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CONTENTS

(continued)

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CONTENTS

(continued)

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Aims and Scope

BERNOULLI is the journal of the Bernoulli Society for Mathematical Statistics and Probability, issued four times per year. The journal provides a comprehensive account of important developments in the fields of statistics and probability, offering an international forum for both theoretical and applied work.

Bernoulli Society for Mathematical Statistics and Probability

The Bernoulli Society was founded in 1973. It is an autonomous Association of the International Statistical Institute, ISI. According to its statutes, the object of the Bernoulli Society is the advancement, through international contacts, of the sciences of probability (including the theory of stochastic processes) and mathematical statistics and of their applications to all those aspects of human endeavour which are directed towards the increase of natural knowledge and the welfare of mankind.

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Adaptive risk bounds in unimodal regression

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We study the statistical properties of the least squares estimator in unimodal sequence estimation. Although closely related to isotonic regression, unimodal regression has not been as extensively studied. We show that the unimodal least squares estimator is adaptive in the sense that the risk scales as a function of the number of values in the true underlying sequence. Such adaptivity properties have been shown for isotonic regression by Chatterjee et al. (*Ann. Statist.* **43** (2015) 1774–1800) and Bellec (Sharp oracle inequalities for Least Squares estimators in shape restricted regression (2016)). A technical complication in unimodal regression is the non-convexity of the underlying parameter space. We develop a general variational representation of the risk that holds whenever the parameter space can be expressed as a finite union of convex sets, using techniques that may be of interest in other settings.

Keywords: isotonic regression; minimax bounds; shape constrained inference; unimodal regression

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1350-7265

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Leading the field: Fortune favors the bold in Thurstonian choice models

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Schools with the highest average student performance are often the smallest schools; localities with the highest rates of some cancers are frequently small; and the effects observed in clinical trials are likely to be largest for the smallest numbers of subjects. Informal explanations of this "small-schools phenomenon" point to the fact that the sample means of smaller samples have higher variances. But this cannot be a complete explanation: If we draw two samples from a diffuse distribution that is symmetric about some point, then the chance that the smaller sample has larger mean is 50%. A particular consequence of results proved below is that if one draws three or more samples of different sizes from the same normal distribution, then the sample mean of the smallest sample is most likely to be highest, the sample mean of the second smallest sample is second most likely to be highest, and so on; this is true even though for any pair of samples, each one of the pair is equally likely to have the larger sample mean. The same effect explains why heteroscedasticity can result in misleadingly small nominal p-values in nonparametric tests of association.

Our conclusions are relevant to certain stochastic choice models, including the following generalization of Thurstone's Law of Comparative Judgment. There are n items. Item i is preferred to item j if $Z_i < Z_j$, where Z is a random n-vector of preference scores. Suppose $\mathbb{P}\{Z_i = Z_j\} = 0$ for $i \neq j$, so there are no ties. Item k is the favorite if $Z_k < \min_{i \neq k} Z_i$. Let p_i denote the chance that item i is the favorite. We characterize a large class of distributions for Z for which $p_1 > p_2 > \cdots > p_n$. Our results are most surprising when $\mathbb{P}\{Z_i < Z_j\} = \mathbb{P}\{Z_i > Z_j\} = \frac{1}{2}$ for $i \neq j$, so neither of any two items is likely to be preferred over the other in a pairwise comparison. Then, under suitable assumptions, $p_1 > p_2 > \cdots > p_n$ when the variability of Z_i decreases with i in an appropriate sense. Our conclusions echo the proverb "Fortune favors the bold."

Keywords: coupling; discrete choice models; extreme value; maximum (or minimum) of random variables; most dangerous equation; order statistic; preference scores; small schools phenomenon; stochastic domination; test of association; Thurstone; winning probability

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1350-7265

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Self-consistent confidence sets and tests of composite hypotheses applicable to restricted parameters

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Frequentist methods, without the coherence guarantees of fully Bayesian methods, are known to yield self-contradictory inferences in certain settings. The framework introduced in this paper provides a simple adjustment to p values and confidence sets to ensure the mutual consistency of all inferences without sacrificing frequentist validity. Based on a definition of the compatibility of a composite hypothesis with the observed data given any parameter restriction and on the requirement of self-consistency, the adjustment leads to the possibility and necessity measures of possibility theory rather than to the posterior probability distributions of Bayesian and fiducial inference.

Keywords: bounded parameter; deductive closure; deductive cogency; empty confidence set; possibility theory; p-value function; ranking function; ranking theory; restricted parameter space; surprise measure

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Rigid stationary determinantal processes in non-Archimedean fields

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Let F be a non-discrete non-Archimedean local field. For any subset $S \subset F$ with finite Haar measure, there is a stationary determinantal point process on F with correlation kernel $\widehat{\mathbb{1}}_S(x-y)$, where $\widehat{\mathbb{1}}_S$ is the Fourier transform of the indicator function $\mathbb{1}_S$. In this note, we give a geometrical condition on the subset S, such that the associated determinantal point process is rigid in the sense of Ghosh and Peres. Our geometrical condition is very different from the Euclidean case.

Keywords: non-Archimedean local field; rigidity; stationary determinantal point processes

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Stein's method and approximating the quantum harmonic oscillator

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Hall *et al.* [*Phys. Rev. X* **4** (2014) 041013] recently proposed that quantum theory can be understood as the continuum limit of a deterministic theory in which there is a large, but finite, number of classical "worlds." A resulting Gaussian limit theorem for particle positions in the ground state, agreeing with quantum theory, was conjectured in Hall *et al.* [*Phys. Rev. X* **4** (2014) 041013] and proven by McKeague and Levin [*Ann. Appl. Probab.* **26** (2016) 2540–2555] using Stein's method. In this article we show how quantum position probability densities for higher energy levels beyond the ground state may arise as distributional fixed points in a new generalization of Stein's method. These are then used to obtain a rate of distributional convergence for conjectured particle positions in the first energy level above the ground state to the (two-sided) Maxwell distribution; new techniques must be developed for this setting where the usual "density approach" Stein solution (see Chatterjee and Shao [*Ann. Appl. Probab.* **21** (2011) 464–483] has a singularity.

Keywords: higher energy levels; interacting particle system; Maxwell distribution; Stein's method

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Verifiable conditions for the irreducibility and aperiodicity of Markov chains by analyzing underlying deterministic models

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We consider Markov chains that obey the following general non-linear state space model: $\Phi_{k+1} = F(\Phi_k, \alpha(\Phi_k, U_{k+1}))$ where the function F is C^1 while α is typically discontinuous and $\{U_k : k \in \mathbb{Z}_{>0}\}$ is an independent and identically distributed process. We assume that for all x, the random variable $\alpha(x, U_1)$ admits a density p_x such that $(x, w) \mapsto p_x(w)$ is lower semi-continuous.

We generalize and extend previous results that connect properties of the underlying deterministic control model to provide conditions for the chain to be φ-irreducible and aperiodic. By building on those results, we show that if a rank condition on the controllability matrix is satisfied for all x, there is equivalence between the existence of a globally attracting state for the control model and φ-irreducibility of the Markov chain. Additionally, under the same rank condition on the controllability matrix, we prove that there is equivalence between the existence of a steadily attracting state and the φ-irreducibility and aperiodicity of the chain. The notion of steadily attracting state is new. We additionally derive practical conditions by showing that the rank condition on the controllability matrix needs to be verified only at a globally attracting state (resp. steadily attracting state) for the chain to be a φ-irreducible T-chain (resp. φ-irreducible aperiodic T-chain).

Those results hold under considerably weaker assumptions on the model than previous ones that would require $(x, u) \mapsto F(x, \alpha(x, u))$ to be C^∞ (while it can be discontinuous here). Additionally the establishment of a *necessary and sufficient* condition on the control model for the φ-irreducibility and aperiodicity without a structural assumption on the control set is novel – even for Markov chains where $(x, u) \mapsto F(x, \alpha(x, u))$ is C^∞.

We illustrate that the conditions are easy to verify on a non-trivial and non-artificial example of Markov chain arising in the context of adaptive stochastic search algorithms to optimize continuous functions in a black-box scenario.

Keywords: aperiodicity; controllability matrix; deterministic control model; Evolution Strategies; globally attracting state; irreducibility; Markov chains; T-chain

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Recovering the Brownian coalescent point process from the Kingman coalescent by conditional sampling

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We consider a continuous population whose dynamics is described by the standard stationary Fleming–Viot process, so that the genealogy of n uniformly sampled individuals is distributed as the Kingman n-coalescent. In this note, we study some genealogical properties of this population when the sample is conditioned to fall entirely into a subpopulation with most recent common ancestor (MRCA) shorter than ε. First, using the comb representation of the total genealogy (Lambert and Uribe Bravo (*P-Adic Numbers Ultrametric Anal. Appl.* **9** (2017) 22–38)), we show that the genealogy of the descendance of the MRCA of the sample on the timescale ε converges as $\varepsilon \to 0$. The limit is the so-called Brownian coalescent point process (CPP) stopped at an independent Gamma random variable with parameter n, which can be seen as the genealogy at a large time of the total population of a rescaled critical birth–death process, biased by the nth power of its size. Second, we show that in this limit the coalescence times of the n sampled individuals are i.i.d. uniform random variables in $(0, 1)$. These results provide a coupling between two standard models for the genealogy of a random exchangeable population: the Kingman coalescent and the Brownian CPP.

Keywords: coalescent point process; conditional sampling; flows of bridges; Kingman coalescent; small time behavior

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Subexponential decay in kinetic Fokker–Planck equation: Weak hypocoercivity

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We consider here quantitative convergence to equilibrium for the kinetic Fokker–Planck equation. We present a weak hypocoercivity approach *à la* Villani, using weak Poincaré inequality, ensuring subexponential convergence to equilibrium in $\mathcal{H}^1$ sense or in L^2 sense.

Keywords: Fokker–Planck equation; hypocoercivity; weak Poincaré inequality

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Pólya urns with immigration at random times

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We study the number of white balls in a classical Pólya urn model with the additional feature that, at random times, a black ball is added to the urn. The number of draws between these random times are i.i.d. and, under certain moment conditions on the inter-arrival distribution, we characterize the limiting distribution of the (properly scaled) number of white balls as the number of draws goes to infinity. The possible limiting distributions obtained in this way vary considerably depending on the inter-arrival distribution and are difficult to describe explicitly. However, we show that the limits are fixed points of certain probabilistic distributional transformations, and this fact provides a proof of convergence and leads to properties of the limits. The model can alternatively be viewed as a preferential attachment random graph model where added vertices initially have a random number of edges, and from this perspective, our results describe the limit of the degree of a fixed vertex.

Keywords: distributional convergence; distributional fixed point equation; Pólya urns; preferential attachment random graph

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Feller property of the multiplicative coalescent with linear deletion

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We modify the definition of Aldous' multiplicative coalescent process (*Ann. Probab.* **25** (1997) 812–854) and introduce the multiplicative coalescent with linear deletion (MCLD). A state of this process is a square-summable decreasing sequence of cluster sizes. Pairs of clusters merge with a rate equal to the product of their sizes and clusters are deleted with a rate linearly proportional to their size. We prove that the MCLD is a Feller process. This result is a key ingredient in the description of scaling limits of the evolution of component sizes of the mean field frozen percolation model (*J. Stat. Phys.* **137** (2009) 459–499) and the so-called rigid representation of such scaling limits (*Electron. J. Probab.* To appear).

Keywords: Feller process; multiplicative coalescent

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Asymptotic power of Rao's score test for independence in high dimensions

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Let $\mathbf{R}$ be the Pearson correlation matrix of m normal random variables. The Rao's score test for the independence hypothesis $H_0 : \mathbf{R} = \mathbf{I}_m$, where $\mathbf{I}_m$ is the identity matrix of dimension m, was first considered by Schott (*Biometrika* **92** (2005) 951–956) in the high dimensional setting. In this paper, we study the exact power function of this test, under an asymptotic regime in which both m and the sample size n tend to infinity with the ratio m/n upper bounded by a constant. In particular, our result implies that the Rao's score test is minimax rate-optimal for detecting the dependency signal $\|\mathbf{R} - \mathbf{I}_m\|_F$ of order $\sqrt{m/n}$, where $\| \cdot \|_F$ is the matrix Frobenius norm.

Keywords: Frobenius norm; high dimensionality; minimax optimality; Pearson correlation; power; Rao's score

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Extreme M-quantiles as risk measures: From L^1 to L^p optimization

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The class of quantiles lies at the heart of extreme-value theory and is one of the basic tools in risk management. The alternative family of expectiles is based on squared rather than absolute error loss minimization. It has recently been receiving a lot of attention in actuarial science, econometrics and statistical finance. Both quantiles and expectiles can be embedded in a more general class of M-quantiles by means of L^p optimization. These generalized L^p-quantiles steer an advantageous middle course between ordinary quantiles and expectiles without sacrificing their virtues too much for $1 < p < 2$. In this paper, we investigate their estimation from the perspective of extreme values in the class of heavy-tailed distributions. We construct estimators of the intermediate L^p-quantiles and establish their asymptotic normality in a dependence framework motivated by financial and actuarial applications, before extrapolating these estimates to the very far tails. We also investigate the potential of extreme L^p-quantiles as a tool for estimating the usual quantiles and expectiles themselves. We show the usefulness of extreme L^p-quantiles and elaborate the choice of p through applications to some simulated and financial real data.

Keywords: asymptotic normality; dependent observations; expectiles; extrapolation; extreme values; heavy tails; L^p optimization; mixing; quantiles; tail risk

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Error bounds for sequential Monte Carlo samplers for multimodal distributions

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In this paper, we provide bounds on the asymptotic variance for a class of sequential Monte Carlo (SMC) samplers designed for approximating multimodal distributions. Such methods combine standard SMC methods and Markov chain Monte Carlo (MCMC) kernels. Our bounds improve upon previous results, and unlike some earlier work, they also apply in the case when the MCMC kernels can move between the modes. We apply our results to the Potts model from statistical physics. In this case, the problem of sharp peaks is encountered. Earlier methods, such as parallel tempering, are only able to sample from it at an exponential (in an important parameter of the model) cost. We propose a sequence of interpolating distributions called *interpolation to independence*, and show that the SMC sampler based on it is able to sample from this target distribution at a polynomial cost. We believe that our method is generally applicable to many other distributions as well.

Keywords: asymptotic variance bound; central limit theorem; metastability; Potts model; scale invariance; sequential Monte Carlo

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On the convex Poincaré inequality and weak transportation inequalities

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We prove that for a probability measure on $\mathbb{R}^n$, the Poincaré inequality for convex functions is equivalent to the weak transportation inequality with a quadratic-linear cost. This generalizes recent results by Gozlan, Roberto, Samson, Shu, Tetali and Feldheim, Marsiglietti, Nayar, Wang, concerning probability measures on the real line.

The proof relies on modified logarithmic Sobolev inequalities of Bobkov–Ledoux type for convex and concave functions, which are of independent interest.

We also present refined concentration inequalities for general (not necessarily Lipschitz) convex functions, complementing recent results by Bobkov, Nayar, and Tetali.

Keywords: concentration of measure; convex functions; Poincaré inequality; weak transport-entropy inequalities

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On the longest gap between power-rate arrivals

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Let L_t be the longest gap before time t in an inhomogeneous Poisson process with rate function λ_t proportional to $t^{\alpha-1}$ for some $\alpha \in (0, 1)$. It is shown that $\lambda_t L_t - b_t$ has a limiting Gumbel distribution for suitable constants b_t and that the distance of this longest gap from t is asymptotically of the form $(t/\log t)E$ for an exponential random variable E. The analysis is performed via weak convergence of related point processes. Subject to a weak technical condition, the results are extended to include a slowly varying term in λ_t.

Keywords: Gumbel distribution; inhomogeneous Poisson process; point processes; records; regular variation; weak convergence

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Nonparametric depth and quantile regression for functional data

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We investigate nonparametric regression methods based on spatial depth and quantiles when the response and the covariate are both functions. As in classical quantile regression for finite dimensional data, regression techniques developed here provide insight into the influence of the functional covariate on different parts, like the center as well as the tails, of the conditional distribution of the functional response. Depth and quantile based nonparametric regression methods are useful to detect heteroscedasticity in functional regression. We derive the asymptotic behavior of the nonparametric depth and quantile regression estimates, which depend on the small ball probabilities in the covariate space. Our nonparametric regression procedures are used to analyze a dataset about the influence of per capita GDP on saving rates for 125 countries, and another dataset on the effects of per capita net disposable income on the sale of cigarettes in some states in the US.

Keywords: Bahadur representation; conditional spread; convergence rates; maximal depth set; spatial depth; spatial quantile

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Estimation and hypotheses testing in boundary regression models

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Consider a nonparametric regression model with one-sided errors and regression function in a general Hölder class. We estimate the regression function via minimization of the local integral of a polynomial approximation. We show uniform rates of convergence for the simple regression estimator as well as for a smooth version. These rates carry over to mean regression models with a symmetric and bounded error distribution. In such a setting, one obtains faster rates for irregular error distributions concentrating sufficient mass near the endpoints than for the usual regular distributions. The results are applied to prove asymptotic $\sqrt{n}$-equivalence of a residual-based (sequential) empirical distribution function to the (sequential) empirical distribution function of unobserved errors in the case of irregular error distributions. This result is remarkably different from corresponding results in mean regression with regular errors. It can readily be applied to develop goodness-of-fit tests for the error distribution. We present some examples and investigate the small sample performance in a simulation study. We further discuss asymptotically distribution-free hypotheses tests for independence of the error distribution from the points of measurement and for monotonicity of the boundary function as well.

Keywords: goodness-of-fit testing; irregular error distribution; one-sided errors; residual empirical distribution function; uniform rates of convergence

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Consistent order estimation for nonparametric hidden Markov models

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We consider the problem of estimating the number of hidden states (the *order*) of a nonparametric hidden Markov model (HMM). We propose two different methods and prove their almost sure consistency without any prior assumption, be it on the order or on the emission distributions. This is the first time a consistency result is proved in such a general setting without using restrictive assumptions such as *a priori* upper bounds on the order or parametric restrictions on the emission distributions. Our main method relies on the minimization of a penalized least squares criterion. In addition to the consistency of the order estimation, we also prove that this method yields rate minimax adaptive estimators of the parameters of the HMM – up to a logarithmic factor. Our second method relies on estimating the rank of a matrix obtained from the distribution of two consecutive observations. Finally, numerical experiments are used to compare both methods and study their ability to select the right order in several situations.

Keywords: hidden Markov model; least squares method; model selection; nonparametric density estimation; order estimation; spectral method

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Central limit theorem for Fourier transform and periodogram of random fields

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In this paper, we show that the limiting distribution of the real and the imaginary part of the Fourier transform of a stationary random field is almost surely an independent vector with Gaussian marginal distributions, whose variance is, up to a constant, the field's spectral density. The dependence structure of the random field is general and we do not impose any restrictions on the speed of convergence to zero of the covariances, or smoothness of the spectral density. The only condition required is that the variables are adapted to a commuting filtration and are regular in some sense. The results go beyond the Bernoulli fields and apply to both short range and long range dependence. They can be easily applied to derive the asymptotic behavior of the periodogram associated to the random field. The method of proof is based on new probabilistic methods involving martingale approximations and also on borrowed and new tools from harmonic analysis. Several examples to linear, Volterra and Gaussian random fields will be presented.

Keywords: central limit theorem; Fourier transform; martingale approximation; random field; spectral density

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A multidimensional analogue of the arcsine law for the number of positive terms in a random walk

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Consider a random walk $S_i = \xi_1 + \cdots + \xi_i$, $i \in \mathbb{N}$, whose increments $\xi_1, \xi_2, \dots$ are independent identically distributed random vectors in $\mathbb{R}^d$ such that ξ_1 has the same law as $-\xi_1$ and $\mathbb{P}[\xi_1 \in H] = 0$ for every affine hyperplane $H \subset \mathbb{R}^d$. Our main result is the distribution-free formula

$$\mathbb{E}\Bigg[\sum_{1\le i_1<\cdots<i_k\le n} \mathbb{1}_{\{0\notin \mathrm{Conv}(S_{i_1},\dots,S_{i_k})\}}\Bigg] = 2\binom{n}{k}\frac{B(k,d-1)+B(k,d-3)+\cdots}{2^k k!},$$

where the $B(k,j)$'s are defined by their generating function $(t+1)(t+3)\dots(t+2k-1) = \sum_{j=0}^{k} B(k,j)t^j$. The expected number of k-tuples above admits the following geometric interpretation: it is the expected number of k-dimensional faces of a randomly and uniformly sampled open Weyl chamber of type B_n that are not intersected by a generic linear subspace $L \subset \mathbb{R}^n$ of codimension d. The case $d = 1$ turns out to be equivalent to the classical discrete arcsine law for the number of positive terms in a one-dimensional random walk with continuous symmetric distribution of increments. We also prove similar results for random bridges with no central symmetry assumption required.

Keywords: absorption probability; arcsine law; convex cone; convex hull; distribution-free probability; finite reflection group; hyperplane arrangement; random linear subspace; random walk; random walk bridge; Weyl chamber

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Limit properties of the monotone rearrangement for density and regression function estimation

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The monotone rearrrangement algorithm was introduced by Hardy, Littlewood and Pólya as a sorting device for functions. Assuming that x is a monotone function and that an estimate x_n of x is given, consider the monotone rearrangement $\hat{x}_n$ of x_n. This new estimator is shown to be uniformly consistent as soon as x_n is. Under suitable assumptions, pointwise limit distribution results for $\hat{x}_n$ are obtained. The framework is general and allows for weakly dependent and long range dependent stationary data. Applications in monotone density and regression function estimation are detailed. Asymptotics for rearrangement estimators with vanishing derivatives are also obtained in these two contexts.

Keywords: density estimation; dependence; limit distributions; monotone rearrangement; regression function estimation

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Sequential Monte Carlo as approximate sampling: bounds, adaptive resampling via ∞-ESS, and an application to particle Gibbs

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Sequential Monte Carlo (SMC) algorithms were originally designed for estimating intractable conditional expectations within state-space models, but are now routinely used to generate approximate samples in the context of general-purpose Bayesian inference. In particular, SMC algorithms are often used as subroutines within larger Monte Carlo schemes, and in this context, the demands placed on SMC are different: control of mean-squared error is insufficient—one needs to control the divergence from the target distribution directly. Towards this goal, we introduce the *conditional* adaptive resampling particle filter, building on the work of Gordon, Salmond, and Smith (1993), Andrieu, Doucet, and Holenstein (2010), and Whiteley, Lee, and Heine (2016). By controlling a novel notion of effective sample size, the ∞-ESS, we establish the efficiency of the resulting SMC sampling algorithm, providing an adaptive resampling extension of the work of Andrieu, Lee, and Vihola (2018). We apply our results to arrive at new divergence bounds for SMC samplers with adaptive resampling as well as an adaptive resampling version of the Particle Gibbs algorithm with the same geometric-ergodicity guarantees as its nonadaptive counterpart.

Keywords: adaptive resampling; effective sample size; geometric ergodicity; particle Gibbs; sequential Monte Carlo; state-space models; uniform ergodicity

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Optimal rates of statistical seriation

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Given a matrix, the seriation problem consists in permuting its rows in such way that all its columns have the same shape, for example, they are monotone increasing. We propose a statistical approach to this problem where the matrix of interest is observed with noise and study the corresponding minimax rate of estimation of the matrices. Specifically, when the columns are either unimodal or monotone, we show that the least squares estimator is optimal up to logarithmic factors and adapts to matrices with a certain natural structure. Finally, we propose a computationally efficient estimator in the monotonic case and study its performance both theoretically and experimentally. Our work is at the intersection of shape constrained estimation and recent work that involves permutation learning, such as graph denoising and ranking.

Keywords: adaptation; matrix estimation; minimax estimation; permutation learning; shape constraints; statistical seriation

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Second order correctness of perturbation bootstrap M-estimator of multiple linear regression parameter

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Consider the multiple linear regression model $y_i = \mathbf{x}_i'\boldsymbol{\beta} + \varepsilon_i$, where ε_i's are independent and identically distributed random variables, $\mathbf{x}_i$'s are known design vectors and $\boldsymbol{\beta}$ is the $p \times 1$ vector of parameters. An effective way of approximating the distribution of the M-estimator $\bar{\boldsymbol{\beta}}_n$, after proper centering and scaling, is the Perturbation Bootstrap Method. In this current work, second order results of this non-naive bootstrap method have been investigated. Second order correctness is important for reducing the approximation error uniformly to $o(n^{-1/2})$ to get better inferences. We show that the classical studentized version of the bootstrapped estimator fails to be second order correct. We introduce an innovative modification in the studentized version of the bootstrapped statistic and show that the modified bootstrapped pivot is second order correct (S.O.C.) for approximating the distribution of the studentized M-estimator. Additionally, we show that the Perturbation Bootstrap continues to be S.O.C. when the errors ε_i's are independent, but may not be identically distributed. These findings establish perturbation Bootstrap approximation as a significant improvement over asymptotic normality in the regression M-estimation.

Keywords: Edgeworth expansion; generalized bootstrap; M-estimation; perturbation bootstrap; residual bootstrap; S.O.C.; Studentization; wild bootstrap

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Random polymers on the complete graph

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Consider directed polymers in a random environment on the complete graph of size N. This model can be formulated as a product of i.i.d. $N \times N$ random matrices and its large time asymptotics is captured by Lyapunov exponents and the Furstenberg measure. We detail this correspondence, derive the long-time limit of the model and obtain a co-variant distribution for the polymer path.

Next, we observe that the model becomes exactly solvable when the disorder variables are located on edges of the complete graph and follow a totally asymmetric stable law of index $\alpha \in (0, 1)$. Then, a certain notion of mean height of the polymer behaves like a random walk and we show that the height function is distributed around this mean according to an explicit law. Large N asymptotics can be taken in this setting, for instance, for the free energy of the system and for the invariant law of the polymer height with a shift. Moreover, we give some perturbative results for environments which are close to the totally asymmetric stable laws.

Keywords: directed polymers; exactly solvable model; product of random matrices; random medium; stable laws

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Sum rules and large deviations for spectral matrix measures

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In the paradigm of random matrices, one of the most classical object under study is the empirical spectral distribution. This random measure is the uniform distribution supported by the eigenvalues of the random matrix. In this paper, we give large deviation theorems for another popular object built on Hermitian random matrices: the spectral measure. This last probability measure is a random weighted version of the empirical spectral distribution. The weights involve the eigenvectors of the random matrix. We have previously studied the large deviations of the spectral measure in the case of scalar weights. Here, we will focus on matrix valued weights. Our probabilistic results lead to deterministic ones called "sum rules" in spectral theory. A sum rule relative to a reference measure on $\mathbb{R}$ is a relationship between the reversed Kullback–Leibler divergence of a positive measure on $\mathbb{R}$ and some non-linear functional built on spectral elements related to this measure. By using only probabilistic tools of large deviations, we extend the sum rules to the case of Hermitian matrix-valued measures.

Keywords: large deviations; matrix-valued measures; orthogonal matrix polynomials; random matrices; sum rules

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Weak subordination of multivariate Lévy processes and variance generalised gamma convolutions

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Subordinating a multivariate Lévy process, the subordinate, with a univariate subordinator gives rise to a pathwise construction of a new Lévy process, provided the subordinator and the subordinate are independent processes. The variance-gamma model in finance was generated accordingly from a Brownian motion and a gamma process. Alternatively, multivariate subordination can be used to create Lévy processes, but this requires the subordinate to have independent components. In this paper, we show that there exists another operation acting on pairs (T, X) of Lévy processes which creates a Lévy process $X \odot T$. Here, T is a subordinator, but X is an arbitrary Lévy process with possibly dependent components. We show that this method is an extension of both univariate and multivariate subordination and provide two applications. We illustrate our methods giving a weak formulation of the variance-$\boldsymbol{\alpha}$-gamma process that exhibits a wider range of dependence than using traditional subordination. Also, the variance generalised gamma convolution class of Lévy processes formed by subordinating Brownian motion with Thorin subordinators is further extended using weak subordination.

Keywords: Brownian motion; gamma process; generalised gamma convolutions; Lévy process; marked point process; subordination; Thorin measure; variance gamma; variance-alpha-gamma

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Estimating the interaction graph of stochastic neural dynamics

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Dedicated to Enza Orlandi, in memoriam

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In this paper, we address the question of statistical model selection for a class of stochastic models of biological neural nets. Models in this class are systems of interacting chains with memory of variable length. Each chain describes the activity of a single neuron, indicating whether it spikes or not at a given time. The spiking probability of a given neuron depends on the time evolution of its *presynaptic neurons* since its last spike time. When a neuron spikes, its potential is reset to a resting level and postsynaptic current pulses are generated, modifying the membrane potential of all its *postsynaptic neurons*. The relationship between a neuron and its pre- and postsynaptic neurons defines an oriented graph, the *interaction graph* of the model. The goal of this paper is to estimate this graph based on the observation of the spike activity of a finite set of neurons over a finite time. We provide explicit exponential upper bounds for the probabilities of under- and overestimating the interaction graph restricted to the observed set and obtain the strong consistency of the estimator. Our result does not require stationarity nor uniqueness of the invariant measure of the process.

Keywords: biological neural nets; graph of interactions; interacting chains of variable memory length; statistical model selection

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