3) in the sense of Definition 2.3.
- The initial trace (\mathcal{S}, ν) is attained in the sense that (7.1) and (7.2) hold.

Our main result about the initial trace problem is Theorem 1.13, which has two parts: non-existence and existence. Most of the work has already been carried out in Sections 5 and 6. Both proofs require the following lemma.

Lemma 7.2. For $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, $u_t^{\infty\mu}(x) = \lim_{\lambda \rightarrow \infty} u_t^{\lambda\mu}(x)$ is a weak solution to (1.3). Furthermore, for all $\rho > 0$ and $z \in \text{supp}(\mu)$,

$$\lim_{t \rightarrow 0} \int_{B(z, \rho)} u_t^{\infty\mu}(x) dx = +\infty, \quad (7.3)$$

and hence the singular set of $u_t^{\infty\mu}$ contains $\text{supp}(\mu)$.

Proof. Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. We must show that $u^{\infty\mu}(t, x)$ satisfies (2.13) for every $\xi \in C_c^{1,2}(Q)$. For $\lambda > 0$, $u^{\lambda\mu}(t, x)$ is a solution to the problem (2.14) with $u_0 = \lambda\mu$, so by definition we have

$$\int_Q (u^{\lambda\mu}(t, x)[- \partial_t \xi(t, x) - \Delta_\alpha \xi(t, x)]) + u^{\lambda\mu}(t, x)^{1+\beta} \xi(t, x) dx dt = 0.$$

Since ξ has compact support, by (1.15) the bound $u^{\lambda\mu}(t, x) \leq u^{\infty\mu}(t, x) \leq U_t$ allows us to apply Dominated Convergence and conclude that $u^{\infty\mu}(t, x)$ satisfies (2.13) for all $\xi \in C_c^{1,2}(Q)$. Similarly, $u^{\infty\mu}(t, x) \leq U_t$ implies that $u^{\infty\mu}(t, x)$ is bounded on $(\epsilon, \infty) \times \mathbb{R}^d$ for all $\epsilon > 0$ and hence $u^{\infty\mu} \in L_{\text{loc}}^{1+\beta}(Q)$. To see the $u_t^{\infty\mu}(x)$ is continuous, we note that for any $t_0 > 0$, $(t, x) \rightarrow u^{\infty\mu}(t_0 + t, x)$ is a weak solution to (1.3) which is globally bounded by U_{t_0} . In particular, (recall Remark 2.8) it is a solution to the integral equation (1.2) with initial data $u_{t_0}^{\infty\mu} \in \mathcal{B}_b^+$ and hence is continuous. It follows that $u_t^{\infty\mu}(x)$ is continuous on $[t_0, \infty) \times \mathbb{R}^d$ for all $t_0 > 0$ and hence on Q . Thus $u_t^{\infty\mu}(x)$ is a weak solution to (1.3).

We now show that (7.3) holds. Let $z \in \text{supp}(\mu)$. Then $\mu(B(z, \rho)) > 0$ for every $\rho > 0$. For $k > 0$ and $\rho > 0$, we have

$$\liminf_{t \rightarrow 0} \int_{B(z, \rho)} u_t^{k\mu}(x) dx \geq k\mu(B(z, \rho/2)).$$

Since $u_t^{\infty\mu} \geq u_t^{k\mu}$ for all $k > 0$, it follows that

$$\lim_{t \rightarrow 0} \int_{B(z, \rho)} u_t^{\infty\mu}(x) dx = +\infty \quad \text{for all } \rho > 0, \quad (7.4)$$

which completes the proof. \square

Our main result concerning flatness and non-existence is the following. We recall from the introduction the space \mathcal{U} of positive functions on Q bounded above by U_t , and that we have restricted our attention to solutions in \mathcal{U} . The case $\mathcal{S} = \mathbb{R}^d$ is excluded in the following, and we remind the reader that $U_t \in \mathcal{U}$ is a solution with singular set \mathbb{R}^d .

Theorem 7.3. *Suppose that $\mathcal{S} \subset \mathbb{R}^d$ is closed and supports a measure $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ for which $\lim_{\lambda \rightarrow \infty} u_t^{\lambda\mu} = U_t$. If \mathcal{S} is contained in the singular set of a solution u to (1.3) in \mathcal{U} , then $u_t = U_t$ (and u_t has singular set \mathbb{R}^d). Hence there is no solution to (1.3) in \mathcal{U} with singular set \mathcal{S} .*

We obtain Theorem 5.2 and Theorem 1.13(a) as corollaries as follows.

Proof of Theorem 5.2. Let $\mu, \nu \in \mathcal{M}_F(\mathbb{R}^d)$ with $\text{supp}(\mu) \subseteq \text{supp}(\nu)$ and $u_t^{\infty\mu} = U_t$. By Lemma 7.2, $u_t^{\infty\nu}$ is a weak solution to (1.3) whose singular set contains $\text{supp}(\nu)$, which itself contains $\text{supp}(\mu)$. Since $u_t^{\infty\mu} = U_t$, Theorem 7.3 implies that $u_t^{\infty\nu}(x) = U_t$. This is equivalent to $\nu(X_t) > 0$ a.s. on $\{X_t \neq 0\}$ (with respect to both $P_{X_0}^X$ and \mathbb{N}_{x_0}) and result follows. \square

Proof of Theorem 1.13(a). If $\mathcal{S} \subset \mathbb{R}^d$ is closed with $\mathcal{H}^{d_{\text{sat}}}(\mathcal{S}) > 0$, then by Frostman's Lemma there exists $\mu \in \mathcal{M}_F(\mathcal{S})$ which satisfies (F1)- d_{sat} . By Theorem 5.1(a), $u_t^{\infty\mu} = U_t$, and the result follows from Theorem 7.3. \square

The proof of Theorem 7.3 requires the following pointwise estimate for weak solutions with a given singular set. (A more general version of this result is proved for classical solutions in [4], where it was called Theorem C.) Recall that $u^\infty(t, x) = \lim_{\lambda \rightarrow \infty} u^{\lambda\delta_0}(t, x)$.

Proposition 7.4. *Suppose that $u(t, x)$ is a weak solution to (1.3) in \mathcal{U} whose singular set contains $\mathcal{S} \subset \mathbb{R}^d$. Then for every $z \in \mathcal{S}$,*

$$u(t, x) \geq u^\infty(t, x - z) \geq c_{7.4} t^{-\frac{1}{\beta}} p_1(t^{-\frac{1}{\alpha}}(x - z))$$

for all $(t, x) \in Q$, where $c_{7.4} > 0$ depends only on (α, β, d) .

Proof. Suppose that $z \in \mathcal{S}$. Let $\lambda > 0$. Since (7.2) holds, there must be sequences $(t_n)_{n \geq 1}$ and $(\rho_n)_{n \geq 1}$ such that $t_n, \rho_n \rightarrow 0$ and

$$\int_{B(z, \rho_n)} u(t_n, x) dx = \lambda.$$

Let $\phi_n(x) = 1_{B(z, \rho_n)}(x)u(t_n, x)$. We then have

$$u(t_n, x) \geq \phi_n(x)$$

for all $x \in \mathbb{R}^d$. Since $u(t_n, \cdot)$ and ϕ_n are both bounded (because $u \in \mathcal{U}$), it follows from the comparison principle (recall Remark 2.10) that

$$u(t_n + t, x) \geq u^{\phi_n}(t, x) \tag{7.5}$$

for all $(t, x) \in Q$. Note that $\phi_n \rightarrow \lambda \delta_z$ in $\mathcal{M}_F(\mathbb{R}^d)$, so by Lemma 2.6(a) and translation invariance, $u^{\phi_n}(t, x) \rightarrow u^\lambda(t, x - z)$. (Recall that $u_t^\lambda(x) = u^{\lambda \delta_0}(t, x)$.) Of course, $\lim_{n \rightarrow \infty} u(t_n + t, x) = u(t, x)$, and so taking $n \rightarrow \infty$ in (7.5) we obtain

$$u(t, x) \geq u^\lambda(t, x - z).$$

Since this holds for all $\lambda > 0$, it follows that $u(t, x) \geq u^\infty(t, x - z)$. The second inequality in the result then follows from (3.5). \square

Proof of Theorem 7.3. Let \mathcal{S} and μ be as in the statement of the theorem. Suppose that $u_t(x)$ is a solution to (1.3) in \mathcal{U} whose singular set contains \mathcal{S} . By Proposition 7.4, for some constant $c > 0$, for $x \in \mathcal{S}$ we have

$$\begin{aligned} u_t(x) &\geq \sup_{z \in \mathcal{S}} c t^{-\frac{1}{\beta}} p_1(t^{-\frac{1}{\alpha}}(x - z)) \\ &= c t^{-\frac{1}{\beta}} p_1(t^{-\frac{1}{\alpha}} d(x, \mathcal{S})), \end{aligned} \tag{7.6}$$

where the second equality holds because \mathcal{S} is closed and p_1 is continuous and radially decreasing. By assumption, \mathcal{S} supports a finite measure μ such that $\lim_{\lambda \rightarrow \infty} u_t^{\lambda \mu} = U_t$. Fix $\lambda > 0$. By (2.16),

$$\begin{aligned} u_t^{\lambda \mu}(x) &\leq \lambda S_t \mu(x) \\ &\leq \lambda \mu(1) p_t(d(x, \mathcal{S})) \\ &= \lambda \mu(1) t^{-\frac{d}{\alpha}} p_1(t^{-\frac{1}{\alpha}} d(x, \mathcal{S})). \end{aligned} \tag{7.7}$$

Since $\beta < \frac{\alpha}{d}$, (7.6) and (7.7) imply that there exists $t_0(\lambda) > 0$ such that

$$u_t^{\lambda \mu}(x) \leq u_t(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } t \in (0, t_0(\lambda)]. \tag{7.8}$$

Applying the comparison principle (as in Remark 2.10) at time $t_0(\lambda)$, it follows that

$$u_t^{\lambda \mu}(x) \leq u_t(x) \quad \text{for all } (t, x) \in Q. \tag{7.9}$$

The above holds for all $\lambda > 0$. Since $\lim_{\lambda \rightarrow \infty} u_t^{\lambda \mu} = U_t$ and $u \in \mathcal{U}$, it follows that $u_t(x) = U_t$. \square

It remains to prove our existence result, Theorem 1.13(b), which states that if $\mathcal{S} \subset \mathbb{R}^d$ is compact and $\mathcal{S} = \text{supp}(\mu)$ for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfying (F2)- s for some $s < d_{\text{sat}}$, then there exists a weak solution of (1.3) with initial trace $(\mathcal{S}, 0)$ if $\beta^*(\alpha, s) < \beta < \frac{\alpha}{d}$. This solution is $u_t^{\infty\mu}$.

Proof of Theorem 1.13(b). Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfy (F2)- s and have compact support \mathcal{S} . By Lemma 7.2, $u_t^{\infty\mu}(x)$ is a weak solution to (1.3) and \mathcal{S} is contained in the singular set of $u_t^{\infty\mu}$. By (1.16), $u_t^{\infty\mu}(x)$ vanishes uniformly on $\{x : d(x, \mathcal{S}) \geq \epsilon\}$ as $t \downarrow 0$ for any $\epsilon > 0$. It follows that for any $\xi \in C_c(\mathcal{S}^c)$,

$$\lim_{t \rightarrow 0} \int \xi(x) u_t^{\infty\mu}(x) dx = 0.$$

Hence the singular set of $u_t^{\infty\mu}$ is no larger than \mathcal{S} and (1.19) holds with measure $\nu = 0$, which implies that $u_t^{\infty\mu}$ has initial trace $(\mathcal{S}, 0)$. \square

Acknowledgements. This work is part of the author's PhD dissertation at the University of British Columbia. The author thanks his supervisor, Ed Perkins, for many useful discussions, including suggestions which led to the proof of Theorem 1.4, and for giving close readings of the manuscript. The author also thanks Leonid Mytnik for encouraging him to read [3], which was the starting point for this work.

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