Abstract. We introduce random interlacements for transient vertex-reinforced jump processes on a general graph $G$. Using increasing finite subgraphs $G_n$ of $G$ with wired boundary conditions, we show convergence of the vertex-reinforced jump process on $G_n$ observed in a finite window to the random interlacement observed in the same window.


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1 Introduction

In this paper, we analyze random interlacements for transient vertex-reinforced jump processes on infinite graphs. This joins two worlds, random interlacements for transient Markov processes and vertex-reinforced jump processes (VRJP). Random interlacements for transient Markov processes are a well studied topic; we describe this theory for transient Markovian jump processes in Section 1.2 as an ingredient for the present work. On the other hand, vertex-reinforced jump processes starting at a given point can be seen as mixture of Markovian jump processes with a mixing measure depending on the starting point. However, controlling the behavior of that mixing measure as the starting point goes to infinity causes a lot of technical problems concerning absolute continuity and uniform integrability. These problems become more tractable if the starting point is fixed on a finite graph growing towards an infinite graph. However, the random jump rates governed by the mixing measure strongly depend on the size of the finite graph; this makes the question more complicated than for classical random walk in a random environment. The purpose of this paper is to show that it is still possible to obtain a corresponding limiting random interlacement. We review the parts of the theory of VRJP that we need in Section 1.1.

The main result of this paper concerns the convergence of loop measures of VRJP on finite pieces of a graph with wired boundary conditions to random interlacements as the pieces grow to the infinite graph. It is stated in Section 1.3.

1.1 Vertex-reinforced jump processes

The vertex-reinforced jump process is a continuous time process $Y = (Y_s)_{s \geq 0}$ taking values in the set $V$ of vertices of a locally finite connected undirected graph $G = (V, E)$ without direct loops. The edges $e = \{x, y\} \in E$ with $x, y \in V$ are assigned conductances

$$C_e = C_{xy} > 0. \quad (1.1)$$

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The process starts in a vertex $o \in V$ and it keeps the memory of the local times $L_x(s)$ spent at any vertex $x \in V$ at time $s$, where we use the convention that initial local times equal $1 = L_x(0)$ for all $x \in V$. Given that $Y$ is at vertex $x$ at time $s$, it jumps to a neighboring vertex $y$ at rate $C_{xy}L_y(s)$. The process was conceived by Werner and first studied by Davis and Volkov in [1] and [2]. In the present paper, we look at VRJP in a different time scale, called exchangeable time scale. We encode it as a process $\tilde{w} = (w,l)$ in discrete time decorated with the waiting times $l = (l(k))_{k \in \mathbb{N}_0}$ at the vertices $w = (w(k))_{k \in \mathbb{N}_0}$. More precisely, the time change is given by

$$t = D(s) = \sum_{x \in V} (L_x(s)^2 - 1). \quad (1.2)$$

We consider the process $Z = (Z_t = Y_{D^{-1}(t)})_{t \geq 0}$; the component $w(k)$ means its location immediately before the $k + 1$-st jump time, and $l(k)$ is the time spent by $Z$ at $w(k)$ between the $k$-th and $k + 1$-st jump time.

In the remainder of the article, we fix a vertex $o \in V$ and make the following assumption:

**Assumption 1.1** VRJP on $G$ starting at $o$ is transient, i.e. almost all paths visit every vertex at most finitely often.

In particular, according to corollary 4 in [9], this assumption is fulfilled for $\mathbb{Z}^d$, $d \geq 3$ and large constant initial weights $C$.

In the following, let $\mathbb{R}_+ = \{a \in \mathbb{R} : a > 0\}$ and $\mathbb{R}_+^0 = \{a \in \mathbb{R} : a \geq 0\}$. We use the convention $C_{xy} = 0$ whenever $\{x, y\}$ is not an edge in $E$. Sabot and Tarrés [9] and Sabot and Zeng [3] showed that the time-changed VRJP $Z = (Z_t)_{t \geq 0}$ starting in the vertex $o$ is a Markov jump process in a random environment. Given the starting point $o \in V$, the random environment can be described by random variables $\beta = (\beta_x)_{x \in V}$, $\beta_x > 0$, having a joint law $\rho_o$, introduced in Definition 2.3 below. We realize $\beta$ as canonical process (identity map) on $\mathbb{R}_+^o$. There are random variables $u_{o,x} \in \mathbb{R}$, $x \in V$, defined in [2.16] below, which are functions of $\beta$, fulfill the normalization $u_{o,o} = 0$, and

$$\beta_x = \frac{1}{2} \sum_{y \in V} C_{xy}e^{u_{o,y} - u_{o,x}} \rho_o\text{-a.s. for all } x \in V. \quad (1.3)$$

The reason is explained in Remark 2.9 below. In a fixed environment, the Markov jump process has jump rates $\frac{1}{2}C_{xy}e^{u_{o,y} - u_{o,x}}$ from $x$ to $y$. Consequently, $\beta_x$ can be interpreted as the total jump rate away from $x$. Although the jump rates are given solely in terms of the variables $u_{o,x}$, it is still convenient to view the family $\beta$ of total jump rates as the basic object because of a coupling needed in Section 2.

Let us describe the Markov jump process in formulas. Given a value of the environment $\beta$, a starting point $z \in V$ (which equals $o$ in most cases but not always), and the corresponding $u_{o,z} = u_{o,x}(\beta)$, we define a probability law $Q_{z,\beta}^G$ on $V^{\mathbb{N}_0} \times \mathbb{R}_+^o$ with canonical process $(w,l)$, encoding a nearest-neighbor continuous time Markov jump process on $G$ with conductances $C$ by the following requirements: $w(0) = z$ holds $Q_{z,\beta}^G$-a.s., and for any $k \in \mathbb{N}_0$, conditionally on $(w(k'))_{0 \leq k' \leq k}$ and $(l(k'))_{0 \leq k' \leq k}$, the joint law of $w(k + 1)$ and $l(k)$ is characterized by

$$Q_{z,\beta}^G(w(k + 1) = x, l(k) > \ell | (w(k'))_{0 \leq k' \leq k}, (l(k'))_{0 \leq k' < k})$$

$$= \frac{C_{xw(k)}e^{u_{o,x} - u_{o,w(k)}}}{\sum_{y \in V} C_{yw(k)}e^{u_{o,y}}} \exp(-\ell \beta_{w(k)})$$

$$= \frac{C_{xw(k)}e^{u_{o,x} - u_{o,w(k)}}}{2\beta_{w(k)}} \exp(-\ell \beta_{w(k)}); \quad (1.4)$$

recall $C_{xy} = 0$ for $\{x, y\} \notin E$. Of course, the measures $Q_{z,\beta}^G$, $\rho_o$, and some other objects introduced below depend also on the choice of the weights $C$. However, this is not displayed in the notation, as we consider $C$ to be fixed.
Fact 1.2 (Variant of theorem 1 (iii) \cite{8}) Let $P_0$ denote the law of the VRJP $(Z_t)_{t \geq 0}$ in exchangeable time scale on the infinite graph $G$ encoded as $\hat{w} = (w, l)$ with starting point $o$. There exists a probability measure $\rho_o$ on $\mathbb{R}_+^V$ such that for any event $A \subseteq V^{N_0} \times \mathbb{R}_0^N$ one has

$$P_o(A) = \int_{\mathbb{R}_+^V} Q_{o,\beta}^V(A) \rho_o(d\beta).$$

More specifically, the probability measure $\rho_o$ on $\mathbb{R}_+^V$ introduced in Definition 2.3 below fulfills this requirement.

The fact that VRJP on infinite graphs is a mixture of Markov jump processes was stated in \cite{8} using the law $\mu^V_n$ on $\beta$ given in section 4 in \cite{7}. However, for our construction it is essential to use the law $\rho_o$ defined in Definition 2.3 below. We remark that $\rho_o$ is not an infinite volume version of $\mu^V_n$. Fact 1.2 is proven at the end of Section 2 below.

A representation similar to Fact 1.2 holds for VRJP on finite subgraphs $G_n$ of $G$ with wired boundary conditions with the same mixing measure $\rho_o$ as is shown in Lemma 2.10 below. However, the laws of the transition probabilities on $G_n$ and on $G$ differ, because the transition probabilities on $G_n$ are given in terms of random variables $u^{(n)}_{o,x}$ given in formula (2.5) below. In general, the $u^{(n)}_{o,x}$ are not equal to $u_{o,x}$.

Comparison of the present approach to the approach in \cite{8}. For the following three reasons, the construction from \cite{8} cannot be used directly to provide a consistent measure on random interlacements.

1. When one uses the infinite volume representation from \cite{8} and then constructs a random interlacement directly given a fixed environment, the object thus obtained is not tractable in terms of finite volume approximations of VRJP. This makes it difficult to observe the properties of that object, in particular its reinforced behavior.

2. The random environment for the VRJP started at the wiring point $\delta_n$ of a finite subgraph $G_n$ of $G$ with wired boundary conditions is described by random variables $\psi^{(n)}(x)$ introduced in lemma 2 in \cite{8}; see also (2.4) below. Uniform integrability of $\psi^{(n)}(x)$, $n \in \mathbb{N}$, is unfortunately unknown, see for instance section 2.6 in \cite{8}. Therefore, without a solution of this open problem, there is no direct way to start the random interlacement process associated to VRJP at infinity.

3. In \cite{8}, the random environment for the VRJP started at the wiring point $\delta_n$ of $G_n$ needs an additional gamma variable $\gamma_{\delta_n}$ associated to $\delta_n$; see formulas (2.4) and (4.2) in \cite{8}. As $G_n$ increases towards $G$, it is unclear how to couple the variables $\gamma_{\delta_n}$, $n \in \mathbb{N}$.

In the present paper, we go around these problems by the following approach. We start VRJP on a finite graph at the given vertex $o$ rather than starting at infinity. Another difference to the classical theory of random interlacements is that the law of the transition probabilities for VRJP on finite subgraphs of $G$, viewed as a mixture of Markov jump processes, depends on the size of the finite subgraph. For an introduction to random interlacements see the textbook \cite{5} by Drewitz, Ráth, and Sapozhnikov.

1.2 Random interlacements

In order to describe random interlacements associated to VRJP, we are interested in Markovian random interlacements in random environments. However, to start with, we describe the theory in a fixed environment first. It is closely linked to the work of Sznitman: Random interlacements were introduced for simple random walks in $\mathbb{Z}^d$, $d \geq 3$, by Sznitman in \cite{9}. In \cite{11}, Teixeira generalized the notion of random interlacements to transient random walks on weighted graphs. Sznitman \cite{10} considered random interlacements associated to transient continuous-time jump processes on weighted graphs. In this paper, we need a variant of this construction, including an initial piece of the jump process starting at a given point rather than starting at infinity. Another difference to the classical theory of random interlacements is that the law of the transition probabilities for VRJP on finite subgraphs of $G$, viewed as a mixture of Markov jump processes, depends on the size of the finite subgraph.
Introduction of random interlacement with initial path. One ingredient for the present paper are Markovian random interlacements in continuous time in a random environment encoded by $\beta$ as above. For the moment, let us take $\beta \in \mathbb{R}^+_1$ fixed such that $u_{\infty} = u_{\infty} (\beta)$ fulfilling the equality in (1.3) exists and such that the Markovian jump process with law $Q_{G,\beta}^G$ is transient.

In the following “path” means nearest-neighbor path. For $I \subseteq \mathbb{Z}$, we define the set of paths in $G$ indexed by $I$ which visit every vertex at most finitely often:

$$ W(I) := \left\{ (w(k))_{k \in I} \in V^I : \{ w(k), w(k+1) \} \in E \text{ if } \{ k, k+1 \} \subseteq I \right. $$

$$ \left. \text{ and } |\{ k \in I : w(k) = j \}| < \infty \text{ for all } j \in V \right\}. $$

(1.6)

We introduce the set of paths decorated with waiting times

$$ \hat{W}(I) := W(I) \times \mathbb{R}^I_+. $$

(1.7)

We endow it with its natural $\sigma$-field $\hat{\mathcal{W}}(I)$. Typical elements of $\hat{W}(I)$ are denoted by $\hat{w} = (w, l) = (\hat{w}(k))_{k \in I}$. We abbreviate $\hat{W}^\rightarrow := \hat{W}([0])$, $\hat{W} := \hat{W}(\mathbb{Z})$, and use similar abbreviations $\hat{W}^\rightarrow$, $\hat{W}$ for the corresponding $\sigma$-fields.

Let $\emptyset \neq K \subseteq V$ be a finite subset. In the spirit of Sznitman [9] [10], we introduce a measure $\hat{Q}_{K,\beta}$ on $(\hat{W}, \hat{\mathcal{W}})$ as follows. We define the event

$$ A_K := \{ w(0) \in K \text{ and } w(k) \notin K \text{ for all } k > 0 \} \subseteq \hat{W}^\rightarrow $$

(1.8)

that the path $w$ visits $K$ for the last time at index 0. The Markov jump process described by $Q_{G,\beta}^G$ is reversible in the sense of Lemma A.1 in the appendix. Motivated by this lemma, we take the unique finite measure $\hat{Q}_{K,\beta}$ on $(\hat{W}, \hat{\mathcal{W}})$ specified by the following requirement: For all $x \in V$, $\ell \geq 0$, and $B_1, B_2 \in \hat{\mathcal{W}}(\mathbb{N})$,

$$ \hat{Q}_{K,\beta}[\hat{w}(-n)_{n \in \mathbb{N}} \in B_1, w(0) = x, l(0) \geq \ell, w|_n \in B_2] = \beta_x e^{-\ell \beta_x} e^{2u_{\infty} x} Q_{x,\beta}^G[w|_n \in B_1, w|_{n_0} \in A_K] Q_{x,\beta}^G[w|_{n} \in B_2]. $$

(1.9)

Note that on the left-hand side of this formula the event $B_1$ is considered for the time reversed path, while on the right-hand side the path is not taken reversed, neither for the event $B_1$ nor for the event $A_K$.

Frequently, we consider elements of $\hat{W}$ modulo time shifts. Therefore we introduce

$$ \hat{W}^* := \hat{W}/\sim, \text{ where } \hat{w} \sim \hat{w}' \Leftrightarrow \exists m \in \mathbb{Z} \forall k \in \mathbb{Z} : \hat{w}(k) = \hat{w}'(k + m). $$

(1.10)

Let $\pi^* : \hat{W} \to \hat{W}^*$ and $\hat{W}^*$ respectively denote the canonical map and the $\sigma$-field on $\hat{W}^*$ induced by $\pi^*$, i.e., a set $B \subseteq \hat{W}^*$ is measurable if and only if its inverse image $(\pi^*)^{-1}(B)$ is measurable. We consider the set of equivalence classes of paths which visit a finite set $K$:

$$ \hat{W}^*_K = \pi^*[\{(w, l) \in \hat{W} : w(0) \in K\}]. $$

(1.11)

The next theorem, proven in the appendix, provides the intensity measure for the Poisson point process of random interlacements in a fixed environment encoded by $\beta$.

Theorem 1.3 (Intensity measure) There exists a unique measure $\hat{\nu}_\beta$ on $(\hat{W}^*, \hat{W}^*_K)$ such that for any finite $K \subseteq V$, one has

$$ 1_{\hat{W}^*_K} \hat{\nu}_\beta = \pi^*[\hat{Q}_{K,\beta}]. $$

(1.12)

It is $\sigma$-finite and it is given by

$$ \hat{\nu}_\beta(A) = \sup_{K \subseteq V_{\text{finite}}} \pi^*[\hat{Q}_{K,\beta}](A) \text{ for all } A \in \hat{W}^*. $$

(1.13)

The measure $\hat{\nu}_\beta$ is not the measure zero: for all finite $\emptyset \neq K \subseteq V$ one has

$$ 0 < \hat{\nu}_\beta(\hat{W}^*_K) = \pi^*[\hat{Q}_{K,\beta}](\hat{W}^*_K) < \infty. $$

(1.14)
We define a suitable set of point measures, where the individual points consist of pairs \((\hat{w}, t)\) with a doubly infinite path \(\hat{w}\) and a time \(t > 0\):

\[
\Omega^+ := \begin{cases} 
\omega \leftrightarrow = \sum_{i \in \mathbb{N}} \delta(\hat{w}^*, t_i) \quad \hat{w}^* \in \hat{W}^* \quad t_i > 0 \quad t_i \neq t_j \text{ for } i \neq j, \\
\text{finite } \emptyset \neq K \subset V \text{ and all } t > 0
\end{cases}
\]

(1.15)

This means that we now have two different time lines: \(l\)-times \(l = (l(k))_{k \in \mathbb{N}_0}\) on the one hand and \(t\)-times \(t, t\) on the other hand. They should not be confused with each other. Local times at vertices in \(V\) are always measured in the \(l\)-time line. Informally speaking, pairs \((t, l)\) should be compared with the lexicographic order, with the \(t\)-time being the coarser scale and the \(l\)-time being the finer scale.

We endow \(\Omega^+\) with the \(\sigma\)-field generated by cylinders. Because of (1.14), there is a Poisson point process with a law \(Q_\beta\), realized as canonical process on \(\Omega\), and having the intensity measure

\[
\hat{\nu}(d\hat{w}^*) \times dt.
\]

(1.16)

It describes random point measures over \(\hat{W}^* \times (0, \infty)\). Moreover, we introduce the product measure

\[
Q_{o, \beta} := Q^G_{o, \beta} \times Q_\beta \quad \text{on} \quad \Omega := \hat{W}^* \times \Omega^+.
\]

(1.17)

The measure \(Q_{o, \beta}\) is intended to model random interlacements with an initial one-sided infinite path starting at \(o\) and then infinitely many two-sided infinite paths in a given environment encoded by the conductances \(C\), cf. [1,1], and \(\beta \in \mathbb{R}_V^+\) as specified at the beginning of this Section 1.2.

Using \(Q^G_{o, \beta}(W^-) = 1\) from transience, we define a probability measure \(P_o\) on \(\Omega\) by

\[
P_o(A) := \int_{\mathbb{R}_V^+} Q_{o, \beta}(A) \rho_o(d\beta)
\]

(1.18)

for any measurable set \(A\) with the measure \(\rho_o\) describing the random environment for VRJP as in Fact [1.2]. It models random interlacements with the measure \(\rho_o\) describing the random environment for VRJP as in Fact [1.2].

Let \(\omega = (\omega_s, \omega^+)\) be distributed according to \(P_o\) with the given \(o \in V\). Here “s” stands for start. Then, note that the initial piece \(\omega_s\) has the same distribution as the trace together with the waiting times of a vertex-reinforced jump process in exchangeable time scale starting in \(o\) with weights \(C\).

We remark that this construction is slightly different from the standard random interlacement in two respects:

- First, in the classical setup all paths are doubly infinite, without a one-sided initial piece. Removing our initial path would require uniform integrability assumptions that have not been shown to hold so far, as discussed in item 2 of the comparison of our approach to the one in [8] at the end of Section 1.1.

- Second, there is no intensity level in the classical sense because of the same lack of known uniform integrability; see Remark 3.5 below.

### 1.3 Approximation of random interlacements by VRJP

VRJP on finite graphs is much better understood than on infinite graphs because of the explicitly known formulas for the random environment described in [6]. Therefore it is natural to compare the random interlacements studied in this paper with VRJP on finite subgraphs. For this purpose, we consider a finite observation window \(K \subset V\) and an additional \(\delta\) “at infinity”. We study the reductions of the processes consisting in an infinite speed up of time whenever the process is not in \(K\).

We use the notation \([a, b]\) and \((a, b]\) not only for real intervals but also for integer ones.
Finite approximations with wired boundary conditions. First, we approximate the infinite graph $G$ by finite subgraphs. Let $V_n \uparrow V$ be an increasing sequence of connected subsets of $V$. We take wired boundary conditions as follows. Let $\delta$ be a new vertex, not contained in $V$. Let $G_n = (\tilde{V}_n, E_n)$ be the graph with vertex set $\tilde{V}_n = V_n \cup \{\delta\}$. There are two types of edges in $\tilde{E}_n$: First, all edges $\{x, y\}$ in $E$ with $x, y \in V_n$ belong to $\tilde{E}_n$ with inherited conductance $C_{xy}^{(n)} = C_{xy}$. Second, for any $x \in V_n$ with $\{y \in V \setminus V_n : \{x, y\} \in E\} \neq \emptyset$ there is an edge $\{x, \delta\} \in \tilde{E}_n$ with conductance $C_{x\delta}^{(n)} = \sum_{y \in V \setminus V_n} C_{xy}$. For convenience of notation, we set $C_{\delta\delta}^{(n)} = 0$ if $\{x, y\} \notin \tilde{E}_n$. Let

$$\tilde{W}_n^{-} = \left\{ (w(k), l(k))_{k \in \mathbb{N}_0} \in (\tilde{V}_n \times \mathbb{R}_+)^{\mathbb{N}_0} : \{w(k), w(k + 1)\} \in \tilde{E}_n \text{ for all } k \in \mathbb{N}_0 \right\}$$

(1.19)

denote the set of decorated paths in $G_n$ that visit every vertex infinitely often.

Let $K \subseteq V$ be a finite set with $\delta \in K$ and set $\tilde{K} := K \cup \{\delta\}$.

$K^+$-reduction on finite graphs. We take $n \in \mathbb{N}$ large enough that $K \subseteq V_n$. Let $\tilde{w} = (w, l) \in \tilde{W}_n^{-}$. By the definition of $W_n^{-}$, one has $w(k) \in K$ for infinitely many $k$ and $w(k) = \delta$ for infinitely many $k$. Consider the subsequence $(w(k_j), l(k_j))_{j \in \mathbb{N}_0}$ of $\tilde{w}$ consisting only of the pairs $(w(k), l(k))$ with $w(k) \in K = \tilde{K} \cup \{\delta\}$. In this subsequence, finitely many (but not infinitely many) consecutive $w(k_j)$ may coincide. We unite these consecutive holding pieces as follows. Recursively, let

$$j_0 := 0 \quad \text{and} \quad j_{m+1} := \min\{j > j_m : w(k_j) \neq w(k_{j-1})\} \text{ for } m \in \mathbb{N}_0.$$

(1.20)

The $K^+$-reduction $\tilde{w}^K$ of $\tilde{w}$ is defined as follows:

$$\tilde{w}^K = (w^K(m), l^K(m))_{m \in \mathbb{N}_0}$$

(1.21)

$$\text{with} \quad w^K(m) = w(k_{j_m}) \quad \text{and} \quad l^K(m) = \sum_{j = j_m}^{j_{m+1} - 1} l(k_j) 1_{(w(k_j) \neq \delta)}.$$

(1.22)

We emphasize that the local time at $\delta$ is not counted in this definition.

With the name $K^+$-reduction we would like to indicate that we observe the process not only in $K$, but a little bit more, namely whenever it is at $\delta$. This in contrast to the $K^+$-reduction on the infinite graph introduced in the next paragraph, where the process is only observed at $K$.

On the finite graph $G_n$, VRJP is recurrent. Hence, a.s. visits the set $K$ infinitely often. On the other hand, we assume VRJP to be transient on the infinite graph $G$. Hence, it visits $K$ at most finitely often a.s. Extending VRJP on the infinite graph by a vertex-reinforced interlacement process this difference disappears, making a direct comparison between the two reductions possible.

$K$-reduction on the infinite graph. Let $\tilde{w} = (w, l) \in \tilde{W}^{-}$. If $w$ does not meet the set $K$ we define $\tilde{w}^K$ to be the empty list. Else we proceed as follows. By the definition of $W^{-}$, one has $w(k) \in K$ for at most finitely many $k$, say for $J + 1$ time points $k$. Similarly to the above, we consider the finite subsequence $(w(k_j), l(k_j))_{j \in [0, J]}$ of $\tilde{w}$ consisting only of the pairs $(w(k), l(k))$ with $w(k) \in K$. In this subsequence, some consecutive $w(k_j)$ may coincide. We unite them as follows. Recursively, let $j_0 := 0$ and $j_{m+1} := \inf\{j \in (j_m, J] : w(k_j) \neq w(k_{j-1})\}$ for $m \in \mathbb{N}_0$. Let $M \in \mathbb{N}_0$ be the largest $m$ with $j_m < \infty$.

The $K$-reduction $w^K$ of $\tilde{w}$ is defined by

$$\tilde{w}^K = (w^K(m), l^K(m))_{m \in [0, M]}$$

(1.23)

$$\text{with} \quad w^K(m) = w(k_{j_m}) \quad \text{and} \quad l^K(m) = \sum_{j = j_m}^{(j_{m+1}-1) \wedge J} l(k_j).$$

We intentionally use the same notation $\tilde{w}^K$ for both the $K^+$-reduction on the finite graph and the $K$-reduction on the infinite graph in order to emphasize the analogy between the two constructions. In any case, it will be clear from the context which of the two notions is understood.
**K+**-reduction for interlacements. Recall the definition of $\hat{W}$ from \[1.11\]. Let

$$\omega = \left(\omega_w, \omega^{++} = \sum_{t \in \mathbb{N}} \delta_{(\hat{w}_t^+, t_i)}\right)$$ (1.24)

be a typical element of $\Omega$, given in \[1.17\]. We consider $(\omega_w, 1_{\hat{W}_K^{*}\times \mathbb{R}_{+}^{++}} \omega^{++})$; the second component involves only the two-sided infinite paths which hit $K$. We write it as

$$1_{\hat{W}_K^{*}\times \mathbb{R}_{+}^{++}} \omega^{++} = \sum_{j \in \mathbb{N}} \delta_{(\hat{w}_j^+, t_i)}$$ (1.25)

with $(i_j)_{j \in \mathbb{N}}$ chosen such that $t_i$ increases with $j$. Given the definition of $\Omega^{++}$ as in \[1.15\], this construction works.

Let $\hat{w}_{i_j} = (w_{i_j}(k), l_{i_j}(k))_{k \in \mathbb{Z}}$ be the representative of $\hat{w}_{i_j}^{*}$ with $w_{i_j}(0) \in K$ and $w_{i_j}(k) \notin K$ for $k < 0$. The $K^{+}$-reduction $\omega^K$ of the interlacement $\omega$ is defined to be the concatenation of

$$\omega^K = \omega^K_\delta$$ (1.26)

In other words, we take the part of the initial piece $\omega^K_\delta$ running through $K$ and then infinitely many loops around $\delta$ obtained from the $K$-reduction of all $\hat{w}_{i_j}$, with holding times at $\delta$ again not being counted.

Let $P^o_n$ denote the law of the vertex-reinforced jump process in exchangeable time scale encoded as $\hat{w} = (w, l) = (\hat{w}(k))_{k \in \mathbb{Z}}$ on the finite graph $G_n$ with weights $C^{(n)}$ and starting point $o$.

**Theorem 1.4 (Main result: Convergence of $K^{+}$-reductions)** Let $K \subset V$ be finite with $o \in K$. The finite-dimensional distributions of the $K^{+}$-reduction of VRJP on $G_n$ converge weakly as $n \to \infty$ to the finite-dimensional distributions of the $K^{+}$-reduction of the random interlacement. More precisely, for all $J \in \mathbb{N}$, it holds

$$\mathcal{L}_{P^o_n} (\hat{w}^{K}_{[0, J]}) \xrightarrow{w^*} \mathcal{L}_{P^o} (\omega^K_{[0, J]}) \quad \text{as } n \to \infty.$$ (1.27)

Intuitively speaking, the theorem means the following. Suppose we have a finite observation window $K \times [0, J]$, where $K$ refers to location and $[0, J]$ refers to the observable number of jumps. On the one hand, we observe the jumping particle of a VRJP on the finite graph $G_n$ whenever it is inside $K$ or at $\delta$. On the other hand, we observe another particle jumping on $K \cup \delta$ described by the $K^{+}$-reduction of the random interlacement. One may imagine time to run infinitely fast whenever the particle is not in $K$ including when it is in $\delta$. Then, according to the theorem, as $n \to \infty$, in the chosen space-time window, we can hardly see any difference between the jumping particle on the finite graph and the jumping particle coming from the interlacement process.

**Remark.** The random environment for VRJP in an appropriate time scaling has a Bayesian conjugate prior property: Conditioned on an initial piece of the path, the future of the path is distributed according to a VRJP with updated weights. We expect this property to be inherited to random interlacements. Working this out in detail is beyond the scope of this paper.

**How this article is organized.** In Section 2, we construct the measure $\rho_o$ describing the random environment for VRJP. We prove the representation of VRJP as a mixture of Markov jump processes on the infinite $G$ stated in Fact 1.2 using an analogous representation on finite approximating subgraphs $G_n$ of $G$ with the same measure $\rho_n$; see Lemma 2.10. This construction uses a martingale discovered by Sabot and Zeng \[8\]. Section 2.2 describes the connection between the representation of VRJP as a mixture of Markov jump processes given by Sabot, Tarres, and Zeng in \[17\] and the measure $\rho_o$.

In Section 3, we study VRJP and the random interlacement reduced to a finite observation window. We describe the transition rates of these different $K^{+}$-reductions and prove convergence of the rates for the $K^{+}$-reduction of VRJP on $G_n$ to the corresponding rates for the $K^{+}$-reduction of the random interlacement. This yields a proof of our main Theorem 1.3.

To make the paper more self-contained, we provide a proof of Theorem 1.3 in Appendix A.
2 Construction of random environments for VRJP

The construction of random environments for VRJP on an infinite graph $G$ with given starting point $o$ is done in two steps. First, the construction is done on the finite approximation $G_n$ of $G$ with wired boundary conditions and wiring point $\delta$. This requires studying the Radon-Nikodym derivative $e^{u^{(n)}_o}$ between the laws of the random environments with starting point $o$ and with starting point $\delta$. This is done in Section 2.1. Second, using a martingale argument, the limit $n \to \infty$ of finite approximations is taken.

2.1 The random environment associated to a fixed reference vertex

The mixing measure on the infinite graph $G = (V, E)$ is constructed through finite volume approximations. Let $G_n$, $n \in \mathbb{N}$, be approximating finite subgraphs as in Section 1.3.

It was shown by Sabot and Tarrès in [6] that the mixing measure for VRJP on $G_n$ can be described in terms of the supersymmetric hyperbolic nonlinear sigma-model $H^{2|2}$ in horospherical coordinates, studied in [4]. We define it here through an alternative random Schrödinger operator construction given in [7]: see proposition 1 in [7] (see also theorem 2.1 in [5]) and the Kolmogorov extension theorem construction used in lemma 1 in [5]. We abbreviate $(\lambda, \beta) = \sum_{x \in V} \lambda_x \beta_x$. There is a probability measure $\rho_\infty$ on $\mathbb{R}_+^V$, depending on the graph $G$ and the conductances $C$, with Laplace transform

$$\int_{\mathbb{R}_+^V} e^{-\langle \lambda, \beta \rangle} \rho_\infty(d\beta) = \exp \left( - \sum_{x, y \in V} C_{xy} \left( \sqrt{\lambda_x + 1} \sqrt{\lambda_y + 1} - 1 \right) \right) \prod_{x \in V} \frac{1}{\sqrt{\lambda_x + 1}}$$

(2.1)

for all $(\lambda_x)_{x \in V} \in (-1, \infty)^V$ having only finitely many nonzero entries. Given $\beta \in \mathbb{R}_+^V$, let

$$\mathcal{H}_\beta \in \mathbb{R}^{V \times V}, \quad (\mathcal{H}_\beta)_{xy} = 2\beta_x 1_{\{x = y\}} - C_{xy};$$

(2.2)

recall the convention $C_{xy} = 0$ if $\{x, y\}$ is not an edge in $G$. For any $n \in \mathbb{N}$, given the finite subset $V_n \subset V$, we introduce the restriction $\mathcal{H}_\beta^{(n)} = (\mathcal{H}_\beta)_{xy} 1_{\{x, y\} \in V_n}$. Let

$$B = \{ \beta \in \mathbb{R}_+^V : \mathcal{H}_\beta^{(n)} \in \mathbb{R}^{V_n \times V_n} \text{ is positive definite for all } n \}. \quad (2.3)$$

Note that $\rho_\infty$-a.e. $\beta$ belongs to $B$ by definition 1 and proposition 1 in [7]. For any $\beta$ such that $\mathcal{H}_\beta^{(n)}$ is positive definite, the vector $(\psi^{(n)}(x))_{x \in V_n}$ and its component-wise logarithm $(u^{(n)}_x)_{x \in V_n}$ are defined by

$$\psi^{(n)}(\delta) = e^{u^{(n)}_\delta} := 1, \quad (\psi^{(n)}(x))_{x \in V_n} = (e^{u^{(n)}_x})_{x \in V_n} := (\mathcal{H}_\beta^{(n)})^{-1} C^{(n)}_{V_n, \delta} \quad (2.4)$$

where $C^{(n)}_{V_n, \delta} = (C_{x\delta})_{x \in V_n}$; indeed all entries in $(\mathcal{H}_\beta^{(n)})^{-1}$ are strictly positive, as was shown in proposition 2 in [7], which allows us to take the logarithms to define $u^{(n)}$. If $\mathcal{H}_\beta^{(n)}$ is not positive definite, we set $u^{(n)}_x = 0$ for $x \in \bar{V}_n$. We also set $u^{(n)}_n = 0$ for all $x \in V \setminus V_n$. For $x \in V \cup \{\delta\}$ and the fixed vertex $o \in V$, we define

$$u^{(n)}_{o, x} = u^{(n)}_x - u^{(n)}_o. \quad (2.5)$$

In particular, $u^{(n)}_{o, o} = 0$. Note that for $x \in V_n$, formula (2.4) implies

$$\beta_x = \frac{1}{2} \sum_{y \in V_n} C^{(n)}_{xy} e^{u^{(n)}_y} - u^{(n)}_x = \frac{1}{2} \sum_{y \in V_n} C^{(n)}_{xy} e^{u^{(n)}_{y, o}} - u^{(n)}_{o, x}. \quad (2.6)$$

For any given $n$, we extend $\beta$ to be also defined at $\delta \in \bar{V}_n$ by

$$\beta_\delta := \beta^{new,n}_\delta := \frac{1}{2} \sum_{y \in V_n} C^{(n)}_{\delta y} e^{u^{(n)}_y} = \frac{1}{2} \sum_{y \in V_n} C^{(n)}_{\delta y} e^{u^{(n)}_{y, o}} - u^{(n)}_{o, \delta}. \quad (2.7)$$
The dependence of $\beta_\delta$ on $n$ is not displayed in the notation. We remark that this quantity is called $\tilde{\beta}_\delta$ in [7]; it does not coincide with what is called $\beta_\delta$ there.

Consider a nearest-neighbor continuous-time Markov jump process on the finite graph $G_n$ endowed with the weights $C^{(n)}$ defined in analogy to (1.4) replacing the weighted graph $(V, E, C)$ by $(\tilde{V}_n, \tilde{E}_n, C^{(n)})$ and $u_o$ by $u^{(n)}_o$. For a starting point $z \in V_n$, the corresponding probability law $Q^{G^{(n)}_{\delta, \beta}}_n$ on $\mathbb{V}^{N_0} \times \mathbb{R}^+_\mathbb{R}$ is defined by the requirements that $w(0) = z$ holds $Q^{G^{(n)}_{\delta, \beta}}_n$-a.s., and for any $k \in N_0$, conditionally on $(w(k'))_{0 \leq k' \leq k}$ and $(l(k'))_{0 \leq k' \leq k}$, the joint law of $w(k+1)$ and $l(k)$ is given by

$$
Q^{G^{(n)}_{\delta, \beta}}_n(w(k+1) = x, l(k) > \ell) = \frac{C^{(n)}_{xw(k)} e^{u^{(n)}_x - u^{(n)}_{o,w(k)}}}{\sum_{y \in \tilde{V}_n} C^{(n)}_{yw(k)} e^{u^{(n)}_k} \exp \left(-\ell \beta w(k)\right)} \exp \left(-\ell \beta w(k)\right),
$$

where we have used the expressions (2.6) and (2.7) for $\beta$ in the last equation.

On $\mathbb{R}^+_V$, we define $F_\infty = \sigma(\beta_x, x \in V)$ and the filtration

$$
F_n = \sigma(\beta_x, x \in V_n), \quad n \in \mathbb{N}.
$$

By (2.4), all $u^{(n)}_x$ are $F_n$-measurable. For any vertex $x \in V_n$, we define a measure $\rho^{n}_x$ on $(\mathbb{R}^+_V, F_n)$ by

$$
d\rho^{n}_x = e^{u^{(n)}_x} d\rho_\infty \big|_{F_n}.
$$

Theorem 3(i) in [7] shows that VRJP on $G_n$ starting from $\delta$ is a mixture of the laws $Q^{G^{(n)}_{\delta, \beta}}_n$ when $\beta$ is drawn randomly with respect to the mixing measure $\rho_\infty$; only the information of $\beta$ encoded in the $\sigma$-field $F_n$ matters here. The next lemma provides an analogous result for VRJP on $G_n$ starting from $o$ rather than from $\delta$:

**Lemma 2.1** VRJP on $G_n$ starting from $o$ is a mixture of the laws $Q^{G^{(n)}_{o, \beta}}_n$ when $\beta$ is drawn randomly with respect to the mixing measure $\rho^{n}_o$.

**Proof.** Formula (3) in theorem 2 of [7] shows that the distribution of $u^{(n)}$ with respect to $\rho_\infty$ equals the distribution of the supersymmetric hyperbolic nonlinear sigma model given in formulas (1.2) and (1.5) of [4]. This model was first introduced by Zirnbauer in [12]. Note that in [4], the point $\delta$ is not explicitly mentioned. The pinning strengths $\varepsilon_x$ of that paper correspond to the weights $C^{(n)}_{x\delta}$.

The effect of changing the reference point in the $H^{2|\varepsilon}$-model on $G_n$ from $\delta$ to $o$ consists of two steps: First the underlying measure, here $\rho_\infty \big|_{F_n}$, gets an additional Radon-Nikodym-derivative $e^{u^{(n)}_o}$. Second, the transformation $u^{(n)} \mapsto u^{(n)}_o$ given in (2.5) changes the normalization from $u^{(n)}_o(0) = 0$ to $u^{(n)}_o(0) = 0$; cf. theorem 2 and section 6 of [6]. Using theorem 3(i) in [7] again, this time with starting point $o$ rather than $\delta$, the claim follows.

The most important case for the vertex $x$ in the following lemma is $x = o$.

**Lemma 2.2** For any vertex $x \in V$, the collection $(\rho^{n}_x)_{n \in \mathbb{N}}$ is a consistent family of probability measures, i.e. $\rho^{n+1}_x \big|_{F_n} = \rho^{n}_x$ for all $n \in \mathbb{N}$.

**Proof.** By formula (5.26) in [3],

$$
\rho^{n}_x(\mathbb{R}^+_V) = \int_{\mathbb{R}^+_V} e^{u^{(n)}_x} d\rho_\infty = 1.
$$
Hence, $\rho_n^x$ is a probability measure. In order to show consistency, take an event $A \in \mathcal{F}_n$. We calculate
\[
\rho_{n+1}^x(A) = \int_A e^{u_x^{(n+1)}} \, d\rho_\infty = \int_A E_{\rho_\infty} \left[ e^{u_x^{(n+1)}} \middle| \mathcal{F}_n \right] \, d\rho_\infty. \tag{2.12}
\]
By proposition 9 in [8], $(e^{u_x^{(n)}})_{n \in \mathbb{N}}$ is a martingale with respect to $\rho_\infty$ and $(\mathcal{F}_n)_{n \in \mathbb{N}}$; see also theorem 2.5 of [3] for a formulation in a notation which is closer to the one used in the present paper. This yields $\rho_\infty$-a.s.
\[
E_{\rho_\infty} \left[ e^{u_x^{(n+1)}} \middle| \mathcal{F}_n \right] = e^{u_x^{(n)}}. \tag{2.13}
\]
Inserting this in (2.12) yields the consistency as follows:
\[
\rho_{n+1}^x(A) = \int_A e^{u_x^{(n)}} \, d\rho_\infty = \rho_n^x(A). \tag{2.14}
\]

**Definition 2.3** For $x \in V$, let $\rho_x$ denote the unique probability measure on $(\mathbb{R}_+^V, \mathcal{F}_\infty)$ with restrictions $\rho_x|\mathcal{F}_n = \rho_n^x$ for all $n \in \mathbb{N}$ given by Kolmogorov’s consistency theorem.

For all $o, x \in V$ and $n \in \mathbb{N}$, it follows from (2.14) that $u_{o,x}^{(n)} = u_x^{(n)} - u_o^{(n)}$ that
\[
\frac{d\rho_x|\mathcal{F}_n}{d\rho_o|\mathcal{F}_n} = e^{u_x^{(n)}}. \tag{2.15}
\]
Recall that $\rho_\infty$ is supported on the set $B$ defined in [3] so that $\rho_o$ is also supported on the same set $B$. Indeed, for any fixed $n$, the restriction $\rho_o|\mathcal{F}_n$ is absolutely continuous with respect to $\rho_\infty|\mathcal{F}_n$.

**Lemma 2.4** For all $o, x \in V$, the process $\left( e^{u_{o,x}^{(n)}} \right)_{n \in \mathbb{N}}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and the measure $\rho_o$. It fulfills $E_{\rho_o} [e^{u_{o,x}^{(n)}}] = 1$.

**Proof.** The claims are consequences of (2.15) and the fact that $\rho_o$ and $\rho_x$ are probability measures. $\blacksquare$

Being a positive martingale, the process $\left( e^{u_{o,x}^{(n)}} \right)_{n \in \mathbb{N}}$ converges $\rho_o$-almost surely to a limit taking values in $[0, \infty)$; cf. Lemma 2.8 below. We define
\[
u_{o,x} := \lim_{n \to \infty} u_{o,x}^{(n)}, \tag{2.16}
\]
whenever this limit exists in $\mathbb{R}$, and $\nu_{o,x} := 0$ otherwise.

Recall the definition (2.3) of the set $B$ of environments.

**Definition 2.5** Let $B'$ denote the set of all $\beta \in B$ such that $u_{o,x}^{(n)} \to \nu_{o,x} \in \mathbb{R}$ as $n \to \infty$ for all $x \in V$ and the Markov jump process on the infinite graph $G$ in the environment $\beta$ with distribution $Q_{o,\beta}^G$ is transient.

### 2.2 Comparison with the approach in [7] and [8]

In this section, we explain the connection between the measure $\rho_o$ and the construction of the mixing measure used by Sabot and Zeng [8], which uses an additional gamma random variable. We use this connection to deduce uniform integrability of $(e^{u_{o,x}^{(n)}})_{n \in \mathbb{N}}$ with respect to $\rho_o$.

Recall that the random variables $\beta_x, x \in V$, denote the canonical projections on $\mathbb{R}_+^V$, and $o \in V$ is fixed. We enlarge the underlying space $\mathbb{R}_+^V$ by an additional component, taking $\mathbb{R}_+^V \times \mathbb{R}_+$. The projection to the last coordinate is denoted by $\gamma_o$, while the projections to the other components are again denoted by $\beta_x$, slightly abusing the notation. We endow $\mathbb{R}_+^V \times \mathbb{R}_+$ with the sigma field generated by the projections and with the probability measure $\rho_o \times \Gamma$, where $\Gamma$ denotes the $\Gamma(1/2, 1)$-distribution.
Fix $n \in \mathbb{N}$. We define
\[ \beta_{x}^{\text{new}} := \beta_{x} + \delta_{x} \gamma_{o} \quad \text{for } x \in V. \tag{2.17} \]
Recall the definition (2.7) of $\beta^{\text{new}}$. For $x \in V_{n}$, we use $\beta_{x}^{\text{new}}$ to be a synonym for $\beta_{x}^{\text{new}}$, and abbreviate $\beta^{\text{new},n}_{x} = (\beta^{\text{new},n}_{x})_{x \in V_{n}}$

Let $\mu_{V_{n}}^{C_{\beta}^{(n)}}$ denote the measure on $\mathbb{R}_{+}^{V_{n}}$ with Laplace transform given by formula (2.1) with the weighted graph $(V, E, C)$ in (2.1) replaced by $(\tilde{V}_{n}, \tilde{E}_{n}, C^{(n)})$. This measure was introduced in [7].

**Lemma 2.6** The distribution of $\beta^{\text{new},n}$ with respect to $\rho_{o} \times \Gamma$ equals the measure $\mu_{\tilde{V}_{n}}^{C_{\beta}^{(n)}}$. In particular, $\mathcal{H}_{\beta^{\text{new},n}} \in \mathbb{R}^{\tilde{V}_{n} \times \tilde{V}_{n}}$ is $\rho_{o} \times \Gamma$-a.s. positive definite. Moreover, the random vector $(u_{o,x}^{(n)})_{x \in \tilde{V}_{n}}$ can be $\rho_{o} \times \Gamma$-a.s. recovered from $\beta^{\text{new},n}$ via $u_{o,x}^{(n)} = 0$ and
\[ (e^{u_{o,x}^{(n)}})_{x \in \tilde{V}_{n} \setminus \{o\}} = ((\mathcal{H}_{\beta^{\text{new},n}})_{\tilde{V}_{n} \setminus \{o\}, \tilde{V}_{n} \setminus \{o\}})^{-1} C_{\tilde{V}_{n} \setminus \{o\}}^{(n)} \beta^{\text{new},n}_{\tilde{V}_{n} \setminus \{o\}}, \quad \text{for all } x \in \tilde{V}_{n} \setminus \{o\} \tag{2.18} \]
or equivalently,
\[ \beta^{\text{new},n}_{x} = \frac{1}{2} \sum_{y \in V_{n}} C_{xy}^{(n)} e^{u_{o,y}^{(n)} - u_{o,x}^{(n)}}, \quad (x \in \tilde{V}_{n} \setminus \{o\}). \tag{2.19} \]

We remark that in this lemma, the probability measure $\rho_{o}$ could also be replaced by its restriction $\rho_{o}^{\tilde{n}}$. **Proof of Lemma 2.6** Using the definition (2.17) of $\beta^{\text{new},n}$, the claim (2.19) is just a combination of the expression (2.6) for $\beta_{x}$, $x \in V_{n}$, and the definition (2.7) of $\beta^{\text{new},n}$.

By Lemma 2.1, $\rho_{o}^{\tilde{n}} = \rho_{o}|_{\tilde{G}_{n}}$ describes the mixing measure for VRJP on $G_{n}$ starting from $o$, with random transition rates expressed in terms of the variables $(u_{o,x}^{(n)})_{x \in \tilde{V}_{n}}$; cf. (2.8). Since these variables satisfy the equations (2.19), Corollary 2 in [7] and the fact $u_{o,o}^{(n)} = 0$ imply that $(\beta^{\text{new},n}_{x})_{x \in \tilde{V}_{n}}$ has distribution $\mu_{\tilde{V}_{n}}^{C_{\beta}^{(n)}}$ with respect to $\rho_{o} \times \Gamma$.

The measure $\mu_{\tilde{V}_{n}}^{C_{\beta}^{(n)}}$ is supported on $\{ \beta \in \mathbb{R}^{\tilde{V}_{n}} : \mathcal{H}_{\beta} \text{ is positive definite} \}$ by its definition, i.e., definition 1 in [7]. Given invertibility of $((\mathcal{H}_{\beta^{\text{new},n}})_{\tilde{V}_{n} \setminus \{o\}, \tilde{V}_{n} \setminus \{o\}})$, formula (2.18) is just another way of writing (2.19).

We remark that the martingale property of $(e^{u_{o,x}^{(n)}})_{n \in \mathbb{N}}$ stated in Lemma 2.4 is written with respect to the measure $\rho_{o}$ without using $\Gamma$, because $u_{o,x}^{(n)}$ does not depend on $\gamma_{o}$.

**Lemma 2.7** For all $o, x \in V$, the sequence $(e^{u_{o,x}^{(n)}})_{n \in \mathbb{N}}$ is uniformly integrable with respect to $\rho_{o}$.

**Proof.** The claimed uniform integrability is essentially contained in Corollary 2 in [8]. Indeed, Sabot and Zeng define a family of random variables $(u^{(n)}(o,x))_{x \in \tilde{V}_{n}}$. As a consequence of Lemma 2.6, its joint law equals $\mathcal{L}_{\rho_{o}}((u_{o,x}^{(n)})_{x \in \tilde{V}_{n}})$. Corollary 2 of [8] implies that for any $x \in V$ the sequence $(e^{u^{(n)}(o,x)})_{n \in \mathbb{N}}$ is uniformly integrable, which allows us to conclude.

### 2.3 The random environment for VRJP on an infinite graph

In this section, we examine the convergence of the environment $e^{u_{o,x}}$ on the finite graph $G_{n}$ to its limit $e^{u_{o,x}}$ and deduce the corresponding representation in infinite volume stated in Fact 1.2.

**Lemma 2.8** For all $o, x \in V$, the limit of $u_{o,x}^{(n)}$ as $n \to \infty$ exists $\rho_{o}$-almost surely in $\mathbb{R}$. In other words, $u_{o,x}$ is $\rho_{o}$-almost surely given by formula (2.16). Moreover, the measures $\rho_{x}$ and $\rho_{o}$ are mutually absolutely continuous with the Radon-Nikodym derivative
\[ \frac{d\rho_{x}}{d\rho_{o}} = e^{u_{o,x}} \quad \rho_{o}\text{-a.s.} \tag{2.20} \]
Furthermore,
\[ e^{u_{o,x}} = E_{\rho_{o}}[e^{u_{o,x}}|\mathcal{F}_{n}] \quad \text{for all } n \quad \text{and} \quad E_{\rho_{o}}[e^{u_{o,x}}] = 1. \tag{2.21} \]
Hence, using Lemma 2.10 and dominated convergence again, we obtain

$$\lim_{n \to \infty} \frac{d\rho_x}{d\rho_o} |_{F_n} = \lim_{n \to \infty} e^{u_{o,x}^{(n)}}$$

(2.22)

exists \( \rho_o \)-almost surely in \( \mathbb{R} \) and in \( L^1 \), and that \( \lim_{n \to \infty} u_{o,x}^{(n)} \in \mathbb{R} \cup \{-\infty\} \) holds \( \rho_o \)-almost surely. We need to exclude \( \rho_o \)-a.s. the value \(-\infty\). Since \( \bigcup_{n \in \mathbb{N}} F_n \) generates \( \sigma(\beta_x, x \in V) \), we conclude \( \rho_x \ll \rho_o \) with the Radon-Nikodym-derivation \( d\rho_x/d\rho_o = \lim_{n \to \infty} e^{u_{o,x}^{(n)}} \). Interchanging \( x \) and \( o \) it follows also that \( \rho_o \ll \rho_x \) and \( d\rho_x/d\rho_o \geq 0 \) holds \( \rho_o \)-a.s. Hence, \( \rho_o \)-a.s., \( \lim_{n \to \infty} u_{o,x}^{(n)} \to -\infty \). This shows that indeed \( u_{o,x} \) is given by formula (2.16) \( \rho_o \)-a.s. and that the claim (2.20) is valid.

We conclude that \((e^{u_{o,x}^{(n)}})_{n \in \mathbb{N}}\) is a uniformly integrable martingale converging to \( e^{u_{o,x}} \) in \( L^1(\rho_o) \) and \( \rho_o \)-a.s. The first equation in (2.21) follows. \( L^1 \)-convergence and Lemma 2.4 imply the last equation in (2.21).

Remark 2.9 We remark that formula (1.3) is a consequence of (2.6) and the \( \rho_o \)-almost sure convergence of \( u_{o,x}^{(n)} \), \( x \in V \), as \( n \to \infty \) to \( u_{o,x} \) stated in Lemma 2.8. In particular, one has \( \rho_o(B') = 1 \) and the equation in (1.3) holds for all \( \beta \in B' \) with \( B' \) given in Definition 2.5.

The VRJP in exchangeable time scale on the finite graph \( G_n \) can not only be described as a mixture of Markov jump processes with respect to \( \rho_o^n \), but also with respect to its extension \( \rho_o \):

Lemma 2.10 For any event \( A \subseteq \tilde{V}_n^N \times \mathbb{R}_+^N \), one has

$$P_o^n(A) = \int_{\mathbb{R}_+^N} Q_{\rho_o,\beta}^G(A) \rho_o(d\beta).$$

(2.23)

Proof. By Lemma 2.1 the claim holds with \( \rho_o \) replaced by \( \rho_o^n \). Since \( Q_{\rho_o,\beta}^G(A) \) is \( F_n \)-measurable and \( \rho_o = \rho_o|_{F_n} \), the claim follows.

Proof of Fact 1.2. It suffices to show

$$E_{P_o^n}[F] \xrightarrow{n \to \infty} \int_{\mathbb{R}_+^N} E_{Q_{\rho_o,\beta}^G}[F] \rho_o(d\beta)$$

(2.24)

for functions \( F(\hat{w}) := f(\hat{w}|_{[0,J]}) \) with any \( J \in \mathbb{N} \) and any bounded measurable function \( f : \hat{W}([0,J]) \to \mathbb{R} \). Let \( \Pi_J \) denote the set of all paths \( \pi = (\pi_0, \pi_1, \ldots, \pi_{J+1}) \in V^{[0,J+1]} \) in \( G \) which start at \( o \). Clearly, \( \Pi_J \) is a finite set. Take \( N \) large enough that any path in \( \Pi_J \) does not leave \( V_N \). Let \( \beta \in B' \). Because \( \{u_{o,x}^{(n)}(\beta) : x \in V_N, n \geq N\} \) for the given \( \beta \) is bounded, dominated convergence yields for \( n \geq N \)

$$E_{Q_{\rho_o,\beta}^G}[F] = \sum_{\pi \in \Pi_J} \int_{[0,J]} f(\pi|_{[0,J]}) \prod_{j=0}^J \frac{C_{\pi_j,\pi_{j+1}}}{2} e^{u_{o,x}^{(n)} - u_{o,x}} e^{-\beta e_j f(j) \ell} d\ell \xrightarrow{n \to \infty} E_{Q_{\rho_o,\beta}^G}[F].$$

(2.25)

Hence, using Lemma 2.10 and dominated convergence again, we obtain

$$E_{P_o^n}[F] = \int_{\mathbb{R}_+^N} E_{Q_{\rho_o,\beta}^G}[F] \rho_o(d\beta) \xrightarrow{n \to \infty} \int_{\mathbb{R}_+^N} E_{Q_{\rho_o,\beta}^G}[F] \rho_o(d\beta).$$

(2.26)
3 Proof of the main result

In this section, we study VRJP in a finite spatial observation window $K$, augmented by the wiring point $\delta$. Recall the various reductions given in the introduction:

- **$K^+$-reduction on finite graphs**: We observe the process in $K$ and at $\delta$, but the local time is not counted at $\delta$.
- **$K$-reduction on the infinite graph**: We observe the process only when it visits $K$.
- **$K^+$-reduction for interlacements**: We take the part of the initial piece running through the finite observation window $K$, followed by infinitely many loops around $\delta$ obtained from the $K$-reduction of the doubly infinite paths of the interlacement. The holding time at $\delta$ is not counted.

We examine two modifications of these reductions including a holding time at $\delta$.

- **Modified $K^+$-reduction on finite graphs**: We augment the process by a rescaled holding time at $\delta$. The rescaling is necessary because of a problem with the unknown uniform integrability described in Remark 3.5 below.
- **Modified $K^+$-reduction for interlacements**: Here, the local time at $\delta$ is derived from the $t$-parameter of the interlacement.

These reductions turn out to be Markov jump processes. We determine their transition rates in Section 3.1. In Section 3.2, we show that the finite-volume transition rates converge to their infinite volume analogues. Finally, in Section 3.3, we use this convergence to deduce our main result Theorem 1.4.

3.1 Transition rates of various reductions

For $\tilde{w} = (w,l) \in \tilde{W}^\rightarrow$ or $\tilde{w} \in \tilde{W}_n^\rightarrow$ for some $n$, the hitting time and the return time of a set $A$ are defined by

$$H_A(\tilde{w}) = \inf\{k \geq 0 : w(k) \in A\},$$

(3.1)

$$\tilde{H}_A(\tilde{w}) = \inf\{k \geq 1 : w(k) \in A\},$$

(3.2)

respectively. If $A = \{y\}$ is a singleton, we write $H_y = H(y)$ and $\tilde{H}_y = \tilde{H}(y)$.

Let $K \subset V$ be a finite set with $o \in K$. Consider $n$ large enough so that $K \subseteq V_n$. We define for $x, y \in K$

$$e_K^n(x) = e_{K,\beta}^n(x) := 1_{\{x \in K\}} e^{2u_{x,y}^n} Q_{x,\beta}^G(H_\delta < \tilde{H}_K),$$

(3.3)

$$q_K^n(x,y) = q_{K,\beta}^n(x,y) := 1_{\{x \in K\}} Q_{x,\beta}^G(1 < \tilde{H}_K = \tilde{H}_y < \tilde{H}_\delta).$$

(3.4)

Note that $\{H_\delta < \tilde{H}_K\}$ means the event to exit $K$ immediately and reach $\delta$ before returning to $K$ and the event $\{1 < \tilde{H}_K = \tilde{H}_y < \tilde{H}_\delta\}$ means that the walk exits $K$ immediately and reenters it at $y$ before hitting $\delta$. The corresponding quantities in infinite volume are given by

$$e_K(x) = e_{K,\beta}(x) := 1_{\{x \in K\}} e^{2u_{x,y}} Q_{x,\beta}^G(\tilde{H}_K = \infty),$$

(3.5)

$$q_K(x,y) = q_{K,\beta}(x,y) := 1_{\{x \in K\}} Q_{x,\beta}^G(1 < \tilde{H}_K = \tilde{H}_y < \infty).$$

(3.6)

Similarly to the above, the event $\{\tilde{H}_K = \infty\}$ means that the walk exits $K$ immediately and never returns to it, and the event $\{1 < \tilde{H}_K = \tilde{H}_y < \infty\}$ means that the walk exits $K$ immediately and reenters it at $y$.

Recall that for any fixed $n \in \mathbb{N}$, the expression $\beta_\delta$ is a synonym for $\beta_{\delta}^{\text{new},n}$, which does not display the dependence on $n$. 

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Lemma 3.1 For all $\beta \in B$, all finite $K$ with $o \in K \subset V$, all $x \in K$, and all $n \in \mathbb{N}$, one has

$$\beta \delta e^{2u_{n,\delta}} Q_{\delta,\beta}^n [\text{first excursion hits } K \text{ first in } x] = \beta x e^n_K(x).$$  \hspace{1cm} (3.7)

Summing over $x \in K$, we have

$$\beta \delta e^{2u_{n,\delta}} Q_{\delta,\beta}^n [\text{first excursion hits } K] = \sum_{x \in K} \beta x e^n_K(x).$$ \hspace{1cm} (3.8)

Proof. We calculate

$$Q_{\delta,\beta}^n [\text{first excursion hits } K \text{ first in } x] = \sum_{\pi \in \Pi_x} Q_{\delta,\beta}^n(\pi),$$ \hspace{1cm} (3.9)

where we sum over the set $\Pi_x$ of all finite paths $\pi$ from $\delta$ to $x$ hitting $K$ for the first time in $x$ and visiting $\delta$ only at the start; the event that the process initially follows $\pi$ is denoted by $\pi$. For $\pi = (\pi_0, \pi_1, \ldots, \pi_m) \in \Pi_x$, one has by the reversibility [A.3] from the appendix

$$\beta \delta e^{2u_{n,\delta}} Q_{\delta,\beta}^n(\pi) = \beta x e^{2u_{n,\delta}} Q_{\delta,\beta}^n(\pi^{+}),$$ \hspace{1cm} (3.10)

where $\pi^+ = (\pi_m, \pi_{m-1}, \ldots, \pi_0)$ denotes the reversed path. Consequently, we conclude

$$\beta \delta e^{2u_{n,\delta}} Q_{\delta,\beta}^n [\text{first excursion hits } K \text{ first in } x] = \beta x e^{2u_{n,\delta}} \sum_{\pi \in \Pi_x} Q_{\delta,\beta}^n(\pi^+) = \beta x e^n_K(x);$$ \hspace{1cm} (3.11)

in the last step we replaced the sum over $Q_{\delta,\beta}^n(\pi^+)$ by the sum of $Q_{\delta,\beta}^n(\pi)$, where $\pi$ runs over all paths from $x$ to $\delta$ which hit $\delta$ only at the end and reach $\delta$ before returning to $K$. \hfill \blacksquare

Lemma 3.2 (Modified $K^+$-reduction of Markov jump processes - finite volume) Let $n \in \mathbb{N}$ and consider a given $\beta \in B$. Let $K \subset V_n$ with $o \in K$. We define a modified $K^+$-reduction $\hat{w}^K_{\text{mod}} = (w^K(m), t^K_{\text{mod}}(m))_{m \in \mathbb{N}_0}$ of the Markov jump process $\hat{w}$ on the finite graph $G_n$ in the environment $\beta$, described by the probability measure $Q_{\delta,\beta}^n$ just as the $K^+$-reduction in formulas [1.20] – [1.22] except that the local time in $\delta$, which was ignored in [1.22], is now counted, but rescaled:

$$t^K_{\text{mod}}(m) = \sum_{j=m}^{j=m+1} l(k_j) \left( e^{-2u_{n,\delta}} 1_{\{w(k_j) = \delta\}} + 1_{\{w(k_j) \neq \delta\}} \right).$$ \hspace{1cm} (3.12)

Then, $\hat{w}^K_{\text{mod}}$ is a Markov jump process on $\hat{K} = K \cup \{\delta\}$ with respect to $Q_{\delta,\beta}^n$. Its rates for transitions $x \rightarrow y$ with different $x, y \in \hat{K}$ are given by

$$Q_{\delta,\beta}^n[w^K(k+1) = y, t^K_{\text{mod}}(k) < \ell + d\ell | w^K(k) = x, t^K_{\text{mod}}(k) \geq \ell]$$

$$= \left\{ \begin{array}{ll}
\frac{1}{2} C_{xy} e^{u_{n,\delta} - u_{n,\delta}^n} + \beta x q^n_{K,\beta}(x, y) \right) d\ell + o(d\ell) & \text{for } x, y \in K, x \neq y \\
\beta x e^{-2u_{n,\delta}} e^n_K(x) d\ell + o(d\ell) & \text{for } x \in K, y = \delta, \\
\beta y e^n_K(y) d\ell + o(d\ell) & \text{for } x = \delta, y \in K
\end{array} \right.$$ \hspace{1cm} (3.13)

as $d\ell \downarrow 0$.

Recall that by definition [1.20] – [1.22], the term $w^K(m)$ is only updated when the walk arrives at a new site $y \in \hat{K}$ different from $x$. We don’t have to take into account the direct returns to $x$ from $K^c$ for the following reason. Let us examine the walk while the local time at $x$ is in the time interval $[\ell, \ell + d\ell]$. Consider the event that in this time interval, it makes both at least one loop from $x$ to $x$ via $K^c$ and
then a transition from $x$ to $y$. Then, it has to leave $x$ at least twice in this time interval, which has a probability bounded by $O((dl)^{2})$. Thus, the term $o(dl)$ includes this probability.

**Proof of Lemma 3.2.** The jumps from $x \in K$ to $y \in K$, $y \neq x$, originate from two sources. Either the original walk jumps along an edge directly from $x$ to $y$, which it does at rate $\frac{1}{2}C_{xy}e^{\omega_{u_{xy}}-\omega_{u_{xy}}}$, or it leaves $K$ at $x$ and reenters at $y$. Conditionally on jumping away from $x$, which occurs at rate $\beta_x$, the random walker leaves $K$ and reenters $K$ at $y$ before hitting $\delta$ with probability $q_{K,\beta}^{n}(x,y)$. This explains the second summand in the first line on the right-hand side of (3.13). The argument for transitions $K \ni x \to \delta$ is similar: The vertex $x \in K$ is left at rate $\beta_x$, and conditionally on leaving it, the probability to exit $K$ immediately and hitting $\delta$ before reentering $K$ equals $e^{-2u_{xy}}e_{K,\beta}^{n}(x)$; the factor $e^{-2u_{xy}}$ removes the normalization $e^{2u_{xy}}$ in the definition (3.3) of $e_{K,\beta}^{n}(x)$. Finally, the rate of the original walk to leave $\delta$, without rescaling local times at $\delta$, equals $\beta_\delta$. The rescaling with the factor $e^{-2u_{xy}}$ yields the modified rate $\beta_\delta e^{2u_{xy}}$. Multiplying it with the probability that the first excursion from $\delta$ hits $K$ first in $y$, formula (3.7) yields the rate $\beta_\delta e_{K,\beta}^{n}(y)$ for transitions from $\delta$ to $y$. ■

Similarly, we can compute the transition rates for the $K$-reduction in infinite volume as done in the following lemma.

**Lemma 3.3 (K-reduction of Markov jump processes – infinite volume).**

Consider $\beta \in B'$, cf. Definition 2.5. Take a finite subset $K \subset V$ with $o \in K$. Consider a Markov jump process with absorption having state space $K \cup \{\perp\}$, where $\perp$ means absorption, with the following jump rates

\[
\frac{1}{2}C_{xy}e^{\omega_{u_{xy}}-\omega_{u_{xy}}} + \beta_x e_{K,\beta}(x, y) \quad \text{for transitions } x \to y \text{ with } x, y \in K, \tag{3.14}
\]

\[
\beta_x e^{-2u_{xy}} e_{K,\beta}(x) \quad \text{for transitions } x \to \perp \text{ with } x \in K. \tag{3.15}
\]

The law $Q^{K}_{z,\beta}$ of this Markov jump process started in any $z \in K$ and stopped immediately before being absorbed equals the law of the $K$-reduction $\bar{w}^{K} = (w^{K}, l^{K})$ with respect to $Q^{G}_{z,\beta}$.

**Proof.** The proof is almost the same as the proof of Lemma 3.2. The jumps from $x$ to $y$ originate from two sources. Either the original walk jumps along an edge directly from $x$ to $y$, which it does at rate $\frac{1}{2}C_{xy}e^{\omega_{u_{xy}}-\omega_{u_{xy}}}$, or it leaves $K$ at $x$ and reenters at $y$. Conditionally on jumping away from $x$, which occurs at rate $\beta_x$, the random walker leaves $K$ and reenters $K$ at $y$ with probability $q_{K,\beta}(x,y)$. This explains the second summand in (3.14). The argument for (3.15) is similar. The factor $e^{-2u_{xy}}$ removes the normalization $e^{2u_{xy}}$ in the definition (3.3) of $e_{K,\beta}(x)$. ■

In order to phrase a slightly stronger version of the main Theorem 1.4 we define also a modified $K^{+}$-reduction for interlacements, which gives rise to the following transition probabilities described in Lemma 3.4.

**Lemma 3.4 (Modified K^{+}-reduction of interlacements).** Let $\beta \in B'$ and let $K \subset V$ be finite with $o \in K$. Define a modified $K^{+}$-reduction $\omega^{K}_{\text{mod}} = (w^{K}(k), l^{K}(k))_{k \in \mathbb{N}_0}$ for interlacements $\omega$ as in (1.24) by the following modified version of (1.26), where the local time at $\delta$ is now counted and equals the increment of the $t$-parameter of the interlacement: $\omega^{K}_{\text{mod}}$ is defined to be the concatenation of $\omega^{K}$ and all $(\delta, t_{i_j - t_{i_{j-1}}}, (\hat{w}_{i_j} |_B)_{K})_{j \text{ running through } 1, 2, 3, \ldots}$, where we use the convention $t_{i_0} := 0$.

The modified $K^{+}$-reduction $\omega^{K}_{\text{mod}}$ is a Markov jump process on $\hat{K}$ with respect to the law $Q_{o,\beta}$ of the interlacement process in the environment $\beta$. Its rates for transitions $x \to y$ with different $x,y \in \hat{K} = K \cup \{\delta\}$ are given by

\[
\begin{cases}
Q_{o,\beta}[w^{K}(k+1) = y, l^{K}(k) < \ell + dl | w^{K}(k) = x, l^{K}(k) \geq \ell] = \\
(\frac{1}{2}C_{xy}e^{\omega_{u_{xy}}-\omega_{u_{xy}}} + \beta_x q_{K,\beta}(x,y)) dl + o(dl), \quad \text{for } x,y \in K, \quad \text{or } x \in K, y = \delta, \\
\beta_x e^{-2u_{xy}} e_{K,\beta}(x) dl + o(dl), \quad \text{for } x \in K, y = \delta, \\
\beta_y e_{K,\beta}(y) dl + o(dl), \quad \text{for } x = \delta, y \in K
\end{cases}
\tag{3.16}
\]

as $dl \downarrow 0$. 

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Remark 3.5 If we do not rescale the time in \((3.12)\), the rate to jump from \(\delta\) to \(y\) in the last line of \((3.13)\) gets an additional factor \(e^{-2u_{\omega,\delta}}\), which has no counterpart in the infinite volume version \((3.16)\). We do not know almost sure convergence of this factor \(e^{-2u_{\omega,\delta}}\) with respect to \(\rho_\omega\), as it is an open question whether this measure is absolutely continuous with respect to \(\rho_\infty\). For this reason, we have ignored the local time at \(\delta\) in the \(K^+\)-reduction. Therefore the random interlacement is not endowed with an intensity level in the classical sense.

Under the assumption that \((e^{u_{\omega,\delta}})_{n \in \mathbb{N}}\) is uniformly integrable with respect to \(\rho_\infty\), which is unknown to hold, the measure \(\rho_\omega\) is absolutely continuous with respect to \(\rho_\infty\). In that case, we have a \(\rho_\omega\)-a.s. limit \(e^{2u_{\omega,\delta}}\) of \((e^{2u_{\omega,\delta}})_{n \in \mathbb{N}}\). Changing then the intensity measure described in \((1.16)\) with the corresponding Radon-Nikodym derivative one could also prove not only convergence of \(K^+\)-reductions, but also of the modified \(K^+\)-reductions, where one takes into account the time spent at \(\delta\). On the other hand, if the two measures \(\rho_\infty\) and \(\rho_\omega\) were known to be mutually absolutely continuous, then we could also get rid of the initial piece \(\omega_s\) in a sample \(\omega\) of the random interlacement, cf. \((1.24)\). Because all this relies on the unknown uniform integrability assumption, we do not work out the details here.

Proof of Lemma 3.4 A typical path of a Markov jump process on \(\hat{K}\) starting in \(o\) consists of an initial piece running from \(o\) to \(\delta\) and then, independent of it, a concatenation of a sequence of i.i.d. pairs, each consisting of an exponential waiting time at \(\delta\) and, again independent of it, a Markovian loop around \(\delta\). The \(K^+\)-reduction \(\omega^K\) is indeed constructed in this way:

- The initial piece is the \(K\)-reduction of \(\omega_s\), which is independent of \(\omega^\star\). According to Lemma 3.3, it is a Markov jump process with transition rates given by \((3.14)\) and \((3.15)\). These rates coincide with the rates claimed in the first two lines of the right-hand side of \((3.16)\).
- The pairs \(((\delta, t_{ij} - t_{ij-1}), (\hat{u}_{ij}|_{[0,\delta]})_K), j \in \mathbb{N}\), are i.i.d. as they are obtained from a decorated Poisson process. Being functions of \(\omega^\star\), they are independent of the initial piece. Consider a given \(j \in \mathbb{N}\). Since the intensity measure of \(1_{\hat{W}_K}^{\star \, \omega^\star} \omega^\star\) is the product measure \(\pi^\star [Q_{K,\beta}] \times dt\), the components \(t_{ij} - t_{ij-1}\) and \((\hat{u}_{ij}|_{[0,\delta]})_K\) are independent.
  - The waiting time \(t_{ij} - t_{ij-1}\) is exponential with the total mass of \(\pi^\star [Q_{K,\beta}]\) as its parameter, i.e. with parameter \(\sum_{x \in K} \beta_x e_{K,\beta}(x)\), see formula \((A.6)\) in the appendix.
  - The two-sided infinite path \(\hat{u}_{ij}\) has the law \(\hat{Q}_{K,\beta}/Q_{K,\beta}(\hat{W})\). Hence, by \((1.9)\), the law of \(\hat{u}_{ij}|_{[0,\delta]}\) is given by

\[
\frac{1}{\hat{Q}_{K,\beta}(\hat{W})} \sum_{x \in K} \beta_x e_{2u_{\omega,\delta}} Q^G_{x,\beta}(A_K) Q^G_{x,\beta} = \sum_{x \in K} \beta_x e_{K,\beta}(x) Q^G_{x,\beta}.
\]

(3.17)

Recall the definition of the measure \(Q^{K+}_{x,\beta}\) given in Lemma 3.3. The \(K\)-reduction of \(\hat{u}_{ij}|_{[0,\delta]}\), which describes the \(j\)-th excursion from \(\delta\), therefore has the law

\[
\sum_{x \in K} \beta_x e_{K,\beta}(x) Q^{K+}_{x,\beta}.
\]

(3.18)

Conditionally on the starting point \(x \in K\), this is just \(Q^{K+}_{x,\beta}\). According to Lemma 3.3, it describes a Markov jump process with rates \((3.14)\)–\((3.15)\) stopped before being absorbed. Note that these transition rates do not depend on the starting point \(x\). They coincide with the ones claimed in the first two lines of the right-hand side in \((3.16)\). Consequently, the law \((3.18)\) describes also a Markov jump process with the same transition rates, but with a random starting point having the law \(\sum_{x \in K} \beta_x e_{K,\beta}(x) \delta_x / \sum_{x \in K} \beta_x e_{K,\beta}(x)\). Summarizing, jumps away from \(\delta\) occur with the total rate \(\sum_{x \in K} \beta_x e_{K,\beta}(x)\). Any such jump arrives in a given \(y \in K\) with probability \(\beta_y e_{K,\beta}(y) / \sum_{x \in K} \beta_x e_{K,\beta}(x)\). Multiplying these two quantities the transition rate from \(\delta\) to \(y\) is given by \(\beta_y e_{K,\beta}(y)\), as claimed.
3.2 Convergence of transition rates

Theorem 3.6 (Infinite-volume limits) For all finite subsets \( K \subset V \) and all \( x, y \in K \), one has \( \rho_o \)-a.s.

\[
\lim_{n \to \infty} e^n_K(x) = e_K(x), \quad \lim_{n \to \infty} e^n_K(x, y) = q_K(x, y), \quad \lim_{n \to \infty} e^{u(n)}_{x,z} = e^{u_o,x}. \tag{3.19}
\]

In particular, the finite-volume transition rates given in (3.13) for the modified \( K \)-reduction of the Markov jump process in the environment \( \beta \) converge \( \rho_o \)-a.s. to the corresponding infinite-volume quantities:

\[
\frac{1}{2} C_{xy} e^{u_{x,z} - u_{o,z}} + \beta_x q_K(x, y) e^{u_{o,x}} + \beta_x q_{K,\beta}(x, y), \quad \beta_x e^{-2u_{o,x}} e_{K,\beta}(x), \quad \beta_x e_{K,\beta}(y) \quad \lim_{n \to \infty} e^n_{K,\beta}(x,y), \tag{3.20, 3.21, 3.22}
\]

Note that just as in the last line of the transition law described in (3.13), there is no factor \( e^{-2u_{o,x}} \) on the left-hand side of the last equation.

The proof needs some preliminary lemmas and is given in the remainder of this subsection. Recall the filtration \( \mathcal{F}_n = \sigma(\beta_x, x \in V_n), n \in \mathbb{N} \). First, Lemma 3.7 describes a uniform martingale property for probabilities to follow a finite path that does not hit \( \delta \), whereas Lemma 3.8 describes the same property for a path which visits \( \delta \) precisely at its endpoint.

Lemma 3.7 Let \( n \in \mathbb{N}, x, y \in V_n \), and let \( \pi = (\pi_0, \pi_1, \ldots, \pi_m) \) be a finite path in \( G_n \) from \( x \) to \( y \) with \( \pi_k \in V_n \) for all \( k \). Then, writing \( \pi \) for the event that the process follows the path \( \pi \) initially, one has \( \rho_o \)-a.s.

\[
Q^G_n(\pi) e^{u_{o,x}} = E_{\rho_o}[Q^G_n(\pi) e^{u_{o,x}} | \mathcal{F}_n]. \tag{3.23}
\]

Consequently, if \( A \) is the union of countably many such events \( \pi \), one has \( \rho_o \)-a.s.

\[
Q^G_n(A) e^{u_{o,x}} = E_{\rho_o}[Q^G_n(A) e^{u_{o,x}} | \mathcal{F}_n]. \tag{3.24}
\]

Note that \( \pi \) on the left-hand side in (3.23) is understood as an event in \( \hat{W}_n \), while on the right-hand side in (3.23) it is understood as an event in \( \hat{W} \).

Proof of Lemma 3.7 Using \( C_{ab}^{(n)} = C_{ab} \) for all \( a, b \in V_n \), we calculate

\[
Q^G_n(\pi) = \prod_{k=0}^{m-1} \frac{C_{\pi_k \pi_{k+1}}^{(n)}}{2\beta_{\pi_k}} e^{u_{\pi_k \pi_{k+1} - u_{\pi_k} \pi_{k+1}}} = e^{u_{\pi} - u_{\pi'-y}} \prod_{k=0}^{m-1} \frac{C_{\pi_k \pi_{k+1}}}{2\beta_{\pi_k}}. \tag{3.25}
\]

Similarly, we obtain

\[
Q^G_n(\pi') = e^{u_{o,x} - u_{o,z}} \prod_{k=0}^{m-1} \frac{C_{\pi_k \pi_{k+1}}}{2\beta_{\pi_k}}. \tag{3.26}
\]

Since all \( \beta_{\pi_k} \) are \( \mathcal{F}_n \)-measurable, the claim (3.23) follows from the equation \( e^{u_{\pi} - u_{\pi'}} = E_{\rho_o}[e^{u_{\pi} - u_{\pi'}} | \mathcal{F}_n] \) given in (2.21).

Taking a countable union \( A = \bigcup_{i \in I} \pi^{(i)} \) with different \( \pi^{(i)} \) and a countable index set \( I \), we may drop all \( \pi^{(i)} \) for which there is another \( \pi^{(j)}, j \neq i \), being an initial piece of \( \pi^{(i)} \). Let \( J \subseteq I \) denote the set of all remaining indices. Then, \( A = \bigcup_{i \in J} \pi^{(i)} \) is a countable union of pairwise disjoint events. The claim (3.24) then follows from (3.23) and monotone convergence. \( \blacksquare \)
Lemma 3.8 Let \( n \in \mathbb{N} \), \( x \in V_n \), and let \( \pi = (\pi_0, \pi_1, \ldots, \pi_m) \) be a finite path in \( G_n \) from \( x \) to \( \delta \) with \( \pi_k \in V_n \) for all \( 0 \leq k \leq m - 1 \). Let \( \Pi_\pi \) denote the set of finite paths in the infinite graph \( G \) of the form \( (\pi_0, \pi_1, \ldots, \pi_{m-1}, y) \) with \( y \notin V_n \). Let \( \Pi_\pi \) denote the event that the process follows a path in \( \Pi_\pi \) initially. Then, one has \( \rho_0 \)-a.s.

\[
Q_{x,\beta}^G(\pi)e^{u_\beta(n)} = E_{\rho_0}[Q_{x,\beta}^G(\Pi_\pi)e^{u_\beta(n)}|F_n].
\] (3.27)

Proof. Similarly to (3.25), we obtain

\[
Q_{x,\beta}^G(\pi)e^{u_\beta(n)} = \left( \prod_{k=0}^{m-1} \frac{1}{2\beta_{\pi_k}} \right) \left( \prod_{k=0}^{m-2} C_{\pi_k\pi_{k+1}} \right) \cdot C_{\pi_{m-1}\delta} \ e^{u_{\delta}(n)}
\] (3.28)

and for any path \( \zeta \in \Pi_\pi \) from \( x \) to \( y \notin V_n \)

\[
Q_{x,\beta}(\zeta)e^{u_\beta(n)} = \left( \prod_{k=0}^{m-1} \frac{1}{2\beta_{\pi_k}} \right) \left( \prod_{k=0}^{m-2} C_{\pi_k\pi_{k+1}} \right) \cdot C_{\pi_{m-1}y} \ e^{u_{\beta}(y)}
\] (3.29)

Since \( \prod_{k=0}^{m-1} \beta_{\pi_k} \) is \( F_n \)-measurable, it follows

\[
E_{\rho_0}[Q_{x,\beta}^G(\Pi_\pi)e^{u_\beta(n)}|F_n] = \sum_{\zeta \in \Pi_\pi} E_{\rho_0}[Q_{x,\beta}(\zeta)e^{u_\beta(n)}|F_n]
\]

\[
= \left( \prod_{k=0}^{m-1} \frac{1}{2\beta_{\pi_k}} \right) \left( \prod_{k=0}^{m-2} C_{\pi_k\pi_{k+1}} \right) \sum_{y \in V \setminus V_n} C_{\pi_{m-1}y} E_{\rho_0}[e^{u_{\beta}(y)}|F_n].
\] (3.30)

Let \( y \in V \setminus V_n \). Using the martingale representation (2.21) and the definition (2.5) of \( u_{\beta}(n) \) and \( u_{\delta}(n) \) together with \( u_{\beta}(n) = u_{\delta}(n) = 0 \), cf. (2.4), yields \( \rho_0 \)-a.s.

\[
E_{\rho_0}[e^{u_{\beta}(y)}|F_n] = e^{u_{\beta}(n)} = e^{u_{\delta}(n)} - u_{\delta}(n) = e^{u_{\beta}(n)} - u_{\delta}(n) = e^{u_{\delta}(n)}.
\] (3.31)

Using the definition of \( C_{\pi_{m-1}\delta} \) described above (1.19), we obtain \( \rho_0 \)-a.s.

\[
\sum_{y \in V \setminus V_n} C_{\pi_{m-1}y} E_{\rho_0}[e^{u_{\beta}(y)}|F_n] = \sum_{y \in V \setminus V_n} C_{\pi_{m-1}y} e^{u_{\beta}(n)} = C_{\pi_{m-1}\delta} e^{u_{\delta}(n)}.
\] (3.32)

Inserting this in (3.30) and comparing the result with (3.28) the claim follows. \( \blacksquare \)

The following general lemma on conditional expectations of monotone sequences is needed in the sequel.

Lemma 3.9 On some probability space, let \( L^1 \ni X_n \geq 0 \), \( n \in \mathbb{N} \), be a decreasing or an increasing sequence with the pointwise limit \( \lim_{n \to \infty} X_n = X \in L^1 \). Let \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) be a filtration such that all \( X_n \) are measurable with respect to \( \sigma(\bigcup_n \mathcal{G}_n) \). Then, one has

\[
\lim_{n \to \infty} E[X_n|\mathcal{G}_n] = X \quad \text{a.s. and in } L^1.
\] (3.33)

Proof. We have \( \|X_n - X\|_1 \overset{n \to \infty}{\longrightarrow} 0 \) by dominated convergence, and hence

\[
\|E[X_n|\mathcal{G}_n] - E[X|\mathcal{G}_n]\|_1 \leq \|X_n - X\|_1 \overset{n \to \infty}{\longrightarrow} 0.
\] (3.34)

Moreover,

\[
\|E[X|\mathcal{G}_n] - X\|_1 \overset{n \to \infty}{\longrightarrow} 0
\] (3.35)
by the martingale convergence theorem. Together, it follows
\[ \| E[X_n | G_n] - X \| \xrightarrow{n \to \infty} 0. \] (3.36)
This proves convergence in $L^1$. Finally, $(E[X_n | G_n])_{n \in \mathbb{N}}$ is a non-negative super- or submartingale, given that $(X_n)_{n \in \mathbb{N}}$ is decreasing or increasing, respectively. Hence it converges a.s. as well. □

**Proof of Theorem 3.6.** Fix a finite set $K \subset V$ and $x, y \in K$. Recall that $u_{o,x} = \lim_{n \to \infty} u_{o,x}^{(n)} \in \mathbb{R}$ holds $\rho_o$-a.s. by Lemma 2.8. In particular, $\lim_{n \to \infty} e^{u_{o,x}^{(n)}} = e^{u_{o,x}}$ $\rho_o$-a.s.

Given $n \in \mathbb{N}$, the event $A = \{ 1 < \tilde{H}_K = \tilde{H}_y < \tilde{H}_x \} \subseteq \tilde{W}_n^{+}$ of returning to $K$ at $y$ restricted to decorated paths starting at $x$ can be written as a countable union of events $\pi$ with finite paths $\pi$ from $x$ to $y$ which do not hit $\delta$. In particular, equation (3.32) holds for it. This yields
\[ q_{K,\delta}^{(n)}(x, y) = Q_{x,\delta}^{G}(1 < \tilde{H}_K = \tilde{H}_y < \tilde{H}_x) \]
\[ = e^{-u_{o,x}^{(n)}} E_{\rho_o}[Q_{x,\delta}^{G}(1 < \tilde{H}_K = \tilde{H}_y < \tilde{H}_x) e^{u_{o,x}} | \mathcal{F}_n]. \] (3.37)

Consider the increasing sequence $X_n = Q_{x,\delta}^{G}(1 < \tilde{H}_K = \tilde{H}_y < \tilde{H}_x) e^{u_{o,x}} \geq 0$, $n \in \mathbb{N}$. Its pointwise limit as $n \to \infty$ is given by
\[ X = Q_{x,\delta}^{G}(1 < \tilde{H}_K = \tilde{H}_y < \infty) e^{u_{o,x}} = q_{K,\delta}(x, y) e^{u_{o,x}}. \] (3.38)
Clearly, all $u_{o,x}$, $z \in V$, are $\mathcal{F}_\infty$-measurable and hence the same is true for all $X_n$. Furthermore, $E_{\rho_o}[e^{u_{o,x}}] = 1 < \infty$ by (2.21). Hence, $X_n, X \in L^1(\rho_o)$. An application of Lemma 3.9 yields $\rho_o$-a.s.
\[ \lim_{n \to \infty} q_{K,\delta}^{(n)}(x, y) = e^{-u_{o,x}} X = q_{K,\delta}(x, y). \] (3.39)

Similarly, the event $\{ H_\delta < \tilde{H}_K \} \subseteq \tilde{W}_n^{+}$ of hitting $\delta$ before returning to $K$ restricted to decorated paths starting at $x$ can be written as a countable union of finite paths from $x$ to $\delta$. Hence, inserting the definition (3.3) of $e^{k}(x)$ and applying Lemma 3.8 and monotone convergence, we obtain
\[ e^{k}(x) = e^{2u_{o,x}} Q_{x,\delta}^{G}(H_\delta < \tilde{H}_K) = e^{u_{o,x}} E_{\rho_o}[Q_{x,\delta}^{G}(H_V \setminus V_n < \tilde{H}_K) e^{u_{o,x}} | \mathcal{F}_n]. \] (3.40)

We apply Lemma 3.9 with the decreasing sequence
\[ X_n = Q_{x,\delta}^{G}(H_V \setminus V_n < \tilde{H}_K) e^{u_{o,x}} \xrightarrow{n \to \infty} Q_{x,\delta}^{G}(\tilde{H}_K = \infty) e^{u_{o,x}} \] (3.41)
in $L^1(\rho_o)$. This yields the following $\rho_o$-a.s., using the definition (3.5) of $e^{k}(x)$:
\[ \lim_{n \to \infty} e^{k}(x) = \lim_{n \to \infty} e^{u_{o,x}} \cdot Q_{x,\delta}^{G}(\tilde{H}_K = \infty) e^{u_{o,x}} = e^{k}(x). \] (3.42)

### 3.3 Proof of Theorem 1.4

The following theorem shows that the finite-dimensional distributions of the modified $K^+$-reduction of VRJP on $G_n$ converge weakly as $n \to \infty$ to the finite-dimensional distributions of the modified $K^+$-reduction of the random interlacement.

**Theorem 3.10 (Convergence of modified $K^+$-reductions)** For any finite $K \subset V$ with $o \in K$ and $J \in \mathbb{N}$, one has
\[ \mathcal{L}_{P_o} \left( \hat{\omega}_{mod}^{K}[0,n] \right) \xrightarrow{w} \mathcal{L}_{\mathcal{F}_n} \left( \omega_{mod}^{K}[0,n] \right) \quad \text{as } n \to \infty. \] (3.43)
Proof. Let $\beta \in B'$ be an infinite-volume environment. For $x, y \in \tilde{K} = K \cup \{\delta\}$, let $r^n_{x,y} = r^n_{x,y}(\beta)$ denote the rate given in (3.13) for the modified $K^+$-reduction of the Markov jump process on $G_n$ and let $r^\infty_{x,y} = r^\infty_{x,y}(\beta)$ be the corresponding rate in infinite volume given in (3.10) for the modified $K^+$-reduction of the interlacement. In particular, this means $r^n_{x,\delta} = 0 = r^\infty_{x,\delta}$. Let $r^n_{x,x} = \sum_{y \in \tilde{K}} r^n_{x,y}$, $n \in \mathbb{N} \cup \{\infty\}$, denote the total rate to jump away from $x$. By Theorem 3.6, the transition rates converge $\rho_n$-a.s.: $\lim_{n\to \infty} r^n_{x,y} = r^\infty_{x,y}$ for all $x, y$ in the finite set $\tilde{K}$. Let $\Pi^K_n$ denote the set of all paths $\pi = (\pi_0, \pi_1, \ldots, \pi_{j+1}) \in \tilde{K}^{[0,j+1]}$ in $K$ which start at $\delta$. Clearly, $\Pi^K_n$ is a finite set. Hence, using Lemma 2.10 for any continuous function $f : (\tilde{K} \times \mathbb{R}_+)^{[0,j]} \to \mathbb{R}$ with compact support $\text{supp} f \subseteq (\tilde{K} \times [M^{-1}, M])^{[0,j]}$ for some $M > 0$, we have

$$E_{\rho_n} \left[ f(\tilde{\omega}_{\text{mod}}^{K} | [0,j]) \right] = \int_{\tilde{\mathbb{R}}^V} \int_{\mathbb{W}^n} f(\tilde{\omega}_{\text{mod}}^{K} | [0,j]) Q^{G,\beta}_{o,\beta} (d\tilde{\omega}) \, d\rho_o = \int_{\tilde{\mathbb{R}}^V} \sum_{\pi \in \Pi^K_n} \int_{\mathbb{R}^{[0,j]}} f(\pi | [0,j], \ell) \prod_{j=0}^{J} r^n_{\pi_j \pi_{j+1}} e^{-r^n_{\pi_j \pi_{j+1}} \ell(j)} \, d\ell \, d\rho_o \xrightarrow{n \to \infty} E_{\rho_o} \left[ f(\omega_{\text{mod}}^{K} | [0,j]) \right];$$

(3.44)

the used dominated convergence is justified by the fact that $f$ is compactly supported together with the following bound which is valid for all $(\pi | [0,j], \ell)$ in the support of $f$:

$$0 \leq r^n_{\pi_j \pi_{j+1}} e^{-r^n_{\pi_j \pi_{j+1}} \ell(j)} - r^n_{\pi_j \pi_{j+1}} e^{-r^n_{\pi_j \pi_{j+1}} \ell(j)} \leq \ell(j)^{-1} \sup_{x > 0} e^{-x} \leq (\ell(j))^{-1} \leq Me^{-1}. \quad (3.45)$$

In other words, $\mathcal{L}_{\rho_o} \left( \tilde{\omega}_{\text{mod}}^{K} | [0,j] \right)$ converges vaguely to $\mathcal{L}_{\rho_o} \left( \omega_{\text{mod}}^{K} | [0,j] \right)$ as $n \to \infty$. Because vague convergence of a sequence of probability measures to a probability measure implies weak convergence, the claim follows.

Proof of Theorem 1.4. Because the original $K^+$-reduction is obtained from its modified version just by ignoring the local times at $\delta$, Theorem 1.4 is an immediate consequence of Theorem 3.10.

A Poisson point process in a fixed environment

In this appendix, we give a proof of Theorem 1.3.

The following lemma uses the set $B'$ of environments introduced in Definition 2.5.

Lemma A.1 (Reversibility) For all $\beta \in B'$, $m \in \mathbb{N}$, and all measurable sets $A \subseteq (V \times \mathbb{R}_+)^{[0,m]}$, one has

$$\sum_{x \in V} \beta_x e^{2u_{x,m}} Q^G_{x,\beta} (\tilde{\omega}(k) | k \in [0,m] \in A) = \sum_{x' \in V} \beta_{x'} e^{2u_{x,m}} Q^G_{x',\beta} (\tilde{\omega}(m-k) | k \in [0,m] \in A). \quad (A.1)$$

In particular, for $x, x' \in V$ and $A \subseteq \{\{x\} \times \mathbb{R}_+ \} \times (V \times \mathbb{R}_+)^{[1,m-1]} \times \{\{x'\} \times \mathbb{R}_+ \}$, one has

$$\beta_x e^{2u_{x,m}} Q^G_{x,\beta} (\tilde{\omega}(k) | k \in [0,m] \in A) = \beta_{x'} e^{2u_{x,m}} Q^G_{x',\beta} (\tilde{\omega}(m-k) | k \in [0,m] \in A). \quad (A.2)$$

An analogous result holds for $\beta \in B$ and the Markov jump process with the law $Q^G_{x,\beta}$ on the graph $G_n$ with weights $G^{(n)}$. In particular, for any $x, x' \in \tilde{V}_n$ and any path $\pi = (\pi_0, \pi_1, \ldots, \pi_m)$ in $G_n$ from $x$ to $x'$ and its reversed version $\pi^{\ast} = (\pi_m, \pi_{m-1}, \ldots, \pi_0)$, one has

$$\beta_{x'} e^{2u_{x,m}} Q^G_{x',\beta} (\pi^{\ast}) = \beta_{x} e^{2u_{x,m}} Q^G_{x,\beta} (\pi^{\ast}). \quad (A.3)$$

Recall that $\pi$ denotes the event that the process initially follows $\pi$.

Proof. The argument is the same for the infinite volume version and the finite volume version. For this reason we describe it only for infinite volume.
It suffices to consider measurable sets of the form $A = \prod_{k=0}^{m-1}(\{x_k\} \times (l_k, \infty))$ with given $x_k \in V$, $l_k \geq 0$ fulfilling $x = x_0$ and $x' = x_m$. Then, the claim boils down to (A.2) for this special $A$. We express the probability on the left-hand side as follows. Writing $B = \prod_{k=0}^{m-1}(l_k, \infty)$, it holds

$$Q_{x, \beta}^{G}((\hat{w}(k))_{k \in [0, m]} \in A) = \int_B \left( \prod_{k=0}^{m-1} e^{-\beta x_k l(k)} \frac{C_{x_k x_{k+1}}}{2} e^{u_0 x_{k+1} - u_0 x_k} \right) \beta_{x_m} e^{-\beta x_m l(m)} \prod_{k=0}^{m} dl(k)$$

$$= e^{-2 u_{0, x}} \cdot \beta_{x'} e^{u_{0, x'} + u_{0, x}} \int_B \left( \prod_{k=0}^{m-1} \frac{C_{x_k x_{k+1}}}{2} \right) \prod_{k=0}^{m} e^{-\beta x_k l(k)} dl(k). \quad (A.4)$$

Indeed, $e^{-\beta x_k l(k)}$ for $k \in [0, m]$ is the probability to remain at $x_k$ at least a time span of length $l(k)$ after arrival at $x_k$. Moreover, $e^{-\beta x_k l(k)} e^{u_0 x_{k+1} - u_0 x_k} dl(k)$ for $k \in [0, m-1]$ denotes the probability to jump from $x_k$ to $x_{k+1}$ in an infinitesimal time span of length $dl(k)$ given that the particle is at $x_k$. Similarly, $\beta_{x_m} dl(m) = \sum_{x \in V} e^{u_{0, x} - u_{0, x_m}} dl(m)$ equals the probability to jump from $x_m$ to another site in an infinitesimal time span of length $dl(m)$ given that the particle is at $x_m$. Using the same argument and the set $B^{=\infty} = \prod_{k=0}^{m} (l_{m-k}, \infty)$, we obtain

$$Q_{x, \beta}^{G}((\hat{w}(m - k))_{k \in [0, m]} \in A)$$

$$= e^{-2 u_{0, x'}} \cdot \beta_{x'} e^{u_{0, x'} + u_{0, x}} \int_B \left( \prod_{k=0}^{m-1} \frac{C_{x_{m-k} x_{m-k-1}}}{2} \right) \prod_{k=0}^{m} e^{-\beta x_m l(k)} dl(k)$$

$$= e^{-2 u_{0, x'}} \cdot \beta_{x'} e^{u_{0, x'} + u_{0, x}} \int_B \left( \prod_{k=0}^{m-1} \frac{C_{x_k x_{k+1}}}{2} \right) \prod_{k=0}^{m} e^{-\beta x_k l(k)} dl(k). \quad (A.5)$$

Comparing this with (A.4) finishes the proof of the reversibility claim (A.2), which enables us to conclude.

Let $\beta \in B'$ and let $K \subseteq V$ be finite with $o \in K$. Recall the definitions (1.9) of $\hat{Q}_{K, \beta}$, (3.1) of $H_K$, and (4.5) of $e_{K, \beta}$. The total mass of $\hat{Q}_{K, \beta}$ equals

$$\hat{Q}_{K, \beta}(\hat{w}) = \sum_{x \in K} \beta_{x} e^{2 u_{0, x}} Q_{x, \beta}^{G}(A_K) = \sum_{x \in K} \beta_{x} e_{K, \beta}(x). \quad (A.6)$$

**Lemma A.2 (Consistency)** Let $\beta \in B'$ and let $\emptyset \neq K \subseteq K' \subseteq V$ be non-empty finite subsets of $V$. For $x \in K$, $x' \in K'$, $l, l' \geq 0$, $m \in \mathbb{N}_0$, $B_1 \in \mathcal{W}([0, m])$, $B_2 \in \mathcal{W}([1, m - 1])$, and $B_3 \in \mathcal{W}([0, \infty))$, one has

$$\hat{Q}_{K, \beta}((\hat{w}(-n))_{n \in \mathbb{N}} \in B_1, w(0) = x', l(0) \geq l', H_K(\hat{w}|_{[0, m]}) = m, \hat{w}|_{[1, m-1]} \in B_2, w(m) = x, l(m) \geq l, (\hat{w}(n + m))_{n \in \mathbb{N}} \in B_3)$$

$$= \hat{Q}_{K, \beta}((\hat{w}(-n - m))_{n \in \mathbb{N}} \in A_{K'}, (\hat{w}(-n - m))_{n \in \mathbb{N}} \in B_1, w(-m) = x', l(-m) \geq l', (\hat{w}(m - n))_{m \in [1, m-1]} \in B_2, w(0) = x, l(0) \geq l, \hat{w}|_{[0, m]} \in B_3). \quad (A.7)$$

We remark that the case $m \in \{0, 1\}$, where $[1, m - 1] = \emptyset$, is included. Note that we need not include the event $\{H_K((\hat{w}(n - m))_{n \in \mathbb{N}}) = m\}$ on the right-hand side in (A.7) because this event holds $\hat{Q}_{K, \beta}$-almost everywhere according to the specification of $\hat{Q}_{K, \beta}$ described in (1.9).

**Proof of Lemma A.2** We consider the case $m = 0$ first. If $x \neq x'$, then both sides of (A.7) vanish, being measures of the empty set. Assume $x = x'$. Given $m = 0$, $\hat{w}|_{[1, m-1]}$ is the empty path. Here, there are only two possibilities:

Case 1: $\{\hat{w}|_{[1, m-1]} \in B_2\}$ could be the impossible event; this case is trivial.

Case 2: Otherwise, $\{\hat{w}|_{[1, m-1]} \in B_2\}$ is the sure event. In this case, using that $\{w(0) = x\} \subseteq \{H_K(\hat{w}|_{[0, m]}) = 0\}$ in the first equality and $\{w(0) = x, \hat{w}|_{[0, m]} \in A_{K'}\} \subseteq \{\hat{w}|_{[0, m]} \in A_K\}$ in the third equality,
we obtain
\begin{align}
\text{r.h.s.} & \quad (A.7) \\
= & \mathcal{Q}_{K'}^{\beta}[\hat{w}(n)]_{n \in \mathbb{N}} \in B_1, w(0) = x, l(0) \geq \max\{\ell, \ell'\}, \hat{w}|_{n \in \mathbb{N}} \in B_3 \\
= & \beta_x e^{\max(\ell, \ell')\beta_x} e^{2u_{n,x}Q_{x,\beta}^G}[\hat{w}|_{n \in \mathbb{N}} \in B_1, \hat{w}|_{n \in \mathbb{N}} \in A_{K'}] Q_{x,\beta}^G[\hat{w}|_{n \in \mathbb{N}} \in B_3 \\
= & \beta_x e^{\max(\ell, \ell')\beta_x} e^{2u_{n,x}Q_{x,\beta}^G}[\hat{w}|_{n \in \mathbb{N}} \in B_1, \hat{w}|_{n \in \mathbb{N}} \in A_{K'}, \hat{w}|_{n \in \mathbb{N}} \in A_K] Q_{x,\beta}^G[\hat{w}|_{n \in \mathbb{N}} \in B_3 \\
= & \mathcal{Q}_{K,\beta}[(\hat{w}(n))_{n \in \mathbb{N}}]_{n \in \mathcal{A}_K}, (\hat{w}(n))_{n \in \mathbb{N}} \in B_1, w(0) = x, l(0) \geq \max\{\ell, \ell'\}, \hat{w}|_{n \in \mathbb{N}} \in B_3 \\
= & \text{l.h.s.} (A.7) \\
\end{align}
(A.8)

Note that according to the specification \([1.9]\) of the measure \(\mathcal{Q}_{K'}^{\beta}\), on the second line of this formula, the event \(B_1\) is considered for the time reversed path, while on the third and fourth line the path is not taken reversed, neither for the event \(B_1\) nor for the event \(A_{K'}\).

Next, we treat the remaining case \(m \geq 1\). Assume \(x' \notin K\). Then, we have \(\{w(0) = x', H_K(\hat{w}|_{n \in \mathbb{N}}) = m\} = \emptyset\), which implies l.h.s. (A.7) = 0. Furthermore, the inclusion \(\{w(-m) = x'\} \subseteq \{\hat{w}(n)\}_{n \in n_0} \notin A_K\) holds. Together with the definition of \(\mathcal{Q}_{K,\beta}\) this implies r.h.s. (A.7) = 0, which proves (A.7) in the case \(m \geq 1, x' \notin K\).

Finally, assume \(x' \in K' \setminus K\). Using the definition of \(\mathcal{Q}_{K',\beta}\), we obtain
\begin{align}
\text{l.h.s.} & \quad (A.7) \\
= & \beta_x e^{-\ell'x'} e^{2u_{n,x'}Q_{x',\beta}^G}[\hat{w}|_{n \in \mathbb{N}} \in B_1, \hat{w}|_{n \in \mathbb{N}} \in A_{K'}] Q_{x',\beta}^G[\hat{w}|_{n \in \mathbb{N}} \in B_3] \\
= & \beta_x e^{-\ell'x'} e^{2u_{n,x'}Q_{x',\beta}^G}[H_K(\hat{w}|_{n \in \mathbb{N}}) = m, \hat{w}|_{[1,m-1]} \in B_2, w(m) = x, l(m) \geq \ell, (\hat{w}(m+n))_{n \in \mathbb{N}} \in B_3]. \\
\end{align}
(A.9)

Note that given \(x' \notin K\), up to modification on the \(Q_{x',\beta}\)-null set \(\{w(0) \neq x'\}\), the event \(H_K(\hat{w}|_{n \in \mathbb{N}}) = m\) is measurable with respect to \(\sigma(\hat{w}|_{n})\) and hence enters only in the last factor on the right-hand side in (A.9). We apply the Markov property at time \(m\) to the last probability in (A.9):
\begin{align}
\text{last factor in } (A.9) \\
= & \mathcal{Q}_{x',\beta}^G[H_K(\hat{w}|_{n \in \mathbb{N}}) = m, \hat{w}|_{[1,m-1]} \in B_2, w(m) = x] e^{-\ell'x'} Q_{x',\beta}^G[\hat{w}|_{n \in \mathbb{N}} \in B_3]. \\
\text{An application of the reversibility formula } (A.2) \text{ in the case } \\
\{((\hat{w}(k))_{k \in [0,m]} \in A\} = \{w(0) = x', H_K(\hat{w}|_{n \in \mathbb{N}}) = m, \hat{w}|_{[1,m-1]} \in B_2, w(m) = x\} \\
= & \{w(0) = x', w(k) \notin K \text{ for } k \in [1,m-1], (\hat{w}(n))_{n \in [1,m-1]} \in B_2, w(m) = x\} \\
\text{yields} \\
Q_{x',\beta}^G[H_K(\hat{w}|_{n \in \mathbb{N}}) = m, \hat{w}|_{[1,m-1]} \in B_2, w(m) = x] \\
= & \beta_x e^{2u_{n,x} - u_{n,x'}} Q_{x',\beta}^G[w(k) \notin K \text{ for } k \in [1,m-1], (\hat{w}(m-n))_{n \in [1,m-1]} \in B_2, w(m) = x'].
\end{align}
(A.10)

We insert this in (A.10) and then the result in (A.9). Afterwards, we use the Markov property again. This yields
\begin{align}
\text{l.h.s.} & \quad (A.7) \\
= & \beta_x e^{-\ell'x'} e^{2u_{n,x'}Q_{x',\beta}^G} [\hat{w}|_{n \in \mathbb{N}} \in B_1, \hat{w}|_{n \in \mathbb{N}} \in A_{K'}] \\
= & \beta_x e^{-\ell'x'} e^{2u_{n,x}Q_{x,\beta}^G} [\hat{w}|_{n \in \mathbb{N}} \in B_1, \hat{w}|_{n \in \mathbb{N}} \in A_{K'}] Q_{x,\beta}^G[\hat{w}|_{n \in \mathbb{N}} \in B_3] \\
= & \beta_x e^{-\ell'x'} e^{2u_{n,x}Q_{x,\beta}^G} [\hat{w}|_{n \in \mathbb{N}} \in B_1, \hat{w}|_{n \in \mathbb{N}} \in A_{K'}] Q_{x,\beta}^G[\hat{w}|_{n \in \mathbb{N}} \in B_3] \\
= & \beta_x e^{-\ell'x'} e^{2u_{n,x}Q_{x,\beta}^G} [\hat{w}|_{n \in \mathbb{N}} \in B_1, \hat{w}|_{n \in \mathbb{N}} \in A_{K'}, \hat{w}|_{n \in \mathbb{N}} \in A_K] Q_{x,\beta}^G[\hat{w}|_{n \in \mathbb{N}} \in B_3] \\
= & \beta_x e^{-\ell'x'} e^{2u_{n,x}Q_{x,\beta}^G} [\hat{w}|_{n \in \mathbb{N}} \in B_1, \hat{w}|_{n \in \mathbb{N}} \in A_{K'}, \hat{w}|_{n \in \mathbb{N}} \in A_K] Q_{x,\beta}^G[\hat{w}|_{n \in \mathbb{N}} \in B_3] \\
= & \text{r.h.s.} (A.7) \\
\end{align}
(A.13)
In the last equality, we have used that \( x' \in K' \setminus K \) and \( x \in K \subseteq K' \) imply
\[
\{ w(-k) \notin K \text{ for all } k \in [1, m - 1], w(-m) = x', (\hat{w}(-n - m))_{n \in \mathbb{N}_0} \in A_{K'}, w(0) = x \}
= \{(\hat{w}(-n - m))_{n \in \mathbb{N}_0} \in A_{K'}, (\hat{w}(-n))_{n \in \mathbb{N}_0} \in A_{K}, w(-m) = x', w(0) = x \}.
\] (A.14)

We conclude that the claim (A.7) holds in all cases. ■

For \( K \subseteq V \) finite, defining
\[
\hat{W}_K = \{(w, l) \in \hat{W} : w(0) \in K, w(k) \notin K \text{ for } k < 0 \},
\] (A.15)
the definition of the event \( \hat{W}_K \) given in (1.11) can be rewritten as \( \hat{W}_K^* = \pi^*[\hat{W}_K] \). Clearly, \( \hat{Q}_{K, \beta} \) is supported on \( \hat{W}_K \) for finite subsets \( K \subseteq K' \) of \( V \), let
\[
\hat{W}_{K',K} = \{(w, l) \in \hat{W}_{K'} : w(m) \in K \text{ for some } m \in \mathbb{Z} \}.
\] (A.16)

Consider the time shift \( \theta_{K',K} : \hat{W}_K \to \hat{W} \) uniquely characterized by range(\( \theta_{K',K} \)) \( \subseteq W_{K',K} \) and \( \pi^*(\theta_{K',K}(\hat{w})) = \pi^*(\hat{w}) \) for \( \hat{w} \in \hat{W}_K \). Thus the map \( \theta_{K',K} \) does nothing but a shift of any \( \hat{w} \) such that its image \( \theta_{K',K}(\hat{w}) \) visits \( K' \) for the first time at time 0. Lemma A.2 may be rephrased in the following form:

**Lemma A.3** For all \( \beta \in B' \) and all non-empty finite \( K \subseteq K' \subseteq V \), one has
\[
\theta_{K',K}[\hat{Q}_{K,\beta}] = 1_{\hat{W}_{K',K}} \hat{Q}_{K',\beta}.
\] (A.17)

As a consequence, we obtain
\[
\pi^*[\hat{Q}_{K,\beta}] = 1_{\hat{W}_K} \pi^*[\hat{Q}_{K',\beta}] \leq \pi^*[\hat{Q}_{K',\beta}].
\] (A.18)

**Proof.** Because \( \hat{Q}_{K,\beta} \) is supported on the domain \( \hat{W}_K \) of the shift \( \theta_{K',K} \), the image measure \( \theta_{K',K}[\hat{Q}_{K,\beta}] \) is indeed well-defined. We consider the \( \sigma \)-fields
\[
\hat{W}_K := \{ A \in \hat{W} : A \subseteq \hat{W}_K \}, \quad \hat{W}_{K',K} := \{ A \in \hat{W} : A \subseteq \hat{W}_{K',K} \}.
\] (A.19)

With parameters as in Lemma A.2, the set \( B_{K',K} \) of events of the form
\[
D = \{(\hat{w}(-n))_{n \in \mathbb{N}_0} \in A_{K'}, (\hat{w}(-n))_{n \in \mathbb{N}_0} \in B_1, w(0) = x', l(0) \geq \ell', H_K(\hat{w}|_{\mathbb{N}_0}) = m, \\
\hat{w}|_{[1, m - 1]} \in B_2, w(m) = x, l(m) \geq \ell, (\hat{w}(n + m))_{n \in \mathbb{N}_0} \in B_3 \}
\] (A.20)
is a generator of \( \hat{W}_{K',K} \), which is stable under intersections. Furthermore, the space \( \hat{W}_{K',K} \) is a countable union of events of this form. Therefore it suffices to prove the claim (A.17) restricted to \( B_{K',K} \).

Note that l.h.s.
\[
\hat{Q}_{K',\beta}(D); \text{ the condition } (\hat{w}(-n))_{n \in \mathbb{N}_0} \in A_{K'} \text{ comes from the definition (1.9) of the measure } \hat{Q}_{K',\beta}.
\]
Since
\[
\theta_{K',K}^{-1}[D] = \{(\hat{w}(-n))_{n \in \mathbb{N}_0} \in A_{K'}, (\hat{w}(-n - m))_{n \in \mathbb{N}_0} \in A_{K'}, \\
(\hat{w}(-n - m))_{n \in \mathbb{N}_0} \in B_1, w(-m) = x', l(-m) \geq \ell', \\
(\hat{w}(n - m))_{n \in [1, m - 1]} \in B_2, w(0) = x, l(0) \geq \ell, \hat{w}|_n \in B_3 \},
\] (A.21)
Lemma A.2 shows that indeed the claim (A.17) holds restricted to \( B_{K',K} \).

Using that the measure \( \hat{Q}_{K',\beta} \) is supported on \( \hat{W}_K \) and \( \hat{W}_K \cap (\pi^*)^{-1}[\hat{W}_K^*] = \hat{W}_{K',K} \), we infer
\[
1_{\hat{W}_K} \pi^*[\hat{Q}_{K',\beta}] = \pi^*[1_{\hat{W}_{K',K}} \hat{Q}_{K',\beta}]\].
Using formula (A.17), the facts that \( \pi^* \circ \theta_{K',K} = \pi^* \) holds on \( W_K \) and that the measure \( \hat{Q}_{K,\beta} \) is supported on \( \hat{W}_K \), we conclude \( \pi^*[1_{\hat{W}_{K',K}} \hat{Q}_{K',\beta}] = \pi^*[\hat{Q}_{K,\beta}] \). This proves the equality in the second claim (A.18). The inequality in (A.18) is clear from \( 1_{\hat{W}_K} \leq 1 \). ■
Proof of Theorem 1.3. Take any increasing sequence of finite sets $K_n \uparrow V$ as $n \to \infty$. From (A.18) we know that $\pi^*\hat{Q}_{K, \beta}(A)$ is monotonic in the set argument $K$. We conclude

$$\hat{\nu}_\beta(A) := \sup_{K \subset V \text{ finite}} \pi^*\hat{Q}_{K, \beta}(A) = \lim_{n \to \infty} \pi^*\hat{Q}_{K_n, \beta}(A).$$ (A.22)

By monotone convergence, this is $\sigma$-additive in $A$. Hence $\hat{\nu}_\beta$ is a measure. The equation (1.12) is an immediate consequence of (A.18). Uniqueness follows from the fact

$$\hat{W}^* = \bigcup_{n \in \mathbb{N}} \hat{W}_{K_n}^*.$$ (A.23)

Because all measures $\pi^*\hat{Q}_{K, \beta}$ are finite, the measure $\hat{\nu}_\beta$ is $\sigma$-finite.

The equality in the claim (1.14) is an immediate consequence of the restriction property (1.12). The finiteness of $\pi^*\hat{Q}_{K, \beta}(W_K^*)$ follows from the definition (1.9) of $\hat{Q}_{K, \beta}$. Finally, given a finite set $K$ with $\emptyset \neq K \subset V$ and $y \in K$, using transience, we take $x \in K$ such that with positive probability the Markov jump process with law $Q_{y, \beta}$ visits $K$ for the last time in $x$. In particular, $Q_{y, \beta}^x(A_K) > 0$. In view of the definition of $\hat{Q}_{K, \beta}$, this implies the remaining claim $\pi^*\hat{Q}_{K, \beta}(W_K^*) > 0$. ■

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