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Editorial: Special Issue on "Nonparametric Inference Under Shape Constraints"

Richard J. Samworth and Bodhisattva Sen

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Some Developments in the Theory of Shape Constrained Inference

Piet Groeneboom and Geurt Jongbloed

Abstract. Shape constraints enter in many statistical models. Sometimes these constraints emerge naturally from the origin of the data. In other situations, they are used to replace parametric models by more versatile models retaining qualitative shape properties of the parametric model. In this paper, we sketch a part of the history of shape constrained statistical inference in a nutshell, using landmark results obtained in this area. For this, we mainly use the prototypical problems of estimating a decreasing probability density on $[0, \infty)$ and the estimation of a distribution function based on current status data as illustrations.

Key words and phrases: Isotonic regression, Grenander estimator, inverse problem, monotonicity, interval censoring, current status regression, single index model, bootstrap, Chernoff's distribution, Airy functions.

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Recent Progress in Log-Concave Density Estimation

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Abstract. In recent years, log-concave density estimation via maximum likelihood estimation has emerged as a fascinating alternative to traditional nonparametric smoothing techniques, such as kernel density estimation, which require the choice of one or more bandwidths. The purpose of this article is to describe some of the properties of the class of log-concave densities on $\mathbb{R}^d$ which make it so attractive from a statistical perspective, and to outline the latest methodological, theoretical and computational advances in the area.

Key words and phrases: Log-concavity, maximum likelihood estimation.

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Shape Constrained Density Estimation Via Penalized Rényi Divergence

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Abstract. Shape constraints play an increasingly prominent role in nonparametric function estimation. While considerable recent attention has been focused on log concavity as a regularizing device in nonparametric density estimation, weaker forms of concavity constraints encompassing larger classes of densities have received less attention but offer some additional flexibility. Heavier tail behavior and sharper modal peaks are better adapted to such weaker concavity constraints. When paired with appropriate maximal entropy estimation criteria, these weaker constraints yield tractable, convex optimization problems that broaden the scope of shape constrained density estimation in a variety of applied subject areas.

In contrast to our prior work, Koenker and Mizera [*Ann. Statist.* **38** (2010) 2998–3027], that focused on the log concave ($\alpha = 1$) and Hellinger ($\alpha = 1/2$) constraints, here we describe methods enabling imposition of even weaker, $\alpha \leq 0$ constraints. An alternative formulation of the concavity constraints for densities in dimension $d \geq 2$ also significantly expands the applicability of our proposed methods for multivariate data. Finally, we illustrate the use of the Rényi divergence criterion for norm-constrained estimation of densities in the absence of a shape constraint.

Key words and phrases: Density estimation, shape constraints, Rényi entropy, convex optimization.

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Shape Constraints in Economics and Operations Research

Andrew L. Johnson and Daniel R. Jiang

Abstract. Shape constraints, motivated by either application-specific assumptions or existing theory, can be imposed during model estimation to restrict the feasible region of the parameters. Although such restrictions may not provide any benefits in an asymptotic analysis, they often improve finite sample performance of statistical estimators and the computational efficiency of finding near-optimal control policies. This paper briefly reviews an illustrative set of research utilizing shape constraints in the economics and operations research literature. We highlight the methodological innovations and applications, with a particular emphasis on utility functions, production economics and sequential decision making applications.

Key words and phrases: Shape constraints, multivariate convex regression, nonparametric regression, production economics, consumer preferences, revealed preferences, approximate dynamic programming, reinforcement learning.

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Limit Theory in Monotone Function Estimation

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Abstract. We give an overview of the different concepts and methods that are commonly used when studying the asymptotic properties of isotonic estimators. After introducing the inverse process, we illustrate its use in establishing weak convergence of the estimators at a fixed point and also weak convergence of global distances, such as the $\mathbb{L}_p$-distance and supremum distance. Furthermore, we discuss the developments on smooth isotonic estimation.

Key words and phrases: Cox model, current status model, isotonic estimation, limit theory, $\mathbb{L}_p$-distance, maximum likelihood estimators, monotone density, monotone failure rate, monotone regression, supremum distance.

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Nonparametric Shape-Restricted Regression

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Abstract. We consider the problem of nonparametric regression under shape constraints. The main examples include isotonic regression (with respect to any partial order), unimodal/convex regression, additive shape-restricted regression and constrained single index model. We review some of the theoretical properties of the least squares estimator (LSE) in these problems, emphasizing on the adaptive nature of the LSE. In particular, we study the behavior of the risk of the LSE, and its pointwise limiting distribution theory, with special emphasis to isotonic regression. We survey various methods for constructing pointwise confidence intervals around these shape-restricted functions. We also briefly discuss the computation of the LSE and indicate some open research problems and future directions.

Key words and phrases: Adaptive risk bounds, bootstrap, Chernoff's distribution, convex regression, isotonic regression, likelihood ratio test, monotone function, order preserving function estimation, projection on a closed convex set, tangent cone.

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A Framework for Estimation and Inference in Generalized Additive Models with Shape and Order Restrictions

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Abstract. Methodology for the partial linear generalized additive model is presented, where components for continuous predictors may be modeled with shape-constrained regression splines, and components for ordinal predictors may have partial orderings. The estimated mean function is obtained through a projection (or iteratively reweighted projections) onto a polyhedral convex cone; this is key for formally derived inference procedures. Pointwise confidence bands and hypothesis tests for the individual components, as well as a model selection method, are proposed. These methods are available in the R package cgam.

Key words and phrases: Monotone, convex, partial linear, confidence interval.

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Methods for Estimation of Convex Sets

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Abstract. In the framework of shape constrained estimation, we review methods and works done in convex set estimation. These methods mostly build on stochastic and convex geometry, empirical process theory, functional analysis, linear programming, extreme value theory, etc. The statistical problems that we review include density support estimation, estimation of the level sets of densities or depth functions, nonparametric regression, etc. We focus on the estimation of convex sets under the Nikodym and Hausdorff metrics, which require different techniques and, quite surprisingly, lead to very different results, in particular in density support estimation. Finally, we discuss computational issues in high dimensions.

Key words and phrases: Convex body, set estimation, Nikodym metric, Hausdorff metric, support function.

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A Conversation with Jon Wellner

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Abstract. Jon August Wellner was born in Portland, Oregon, in August 1945. He received his Bachelor's degree from the University of Idaho in 1968 and his PhD degree from the University of Washington in 1975. From 1975 until 1983, he was an Assistant Professor and Associate Professor at the University of Rochester. In 1983, he returned to the University of Washington, and has remained at the UW as a faculty member since that time. Over the course of a long and distinguished career, Jon has made seminal contributions to a variety of areas including empirical processes, semiparametric theory and shape-constrained inference, and has co-authored a number of extremely influential books. He has been honored as the Le Cam lecturer by both the IMS (2015) and the French Statistical Society (2017). He is a Fellow of the IMS, the ASA and the AAAS, and an elected member of the International Statistical Institute. He has served as co-Editor of *The Annals of Statistics* (2001–2003) and Editor of *Statistical Science* (2010–2013), and President of IMS (2016–2017). In 2010, he was made a Knight of the Order of the Netherlands Lion. In his free time, Jon enjoys mountain climbing and backcountry skiing in the Cascades and British Columbia.

Key words and phrases: Conversation, empirical processes, semiparametric theory, shape-constrained inference, University of Washington.

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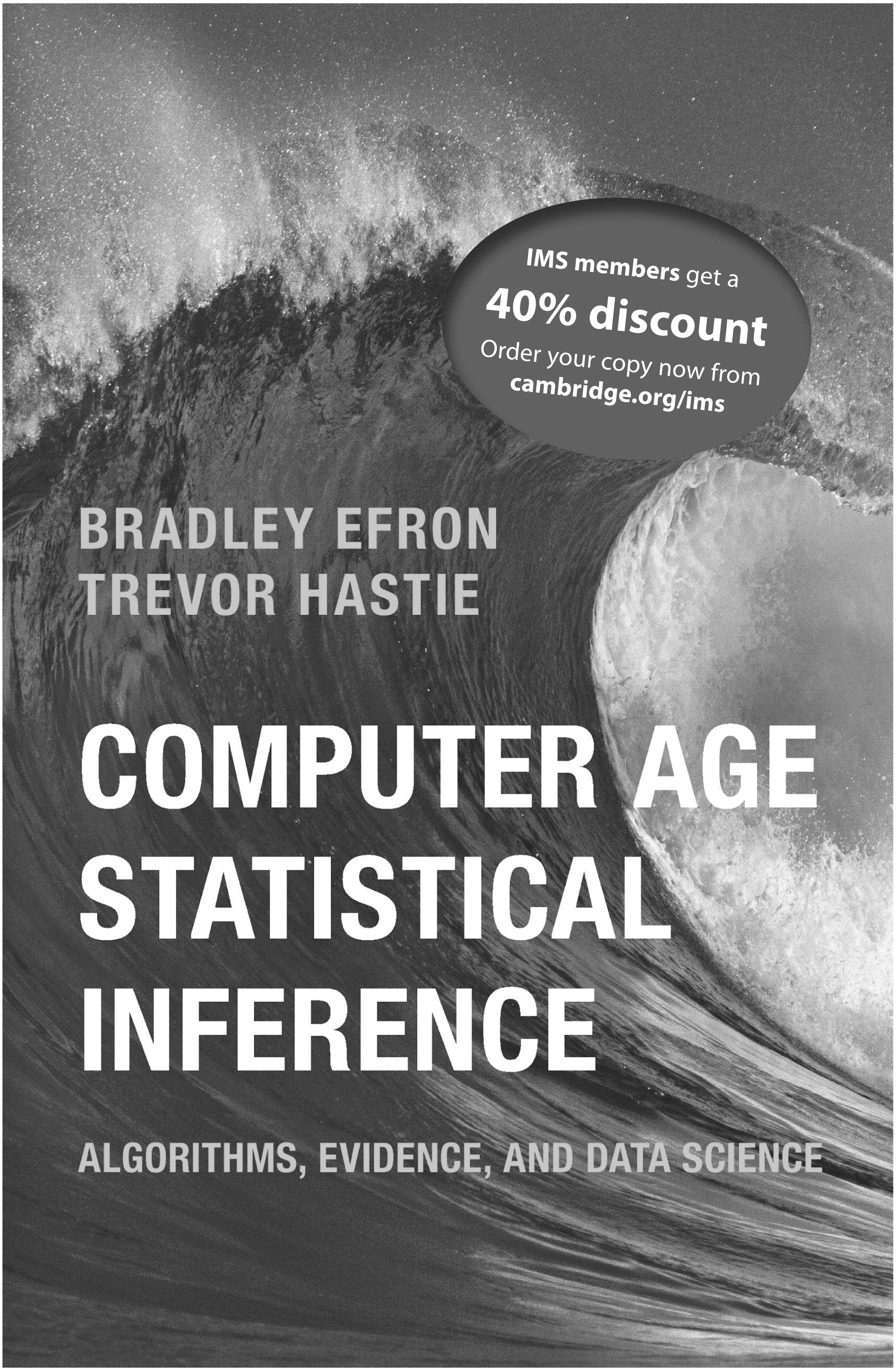
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