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Convergence of sequential quasi-Monte Carlo smoothing algorithms

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Gerber and Chopin [*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **77** (2015) 509–579] recently introduced Sequential quasi-Monte Carlo (SQMC) algorithms as an efficient way to perform filtering in state–space models. The basic idea is to replace random variables with low-discrepancy point sets, so as to obtain faster convergence than with standard particle filtering. Gerber and Chopin (2015) describe briefly several ways to extend SQMC to smoothing, but do not provide supporting theory for this extension. We discuss more thoroughly how smoothing may be performed within SQMC, and derive convergence results for the so-obtained smoothing algorithms. We consider in particular SQMC equivalents of forward smoothing and forward filtering backward sampling, which are the most well-known smoothing techniques. As a preliminary step, we provide a generalization of the classical result of Hlawka and Mück [*Computing (Arch. Elektron. Rechnen)* **9** (1972) 127–138] on the transformation of QMC point sets into low discrepancy point sets with respect to non uniform distributions. As a corollary of the latter, we note that we can slightly weaken the assumptions to prove the consistency of SQMC.

Keywords: hidden Markov models; low discrepancy; particle filtering; quasi-Monte Carlo; sequential quasi-Monte Carlo; smoothing; state–space models

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Baxter's inequality and sieve bootstrap for random fields

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The concept of the autoregressive (AR) sieve bootstrap is investigated for the case of spatial processes in $\mathbb{Z}^2$. This procedure fits AR models of increasing order to the given data and, via resampling of the residuals, generates bootstrap replicates of the sample. The paper explores the range of validity of this resampling procedure and provides a general check criterion which allows to decide whether the AR sieve bootstrap asymptotically works for a specific statistic of interest or not. The criterion may be applied to a large class of stationary spatial processes. As another major contribution of this paper, a weighted Baxter-inequality for spatial processes is provided. This result yields a rate of convergence for the finite predictor coefficients, i.e. the coefficients of finite-order AR model fits, towards the autoregressive coefficients which are inherent to the underlying process under mild conditions. The developed check criterion is applied to some particularly interesting statistics like sample autocorrelations and standardized sample variograms. A simulation study shows that the procedure performs very well compared to normal approximations as well as block bootstrap methods in finite samples.

Keywords: autoregression; bootstrap; random fields

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Testing the maximal rank of the volatility process for continuous diffusions observed with noise

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In this paper, we present a test for the maximal rank of the volatility process in continuous diffusion models observed with noise. Such models are typically applied in mathematical finance, where latent price processes are corrupted by microstructure noise at ultra high frequencies. Using high frequency observations, we construct a test statistic for the maximal rank of the time varying stochastic volatility process. Our methodology is based upon a combination of a matrix perturbation approach and pre-averaging. We will show the asymptotic mixed normality of the test statistic and obtain a consistent testing procedure. We complement the paper with a simulation and an empirical study showing the performances on finite samples.

Keywords: continuous Itô semimartingales; high frequency data; microstructure noise; rank testing; stable convergence

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Distribution of linear statistics of singular values of the product of random matrices

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In this paper we consider the product of two independent random matrices $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$. Assume that $X_{jk}^{(q)}, 1 \le j,k \le n, q = 1,2$, are i.i.d. random variables with $\mathbb{E}X_{jk}^{(q)} = 0, \operatorname{Var} X_{jk}^{(q)} = 1$. Denote by $s_1(\mathbf{W}),\dots,s_n(\mathbf{W})$ the singular values of $\mathbf{W} := \frac{1}{n}\mathbf{X}^{(1)}\mathbf{X}^{(2)}$. We prove the central limit theorem for linear statistics of the squared singular values $s_1^2(\mathbf{W}),\dots,s_n^2(\mathbf{W})$ showing that the limiting variance depends on $\kappa_4 := \mathbb{E}(X_{11}^{(1)})^4 - 3$.

Keywords: central limit theorem; characteristic functions; Fuss–Catalan distributions; products of random matrices

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Studentized U-quantile processes under dependence with applications to change-point analysis

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Many popular robust estimators are U-quantiles, most notably the Hodges–Lehmann location estimator and the Q_n scale estimator. We prove a functional central limit theorem for the U-quantile process without any moment assumptions and under weak short-range dependence conditions. We further devise an estimator for the long-run variance and show its consistency, from which the convergence of the studentized version of the U-quantile process to a standard Brownian motion follows. This result can be used to construct CUSUM-type change-point tests based on U-quantiles, which do not rely on bootstrapping procedures. We demonstrate this approach in detail with the example of the Hodges–Lehmann estimator for robustly detecting changes in the central location. A simulation study confirms the very good efficiency and robustness properties of the test. Two real-life data sets are analyzed.

Keywords: CUSUM test; Hodges–Lehmann estimator; long-run variance; median; near epoch dependence; robustness; weak invariance principle

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"Building" exact confidence nets

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Confidence nets, that is, collections of confidence intervals that fill out the parameter space and whose exact parameter coverage can be computed, are familiar in nonparametric statistics. Here, the distributional assumptions are based on invariance under the action of a finite reflection group. Exact confidence nets are exhibited for a single parameter, based on the root system of the group. The main result is a formula for the generating function of the coverage interval probabilities. The proof makes use of the theory of "buildings" and the Chevalley factorization theorem for the length distribution on Cayley graphs of finite reflection groups.

Keywords: buildings; confidence intervals; confidence nets; Coxeter groups; nonparametrics

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Eigen structure of a new class of covariance and inverse covariance matrices

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There is a one to one mapping between a p dimensional strictly positive definite covariance matrix Σ and its matrix logarithm L. We exploit this relationship to study the structure induced on Σ through a sparsity constraint on L. Consider L as a random matrix generated through a basis expansion, with the support of the basis coefficients taken as a simple random sample of size $s = s^*$ from the index set $[p(p+1)/2] = \{1, \dots, p(p+1)/2\}$. We find that the expected number of non-unit eigenvalues of Σ, denoted $\mathbb{E}[|\mathcal{A}|]$, is approximated with near perfect accuracy by the solution of the equation

$$\frac{4p + p(p-1)}{2(p+1)}\left[\log\left(\frac{p}{p-d}\right) - \frac{d}{2p(p-d)}\right] - s^* = 0.$$

Furthermore, the corresponding eigenvectors are shown to possess only $p - |\mathcal{A}^c|$ non-zero entries. We use this result to elucidate the precise structure induced on Σ and Σ^{-1}. We demonstrate that a positive definite symmetric matrix whose matrix logarithm is sparse is significantly less sparse in the original domain. This finding has important implications in high dimensional statistics where it is important to exploit structure in order to construct consistent estimators in non-trivial norms. An estimator exploiting the structure of the proposed class is presented.

Keywords: covariance matrix; matrix logarithm; precision matrix; spectral theory

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Influence functions for penalized M-estimators

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We study the local robustness properties of general nondifferentiable penalized M-estimators via the influence function. More precisely, we propose a framework that allows us to define rigorously the influence function as the limiting influence function of a sequence of approximating estimators. We show that it can be used to characterize the robustness properties of a wide range of sparse estimators and we derive its form for general penalized M-estimators including lasso and adaptive lasso type estimators. We prove that our influence function is equivalent to a derivative in the sense of distribution theory.

Keywords: distribution theory; implicit function theorem; lasso; regularization; robust statistics

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On predictive density estimation for location families under integrated absolute error loss

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This paper is concerned with estimating a predictive density under integrated absolute error (L_1) loss. Based on a spherically symmetric observable $X \sim p_X(\|x-\mu\|^2)$, $x, \mu \in \mathbb{R}^d$, we seek to estimate the (unimodal) density of $Y \sim q_Y(\|y-\mu\|^2)$, $y \in \mathbb{R}^d$. We focus on the benchmark (and maximum likelihood for unimodal p) plug-in density estimator $q_Y(\|y-X\|^2)$ and, for $d \ge 4$, we establish its inadmissibility, as well as provide plug-in density improvements, as measured by the frequentist risk taken with respect to X. Sharper results are obtained for the subclass of scale mixtures of normal distributions which include the normal case. The findings rely on the duality between the predictive density estimation problem with a point estimation problem of estimating μ under a loss which is a concave function of $\|\hat{\mu}-\mu\|^2$, Stein estimation results and techniques applicable to such losses, and further properties specific to scale mixtures of normal distributions. Finally, (i) we address univariate implications for cases where there exist parametric restrictions on μ, and (ii) we show quite generally for logconcave q_Y that improvements on the benchmark mle can always be found among the scale expanded predictive densities $\frac{1}{c} q_Y(\frac{(y-x)^2}{c^2})$, with $c-1$ positive but not too large.

Keywords: concave loss; dominance; frequentist risk; inadmissibility; L_1 loss; multivariate normal; plug-in; predictive density; restricted parameter space; scale mixture of normals; Stein estimation

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Transportation and concentration inequalities for bifurcating Markov chains

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We investigate the transportation inequality for bifurcating Markov chains which are a class of processes indexed by a regular binary tree. Fitting well models like cell growth when each individual gives birth to exactly two offsprings, we use transportation inequalities to provide useful concentration inequalities. We also study deviation inequalities for the empirical means under relaxed assumptions on the Wasserstein contraction for the Markov kernels. Applications to bifurcating nonlinear autoregressive processes are considered for point-wise estimates of the non-linear autoregressive function.

Keywords: bifurcating Markov chains; deviation inequalities; geometric ergodicity; transportation inequalities; Wasserstein distance

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Pólya urn schemes with infinitely many colors

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In this work, we introduce a class of *balanced* urn schemes with infinitely many colors indexed by $\mathbb{Z}^d$, where the replacement schemes are given by the transition matrices associated with bounded increment random walks. We show that the color of the nth selected ball follows a Gaussian distribution on $\mathbb{R}^d$ after $\mathcal{O}(\log n)$ centering and $\mathcal{O}(\sqrt{\log n})$ scaling irrespective of whether the underlying walk is null recurrent or transient. We also provide finer asymptotic similar to local limit theorems for the expected configuration of the urn. The proofs are based on a novel representation of the color of the nth selected ball as "slowed down" version of the underlying random walk.

Keywords: central limit theorem; infinite color urn; local limit theorem; random walk; reinforcement processes; urn models

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Approximate local limit theorems with effective rate and application to random walks in random scenery

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We show that the Bernoulli part extraction method can be used to obtain approximate forms of the local limit theorem for sums of independent lattice valued random variables, with effective error term. That is with explicit parameters and universal constants. We also show that our estimates allow us to recover Gnedenko and provide a version with effective bounds of Gamkrelidze's local limit theorem. We further establish by this method a local limit theorem with effective remainder for random walks in random scenery.

Keywords: Bernoulli part; effective remainder; independent random variables; lattice distributed; local limit theorem; random walk in random scenery

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On Stein's method for products of normal random variables and zero bias couplings

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In this paper, we extend Stein's method to the distribution of the product of n independent mean zero normal random variables. A Stein equation is obtained for this class of distributions, which reduces to the classical normal Stein equation in the case $n = 1$. This Stein equation motivates a generalisation of the zero bias transformation. We establish properties of this new transformation, and illustrate how they may be used together with the Stein equation to assess distributional distances for statistics that are asymptotically distributed as the product of independent central normal random variables. We end by proving some product normal approximation theorems.

Keywords: coupling; distributional transformation; products of normal random variables; Stein's method; zero biasing

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Weak convergence of empirical copula processes indexed by functions

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Weak convergence of the empirical copula process indexed by a class of functions is established. Two scenarios are considered in which either some smoothness of these functions or smoothness of the underlying copula function is required.

A novel integration by parts formula for multivariate, right-continuous functions of bounded variation, which is perhaps of independent interest, is proved. It is a key ingredient in proving weak convergence of a general empirical process indexed by functions of bounded variation.

Keywords: Donsker classes; empirical copula process; integration by parts; multivariate functions of bounded variation; weak convergence

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Sieve maximum likelihood estimation for a general class of accelerated hazards models with bundled parameters

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In semiparametric hazard regression, nonparametric components may involve unknown regression parameters. Such intertwining effects make model estimation and inference much more difficult than the case in which the parametric and nonparametric components can be separated out. We study the sieve maximum likelihood estimation for a general class of hazard regression models, which include the proportional hazards model, the accelerated failure time model, and the accelerated hazards model. Coupled with the cubic B-spline, we propose semiparametric efficient estimators for the parameters that are bundled inside the nonparametric component. We overcome the challenges due to intertwining effects of the bundled parameters, and establish the consistency and asymptotic normality properties of the estimators. We carry out simulation studies to examine the finite-sample properties of the proposed method, and demonstrate its efficiency gain over the conventional estimating equation approach. For illustration, we apply our proposed method to a study of bone marrow transplantation for patients with acute leukemia.

Keywords: accelerated failure time model; B-spline; proportional hazards model; semiparametric efficiency bound; sieve maximum likelihood estimator; survival data

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Integrated empirical processes in L^p with applications to estimate probability metrics

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We discuss the convergence in distribution of the r-fold (reverse) integrated empirical process in the space L^p, for $1 \le p \le \infty$. In the case $1 \le p < \infty$, we find the necessary and sufficient condition on a positive random variable X so that this process converges weakly in L^p. This condition defines a Lorentz space and can be also characterized in terms of several integrability conditions related to the process $\{(X-t)_+^r : t \ge 0\}$. For $p = \infty$, we obtain an integrability requirement on X guaranteeing the convergence of the integrated empirical process. In particular, these results imply a limit theorem for the stop-loss distance between the empirical and the true distribution. As an application, we derive the asymptotic distribution of an estimator of the Zolotarev distance between two probability distributions. The connections of the involved processes with equilibrium distributions and stochastic integrals with respect to the Brownian bridge are also briefly explained.

Keywords: distributional limit theorems; integrated Brownian bridge; integrated empirical process; Lorentz spaces; probability metrics; stochastic integral; stop-loss distance; Zolotarev metric

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Efficiency and bootstrap in the promotion time cure model

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In this paper, we consider a semiparametric promotion time cure model and study the asymptotic properties of its nonparametric maximum likelihood estimator (NPMLE). First, by relying on a profile likelihood approach, we show that the NPMLE may be computed by a single maximization over a set whose dimension equals the dimension of the covariates plus one. Next, using Z-estimation theory for semiparametric models, we derive the asymptotics of both the parametric and nonparametric components of the model and show their efficiency. We also express the asymptotic variance of the estimator of the parametric component. Since the variance is difficult to estimate, we develop a weighted bootstrap procedure that allows for a consistent approximation of the asymptotic law of the estimators. As in the Cox model, it turns out that suitable tools are the martingale theory for counting processes and the infinite dimensional Z-estimation theory. Finally, by means of simulations, we show the accuracy of the bootstrap approximation.

Keywords: asymptotic inference; bootstrap; Cox model; promotion time cure model; semiparametric efficiency

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Non-central limit theorems for random fields subordinated to gamma-correlated random fields

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A reduction theorem is proved for functionals of Gamma-correlated random fields with long-range dependence in d-dimensional space. As a particular case, integrals of non-linear functions of chi-squared random fields, with Laguerre rank being equal to one and two, are studied. When the Laguerre rank is equal to one, the characteristic function of the limit random variable, given by a Rosenblatt-type distribution, is obtained. When the Laguerre rank is equal to two, a multiple Wiener–Itô stochastic integral representation of the limit distribution is derived and an infinite series representation, in terms of independent random variables, is obtained for the limit.

Keywords: Hermite expansion; Laguerre expansion; multiple Wiener–Itô stochastic integrals; non-central limit results; reduction theorems; series expansions

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Asymptotic expansions and hazard rates for compound and first-passage distributions

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A general theory which provides asymptotic tail expansions for density, survival, and hazard rate functions is developed for both absolutely continuous and integer-valued distributions. The expansions make use of Tauberian theorems which apply to moment generating functions (MGFs) with boundary singularities that are of gamma-type or log-type. Standard Tauberian theorems from Feller [*An Introduction to Probability Theory and Its Applications* **II** (1971) Wiley] can provide a limited theory but these theorems do not suffice in providing a complete theory as they are not capable of explaining tail behaviour for compound distributions and other complicated distributions which arise in stochastic modelling settings. Obtaining such a complete theory for absolutely continuous distributions requires introducing new "Ikehara" conditions based upon Tauberian theorems whose development and application have been largely confined to analytic number theory. For integer-valued distributions, a complete theory is developed by applying Darboux's theorem used in analytic combinatorics. Characterizations of asymptotic hazard rates for both absolutely continuous and integer-valued distributions are developed in conjunction with these expansions. The main applications include the ruin distribution in the Cramér–Lundberg and Sparre Andersen models, more general classes of compound distributions, and first-passage distributions in finite-state semi-Markov processes. Such first-passage distributions are shown to have exponential-like/geometric-like tails which mimic the behaviour of first-passage distributions in Markov processes even though the holding-time MGFs involved with such semi-Markov processes are typically not rational.

Keywords: asymptotic hazard rate; compound distribution; Cramér–Lundberg approximation; Darboux's theorem; first-passage distribution; Ikehara–Delange theorem; Ikehara–Wiener theorem; semi-Markov process; Sparre Andersen model; Tauberian theory

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Efficient Bayesian estimation and uncertainty quantification in ordinary differential equation models

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Often the regression function is specified by a system of ordinary differential equations (ODEs) involving some unknown parameters. Typically analytical solution of the ODEs is not available, and hence likelihood evaluation at many parameter values by numerical solution of equations may be computationally prohibitive. Bhaumik and Ghosal (*Electron. J. Stat.* **9** (2015) 3124–3154) considered a Bayesian two-step approach by embedding the model in a larger nonparametric regression model, where a prior is put through a random series based on B-spline basis functions. A posterior on the parameter is induced from the regression function by minimizing an integrated weighted squared distance between the derivative of the regression function and the derivative suggested by the ODEs. Although this approach is computationally fast, the Bayes estimator is not asymptotically efficient. In this paper, we suggest a modification of the two-step method by directly considering the distance between the function in the nonparametric model and that obtained from a four stage Runge–Kutta (RK4) method. We also study the asymptotic behavior of the posterior distribution of $\boldsymbol{\theta}$ based on an approximate likelihood obtained from an RK4 numerical solution of the ODEs. We establish a Bernstein–von Mises theorem for both methods which assures that Bayesian uncertainty quantification matches with the frequentist one and the Bayes estimator is asymptotically efficient.

Keywords: approximate likelihood; Bayesian inference; Bernstein–von Mises theorem; ordinary differential equation; Runge–Kutta method; spline smoothing

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Fractional Brownian motion satisfies two-way crossing

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We prove the following result: For $(Z_t)_{t\in\mathbf{R}}$ a fractional Brownian motion with arbitrary Hurst parameter, for any stopping time τ, there exist arbitrarily small $\varepsilon > 0$ such that $Z_{\tau+\varepsilon} < Z_\tau$, with asymptotic behaviour when $\varepsilon \searrow 0$ satisfying a bound of iterated logarithm type. As a consequence, fractional Brownian motion satisfies the "two-way crossing" property, which has important applications in financial mathematics.

Keywords: fractional Brownian motion; law of the iterated logarithm; stopping time; two-way crossing

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Adaptive estimation for bifurcating Markov chains

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In a first part, we prove Bernstein-type deviation inequalities for bifurcating Markov chains (BMC) under a geometric ergodicity assumption, completing former results of Guyon and Bitseki Penda, Djellout and Guillin. These preliminary results are the key ingredient to implement nonparametric wavelet thresholding estimation procedures: in a second part, we construct nonparametric estimators of the transition density of a BMC, of its mean transition density and of the corresponding invariant density, and show smoothness adaptation over various multivariate Besov classes under L^p-loss error, for $1 \leq p < \infty$. We prove that our estimators are (nearly) optimal in a minimax sense. As an application, we obtain new results for the estimation of the splitting size-dependent rate of growth-fragmentation models and we extend the statistical study of bifurcating autoregressive processes.

Keywords: bifurcating autoregressive process; bifurcating Markov chains; binary trees; deviations inequalities; growth-fragmentation processes; minimax rates of convergence; nonparametric adaptive estimation

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A proof of the Shepp–Olkin entropy concavity conjecture

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We prove the Shepp–Olkin conjecture, which states that the entropy of the sum of independent Bernoulli random variables is concave in the parameters of the individual random variables. Our proof refines an argument previously presented by the same authors, which resolved the conjecture in the monotonic case (where all the parameters are simultaneously increasing). In fact, we show that the monotonic case is the worst case, using a careful analysis of concavity properties of the derivatives of the probability mass function. We propose a generalization of Shepp and Olkin's original conjecture, to consider Rényi and Tsallis entropies.

Keywords: Bernoulli sums; concavity; entropy; Poisson binomial distribution; transportation of measure

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The failure of the profile likelihood method for a large class of semi-parametric models

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We consider a semi-parametric model for recurrent events. The model consists of an unknown hazard rate function, the infinite-dimensional parameter of the model, and a parametrically specified effective age function. We will present a condition on the family of effective age functions under which the profile likelihood function evaluated at the parameter vector $\boldsymbol{\theta}$, say, exceeds the profile likelihood function evaluated at the parameter vector $\tilde{\boldsymbol{\theta}}$, say, with probability p. From this we derive a condition under which profile likelihood inference for the finite-dimensional parameter of the model leads to inconsistent estimates. Examples will be presented. In particular, we will provide an example where the profile likelihood function is monotone with probability one regardless of the true data generating process. We also discuss the relation of our results to other semi-parametric models like the accelerated failure time model and Cox's proportional hazards model.

Keywords: accelerated failure time model; Cox's proportional hazards model; effective age process; profile likelihood inference; recurrent event data; semi-parametric statistical model; virtual age process

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Some monotonicity properties of parametric and nonparametric Bayesian bandits

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One of two independent stochastic processes (arms) is to be selected at each of n stages. The selection is sequential and depends on past observations as well as the prior information. The objective is to maximize the expected future-discounted sum of the n observations. We study structural properties of this classical bandit problem, in particular how the maximum expected payoff and the optimal strategy vary with the priors, in two settings: (a) observations from each arm have an exponential family distribution and different arms are assigned independent conjugate priors; (b) observations from each arm have a nonparametric distribution and different arms are assigned independent Dirichlet process priors. In both settings, we derive results of the following type: (i) for a particular arm and a fixed prior weight, the maximum expected payoff increases as the prior mean yield increases; (ii) for a fixed prior mean yield, the maximum expected payoff increases as the prior weight decreases. Specializing to the one-armed bandit, the second result captures the intuition that, given the same immediate payoff, the less one knows about an arm, the more desirable it becomes because there remains more information to be gained when selecting that arm. In the parametric case, our results extend those of (*Ann. Statist.* **20** (1992) 1625–1636) concerning Bernoulli and normal bandits (see also (In *Time Series and Related Topics* (2006) pp. 284–294 IMS)). In the nonparametric case, we extend those of (*Ann. Statist.* **13** (1985) 1523–1534). A key tool in the derivation is stochastic orders.

Keywords: Bernoulli bandits; convex order; Dirichlet bandits; log-concavity; optimal stopping; sequential decision; two-armed bandits

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Accelerated Gibbs sampling of normal distributions using matrix splittings and polynomials

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Standard Gibbs sampling applied to a multivariate normal distribution with a specified precision matrix is equivalent in fundamental ways to the Gauss–Seidel iterative solution of linear equations in the precision matrix. Specifically, the iteration operators, the conditions under which convergence occurs, and geometric convergence factors (and rates) are identical. These results hold for arbitrary matrix splittings from classical iterative methods in numerical linear algebra giving easy access to mature results in that field, including existing convergence results for antithetic-variable Gibbs sampling, REGS sampling, and generalizations. Hence, efficient deterministic stationary relaxation schemes lead to efficient generalizations of Gibbs sampling. The technique of polynomial acceleration that significantly improves the convergence rate of an iterative solver derived from a *symmetric* matrix splitting may be applied to accelerate the equivalent generalized Gibbs sampler. Identicality of error polynomials guarantees convergence of the inhomogeneous Markov chain, while equality of convergence factors ensures that the optimal solver leads to the optimal sampler. Numerical examples are presented, including a Chebyshev accelerated SSOR Gibbs sampler applied to a stylized demonstration of low-level Bayesian image reconstruction in a large 3-dimensional linear inverse problem.

Keywords: Bayesian inference; Gaussian Markov random field; Gibbs sampling; matrix splitting; multivariate normal distribution; non-stationary stochastic iteration; polynomial acceleration

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Simulation of hitting times for Bessel processes with non-integer dimension

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In this paper, we complete and improve the study of the simulation of the hitting times of some given boundaries for Bessel processes. These problems are of great interest in many application fields as finance and neurosciences. In a previous work (*Ann. Appl. Probab.* **23** (2013) 2259–2289), the authors introduced a new method for the simulation of hitting times for Bessel processes with integer dimension. The method, called walk on moving spheres algorithm (WoMS), was based mainly on the explicit formula for the distribution of the hitting time and on the connection between the Bessel process and the Euclidean norm of the Brownian motion. This method does not apply anymore for a non-integer dimension. In this paper we consider the simulation of the hitting time of Bessel processes with non-integer dimension $\delta \geq 1$ and provide a new algorithm by using the additivity property of the laws of squared Bessel processes. We split each simulation step of the algorithm in two parts: one is using the integer dimension case and the other one considers hitting time of a Bessel process starting from zero.

Keywords: Bessel processes with non-integer dimension; hitting time; numerical algorithm

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