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Importance sampling and statistical Romberg method

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The efficiency of Monte Carlo simulations is significantly improved when implemented with variance reduction methods. Among these methods, we focus on the popular importance sampling technique based on producing a parametric transformation through a shift parameter θ. The optimal choice of θ is approximated using Robbins–Monro procedures, provided that a nonexplosion condition is satisfied. Otherwise, one can use either a constrained Robbins–Monro algorithm (see, e.g., Arouna (*Monte Carlo Methods Appl.* **10** (2004) 1–24) and Lelong (*Statist. Probab. Lett.* **78** (2008) 2632–2636)) or a more astute procedure based on an unconstrained approach recently introduced by Lemaire and Pagès in (*Ann. Appl. Probab.* **20** (2010) 1029–1067). In this article, we develop a new algorithm based on a combination of the statistical Romberg method and the importance sampling technique. The statistical Romberg method introduced by Kebaier in (*Ann. Appl. Probab.* **15** (2005) 2681–2705) is known for reducing efficiently the complexity compared to the classical Monte Carlo one. In the setting of discritized diffusions, we prove the almost sure convergence of the constrained and unconstrained versions of the Robbins–Monro routine, towards the optimal shift θ^* that minimizes the variance associated to the statistical Romberg method. Then, we prove a central limit theorem for the new algorithm that we called adaptive statistical Romberg method. Finally, we illustrate by numerical simulation the efficiency of our method through applications in option pricing for the Heston model.

Keywords: central limit theorem; Euler scheme; Heston model; Robbins–Monro; statistical Romberg method; stochastic algorithm; variance reduction

References

[1] Andrieu, C., Moulines, É. and Priouret, P. (2005). Stability of stochastic approximation under verifiable conditions. *SIAM J. Control Optim*. **44** 283–312. MR2177157

[2] Arouna, B. (2004). Adaptative Monte Carlo method, a variance reduction technique. *Monte Carlo Methods Appl*. **10** 1–24. MR2054568

[3] Bally, V. and Talay, D. (1996). The law of the Euler scheme for stochastic differential equations. I. Convergence rate of the distribution function. *Probab. Theory Related Fields* **104** 43–60. MR1367666

[4] Benveniste, A., Métivier, M. and Priouret, P. (1990). *Adaptive Algorithms and Stochastic Approximations*. *Applications of Mathematics* (*New York*) **22**. Berlin: Springer. Translated from the French by Stephen S. Wilson. MR1082341

[5] Bouleau, N. and Lépingle, D. (1994). *Numerical Methods for Stochastic Processes*. *Wiley Series in Probability and Mathematical Statistics*: *Applied Probability and Statistics*. New York: Wiley. MR1274043

[6] Chen, H.F., Lei, G. and Gao, A.J. (1988). Convergence and robustness of the Robbins–Monro algorithm truncated at randomly varying bounds. *Stochastic Process*. *Appl*. **27** 217–231. MR0931029

[7] Chen, H.F. and Zhu, Y.M. (1986). Stochastic approximation procedures with randomly varying truncations. *Sci*. *Sinica Ser. A* **29** 914–926. MR0869196

[8] Duflo, M. (1996). *Algorithmes Stochastiques*. *Mathématiques & Applications* (*Berlin*) [*Mathematics & Applications*] **23**. Berlin: Springer. MR1612815

[9] Giles, M.B. (2008). Multilevel Monte Carlo path simulation. *Oper. Res*. **56** 607–617. MR2436856

[10] Heston, S.L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* **6** 327–343.

[11] Jacod, J. and Protter, P. (1998). Asymptotic error distributions for the Euler method for stochastic differential equations. *Ann*. *Probab*. **26** 267–307. MR1617049

[12] Kebaier, A. (2005). Statistical Romberg extrapolation: A new variance reduction method and applications to option pricing. *Ann*. *Appl*. *Probab*. **15** 2681–2705. MR2187308

[13] Kloeden, P.E. and Platen, E. (1995). Numerical methods for stochastic differential equations. In *Nonlinear Dynamics and Stochastic Mechanics*. *CRC Math*. *Model*. *Ser*. 437–461. Boca Raton, FL: CRC. MR1337938

[14] Kunita, H. (1997). *Stochastic Flows and Stochastic Differential Equations*. *Cambridge Studies in Advanced Mathematics* **24**. Cambridge: Cambridge Univ. Press. Reprint of the 1990 original. MR1472487

[15] Kushner, H.J. and Yin, G.G. (2003). *Stochastic Approximation and Recursive Algorithms and Applications*: *Stochastic Modelling and Applied Probability*, 2nd ed. *Applications of Mathematics* (*New York*) **35**. New York: Springer. MR1993642

[16] Lapeyre, B. and Lelong, J. (2011). A framework for adaptive Monte Carlo procedures. *Monte Carlo Methods Appl*. **17** 77–98. MR2784744

[17] Lelong, J. (2008). Almost sure convergence for randomly truncated stochastic algorithms under verifiable conditions. *Statist*. *Probab. Lett*. **78** 2632–2636. MR2542461

[18] Lemaire, V. and Pagès, G. (2010). Unconstrained recursive importance sampling. *Ann*. *Appl*. *Probab*. **20** 1029–1067. MR2680557

[19] Pelletier, M. (2000). Asymptotic almost sure efficiency of averaged stochastic algorithms. *SIAM J*. *Control Optim*. **39** 49–72 (electronic). MR1780908

[20] Talay, D. and Tubaro, L. (1990). Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic Anal*. *Appl*. **8** 483–509. MR1091544

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Pointwise adaptive estimation of a multivariate density under independence hypothesis

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In this paper, we study the problem of pointwise estimation of a multivariate density. We provide a data-driven selection rule from the family of kernel estimators and derive for it a pointwise oracle inequality. Using the latter bound, we show that the proposed estimator is minimax and minimax adaptive over the scale of anisotropic Nikolskii classes. It is important to emphasize that our estimation method adjusts automatically to eventual independence structure of the underlying density. This, in its turn, allows to reduce significantly the influence of the dimension on the accuracy of estimation (curse of dimensionality). The main technical tools used in our considerations are pointwise uniform bounds of empirical processes developed recently in Lepski [*Math. Methods Statist.* **22** (2013) 83–99].

Keywords: adaptation; density estimation; independence structure; oracle inequality; upper function

References

[1] Birgé, L. (2008). Model selection for density estimation with $\mathbb{L}_2$-loss. Available at arXiv:0808.1416v2.
[2] Bretagnolle, J. and Huber, C. (1979). Estimation des densités: Risque minimax. *Z. Wahrsch. Verw. Gebiete* **47** 119–137. MR0523165
[3] Brown, L.D. and Low, M.G. (1996). A constrained risk inequality with applications to nonparametric functional estimation. *Ann. Statist.* **24** 2524–2535. MR1425965
[4] Butucea, C. (2000). Two adaptive rates of convergence in pointwise density estimation. *Math. Methods Statist.* **9** 39–64. MR1772224
[5] Chacón, J.E. and Duong, T. (2010). Multivariate plug-in bandwidth selection with unconstrained pilot bandwidth matrices. *TEST* **19** 375–398. MR2677734
[6] Comte, F. and Lacour, C. (2013). Anisotropic adaptive kernel deconvolution. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 569–609. MR3088382
[7] Devroye, L. and Györfi, L. (1985). *Nonparametric Density Estimation: The L_1 View. Wiley Series in Probability and Mathematical Statistics: Tracts on Probability and Statistics*. New York: Wiley. MR0780746
[8] Devroye, L. and Lugosi, G. (1996). A universally acceptable smoothing factor for kernel density estimates. *Ann. Statist.* **24** 2499–2512. MR1425963
[9] Devroye, L. and Lugosi, G. (1997). Nonasymptotic universal smoothing factors, kernel complexity and Yatracos classes. *Ann. Statist.* **25** 2626–2637. MR1604428
[10] Devroye, L. and Lugosi, G. (2001). *Combinatorial Methods in Density Estimation. Springer Series in Statistics*. New York: Springer. MR1843146

[11] Donoho, D.L., Johnstone, I.M., Kerkyacharian, G. and Picard, D. (1996). Density estimation by wavelet thresholding. *Ann. Statist.* **24** 508–539. MR1394974
[12] Donoho, D.L. and Low, M.G. (1992). Renormalization exponents and optimal pointwise rates of convergence. *Ann. Statist.* **20** 944–970. MR1165601
[13] Efromovich, S. (2008). Adaptive estimation of and oracle inequalities for probability densities and characteristic functions. *Ann. Statist.* **36** 1127–1155. MR2418652
[14] Efromovich, S.Y. (1985). Non parametric estimation of a density of unknown smoothness. *Theory Probab. Appl.* **30** 557–568.
[15] Giné, E. and Guillou, A. (2002). Rates of strong uniform consistency for multivariate kernel density estimators. *Ann. Inst. Henri Poincaré Probab. Stat.* **38** 907–921. MR1955344
[16] Giné, E. and Nickl, R. (2009). An exponential inequality for the distribution function of the kernel density estimator, with applications to adaptive estimation. *Probab. Theory Related Fields* **143** 569–596. MR2475673
[17] Goldenshluger, A. and Lepski, O. (2013). On adaptive minimax density estimation on $\mathbb{R}^D$. *Probab. Theory Related Fields*. To appear. Published online 13 July 2013.
[18] Goldenshluger, A. and Lepski, O. (2011). Bandwidth selection in kernel density estimation: Oracle inequalities and adaptive minimax optimality. *Ann. Statist.* **39** 1608–1632. MR2850214
[19] Golubev, G.K. (1992). Nonparametric estimation of smooth densities of a distribution in L_2. *Problemy Peredachi Informatsii* **28** 52–62. MR1163140
[20] Hasminskii, R. and Ibragimov, I. (1990). On density estimation in the view of Kolmogorov's ideas in approximation theory. *Ann. Statist.* **18** 999–1010. MR1062695
[21] Ibragimov, I.A. and Has'minskiĭ, R.Z. (1980). An estimate of the density of a distribution. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **98** 61–85, 161–162, 166. MR0591862
[22] Ibragimov, I.A. and Khas'minskiĭ, R.Z. (1981). More on estimation of the density of a distribution. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **108** 72–88, 194, 198. MR0629401
[23] Juditsky, A. and Lambert-Lacroix, S. (2004). On minimax density estimation on $\mathbb{R}$. *Bernoulli* **10** 187–220. MR2046772
[24] Kerkyacharian, G., Lepski, O. and Picard, D. (2001). Nonlinear estimation in anisotropic multi-index denoising. *Probab. Theory Related Fields* **121** 137–170. MR1863916
[25] Kerkyacharian, G., Lepski, O. and Picard, D. (2007). Nonlinear estimation in anisotropic multi-index denoising. Sparse case. *Theory Probab. Appl.* **52** 150–171.
[26] Kerkyacharian, G., Picard, D. and Tribouley, K. (1996). L^p adaptive density estimation. *Bernoulli* **2** 229–247. MR1416864
[27] Kluchnikoff, N. (2005). On adaptive estimation of anisotropic functions. Ph.D Thesis, Aix-Marseille 1.
[28] Lepski, O. (2013). Upper functions for positive random functionals. II. Application to the empirical processes theory, part 1. *Math. Methods Statist.* **22** 83–99. MR3071956
[29] Lepski, O. (2013). Multivariate density estimation under sup-norm loss: Oracle approach, adaptation and independence structure. *Ann. Statist.* **41** 1005–1034. MR3099129
[30] Lepski, O.V., Mammen, E. and Spokoiny, V.G. (1997). Optimal spatial adaptation to inhomogeneous smoothness: An approach based on kernel estimates with variable bandwidth selectors. *Ann. Statist.* **25** 929–947. MR1447734
[31] Lepskiĭ, O.V. (1991). A problem of adaptive estimation in Gaussian white noise. *Theory Probab. Appl.* **35** 454–466.
[32] Mason, D.M. (2009). Risk bounds for kernel density estimators. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **363** 66–104, 183. MR2749118
[33] Massart, P. (2007). *Concentration Inequalities and Model Selection. Lecture Notes in Math.* **1896**. Berlin: Springer. MR2319879

[34] Nikol'skiĭ, S.M. (1977). *Priblizhenie Funktsii Mnogikh Peremennykh i Teoremy Vlozheniya*, 2nd ed. Moscow: Nauka. MR0506247
[35] Parzen, E. (1962). On estimation of a probability density function and mode. *Ann. Math. Statist.* **33** 1065–1076. MR0143282
[36] Rigollet, P. (2006). Adaptive density estimation using the blockwise Stein method. *Bernoulli* **12** 351–370. MR2218559
[37] Rosenblatt, M. (1956). Remarks on some nonparametric estimates of a density function. *Ann. Math. Statist.* **27** 832–837. MR0079873
[38] Samarov, A. and Tsybakov, A. (2007). Aggregation of density estimators and dimension reduction. In *Advances in Statistical Modeling and Inference. Ser. Biostat.* **3** 233–251. Hackensack, NJ: World Sci. Publ. MR2416118
[39] Scott, D.W. (1992). *Multivariate Density Estimation: Theory, Practice, and Visualization. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics*. New York: Wiley. MR1191168
[40] Silverman, B.W. (1986). *Density Estimation for Statistics and Data Analysis. Monographs on Statistics and Applied Probability*. London: Chapman & Hall. MR0848134
[41] Tsybakov, A.B. (1998). Pointwise and sup-norm sharp adaptive estimation of functions on the Sobolev classes. *Ann. Statist.* **26** 2420–2469. MR1700239

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Local asymptotic mixed normality property for nonsynchronously observed diffusion processes

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We prove the local asymptotic mixed normality (LAMN) property for a family of probability measures defined by parametrized diffusion processes with nonsynchronous observations. We assume that observation times of processes are independent of processes and we will study asymptotics when the maximum length of observation intervals goes to zero in probability. We also prove that the quasi-maximum likelihood estimator and the Bayes-type estimator proposed in Ogihara and Yoshida (*Stochastic Process. Appl.* **124** (2014) 2954–3008) are asymptotically efficient.

Keywords: asymptotic efficiency; Bayes-type estimators; diffusion processes; local asymptotic mixed normality property; Malliavin calculus; nonsynchronous observations; parametric estimation; quasi-maximum likelihood estimators

References

[1] Adams, R.A. (1975). *Sobolev Spaces*. *Pure and Applied Mathematics* **65**. New York: Academic Press. MR0450957

[2] Adams, R.A. and Fournier, J.J.F. (2003). *Sobolev Spaces*, 2nd ed. *Pure and Applied Mathematics* (*Amsterdam*) **140**. Amsterdam: Elsevier/Academic Press. MR2424078

[3] Aït-Sahalia, Y., Fan, J. and Xiu, D. (2010). High-frequency covariance estimates with noisy and asynchronous financial data. *J. Amer. Statist. Assoc.* **105** 1504–1517. MR2796567

[4] Aldous, D.J. and Eagleson, G.K. (1978). On mixing and stability of limit theorems. *Ann. Probab.* **6** 325–331. MR0517416

[5] Aronson, D.G. (1967). Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.* **73** 890–896. MR0217444

[6] Barndorff-Nielsen, O.E., Hansen, P.R., Lunde, A. and Shephard, N. (2011). Multivariate realised kernels: Consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading. *J. Econometrics* **162** 149–169. MR2795610

[7] Bibinger, M., Hautsch, N., Malec, P. and Reiß, M. (2014). Estimating the quadratic covariation matrix from noisy observations: Local method of moments and efficiency. *Ann. Statist.* **42** 1312–1346. MR3226158

[8] Christensen, K., Kinnebrock, S. and Podolskij, M. (2010). Pre-averaging estimators of the ex-post covariance matrix in noisy diffusion models with non-synchronous data. *J. Econometrics* **159** 116–133. MR2720847

[9] Dohnal, G. (1987). On estimating the diffusion coefficient. *J. Appl. Probab.* **24** 105–114. MR0876173

[10] Gobet, E. (2001). Local asymptotic mixed normality property for elliptic diffusion: A Malliavin calculus approach. *Bernoulli* **7** 899–912. MR1873834

[11] Gobet, E. (2002). LAN property for ergodic diffusions with discrete observations. *Ann. Inst. Henri Poincaré Probab. Stat.* **38** 711–737. MR1931584

[12] Hayashi, T. and Yoshida, N. (2005). On covariance estimation of non-synchronously observed diffusion processes. *Bernoulli* **11** 359–379. MR2132731

[13] Ibragimov, I.A. and Has'minskiĭ, R.Z. (1981). *Statistical Estimation: Asymptotic Theory. Applications of Mathematics* **16**. New York: Springer. MR0620321

[14] Jacod, J. (1997). On continuous conditional Gaussian martingales and stable convergence in law. In *Séminaire de Probabilités, XXXI. Lecture Notes in Math.* **1655** 232–246. Berlin: Springer. MR1478732

[15] Jeganathan, P. (1983). Some asymptotic properties of risk functions when the limit of the experiment is mixed normal. *Sankhyā Ser. A* **45** 66–87. MR0749355

[16] Malliavin, P. and Mancino, M.E. (2002). Fourier series method for measurement of multivariate volatilities. *Finance Stoch.* **6** 49–61. MR1885583

[17] Nualart, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications (New York)*. Berlin: Springer. MR2200233

[18] Nualart, D. and Pardoux, É. (1988). Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields* **78** 535–581. MR0950346

[19] Ogihara, T. and Yoshida, N. (2012). Quasi-likelihood analysis for stochastic regression models with nonsynchronous observations. Available at arXiv:1212.4911.

[20] Ogihara, T. and Yoshida, N. (2014). Quasi-likelihood analysis for nonsynchronously observed diffusion processes. *Stochastic Process. Appl.* **124** 2954–3008. MR3217430

[21] Protter, P. (1990). *Stochastic Integration and Differential Equations: A New Approach. Applications of Mathematics (New York)* **21**. Berlin: Springer. MR1037262

[22] Uchida, M. and Yoshida, N. (2013). Quasi likelihood analysis of volatility and nondegeneracy of statistical random field. *Stochastic Process. Appl.* **123** 2851–2876. MR3054548

[23] Yoshida, N. (2011). Polynomial type large deviation inequalities and quasi-likelihood analysis for stochastic differential equations. *Ann. Inst. Statist. Math.* **63** 431–479. MR2786943

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Some remarks on MCMC estimation of spectra of integral operators

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We prove a law of large numbers for empirical approximations of the spectrum of a kernel integral operator by the spectrum of random matrices based on a sample drawn from a Markov chain, which complements the results by V. Koltchinskii and E. Giné for i.i.d. sequences. In a special case of Mercer's kernels and geometrically ergodic chains, we also provide exponential inequalities, quantifying the speed of convergence.

Keywords: approximation of spectra; kernel operators; MCMC algorithms; random matrices

References

[1] Aaronson, J., Burton, R., Dehling, H., Gilat, D., Hill, T. and Weiss, B. (1996). Strong laws for L- and U-statistics. *Trans. Amer. Math. Soc.* **348** 2845–2866. MR1363941
[2] Adamczak, R. (2008). A tail inequality for suprema of unbounded empirical processes with applications to Markov chains. *Electron. J. Probab.* **13** 1000–1034. MR2424985
[3] Adamczak, R. and Bednorz, W. (2013). Exponential concentration inequalities for additive functionals of Markov chains. Available at arXiv:1201.3569.
[4] Andrieu, C., Jasra, A., Doucet, A. and Del Moral, P. (2011). On nonlinear Markov chain Monte Carlo. *Bernoulli* **17** 987–1014. MR2817614
[5] Arcones, M.A. (1998). The law of large numbers for U-statistics under absolute regularity. *Electron. Commun. Probab.* **3** 13–19 (electronic). MR1624866
[6] Athreya, K.B. and Ney, P. (1978). A new approach to the limit theory of recurrent Markov chains. *Trans. Amer. Math. Soc.* **245** 493–501. MR0511425
[7] Baxendale, P.H. (2005). Renewal theory and computable convergence rates for geometrically ergodic Markov chains. *Ann. Appl. Probab.* **15** 700–738. MR2114987
[8] Bertail, P. and Clémençon, S. (2011). A renewal approach to Markovian U-statistics. *Math. Methods Statist.* **20** 79–105. MR2882153
[9] Bhatia, R. and Elsner, L. (1994). The Hoffman–Wielandt inequality in infinite dimensions. *Proc. Indian Acad. Sci. Math. Sci.* **104** 483–494. MR1314392
[10] Chen, X. (1999). Limit theorems for functionals of ergodic Markov chains with general state space. *Mem. Amer. Math. Soc.* **139** xiv+203. MR1491814
[11] Douc, R., Fort, G., Moulines, E. and Soulier, P. (2004). Practical drift conditions for subgeometric rates of convergence. *Ann. Appl. Probab.* **14** 1353–1377. MR2071426
[12] Douc, R., Guillin, A. and Moulines, E. (2008). Bounds on regeneration times and limit theorems for subgeometric Markov chains. *Ann. Inst. H. Poincaré Probab. Statist.* **44** 239–257. MR2446322
[13] Horn, R.A. and Johnson, C.R. (2013). *Matrix Analysis*, 2nd ed. Cambridge: Cambridge Univ. Press. MR2978290

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[14] Koltchinskii, V. and Giné, E. (2000). Random matrix approximation of spectra of integral operators. *Bernoulli* **6** 113–167. MR1781185

[15] Mendelson, S. and Pajor, A. (2005). Ellipsoid approximation using random vectors. In *Learning Theory*. *Lecture Notes in Computer Science* **3559** 429–443. Berlin: Springer. MR2203278

[16] Mendelson, S. and Pajor, A. (2006). On singular values of matrices with independent rows. *Bernoulli* **12** 761–773. MR2265341

[17] Meyn, S. and Tweedie, R.L. (2009). *Markov Chains and Stochastic Stability*, 2nd ed. Cambridge: Cambridge Univ. Press. MR2509253

[18] Minh, H.Q., Niyogi, P. and Yao, Y. (2006). Mercer's theorem, feature maps, and smoothing. In *Learning Theory*. *Lecture Notes in Computer Science* **4005** 154–168. Berlin: Springer. MR2280604

[19] Montgomery-Smith, S.J. (1993). Comparison of sums of independent identically distributed random vectors. *Probab. Math. Statist.* **14** 281–285 (1994). MR1321767

[20] Nummelin, E. (1978). A splitting technique for Harris recurrent Markov chains. *Z. Wahrsch. Verw. Gebiete* **43** 309–318. MR0501353

[21] Nummelin, E. (1984). *General Irreducible Markov Chains and Nonnegative Operators*. *Cambridge Tracts in Mathematics* **83**. Cambridge: Cambridge Univ. Press. MR0776608

[22] Roberts, G.O. and Rosenthal, J.S. (2004). General state space Markov chains and MCMC algorithms. *Probab. Surv.* **1** 20–71. MR2095565

[23] Rosasco, L., Belkin, M. and De Vito, E. (2010). On learning with integral operators. *J. Mach. Learn. Res.* **11** 905–934. MR2600634

[24] Shawe-Taylor, J., Williams, C.K.I., Cristianini, N. and Kandola, J. (2005). On the eigenspectrum of the Gram matrix and the generalization error of kernel-PCA. *IEEE Trans. Inform. Theory* **51** 2510–2522. MR2246374

[25] Smale, S. and Zhou, D.-X. (2009). Geometry on probability spaces. *Constr. Approx.* **30** 311–323. MR2558684

[26] Sun, H. (2005). Mercer theorem for RKHS on noncompact sets. *J. Complexity* **21** 337–349. MR2138444

[27] Vershynin, R. (2000). On large random almost Euclidean bases. *Acta Math. Univ. Comenian. (N.S.)* **69** 137–144. MR1819516

[28] von Luxburg, U., Belkin, M. and Bousquet, O. (2008). Consistency of spectral clustering. *Ann. Statist.* **36** 555–586. MR2396807

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An empirical likelihood approach for symmetric α-stable processes

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Empirical likelihood approach is one of non-parametric statistical methods, which is applied to the hypothesis testing or construction of confidence regions for pivotal unknown quantities. This method has been applied to the case of independent identically distributed random variables and second order stationary processes. In recent years, we observe heavy-tailed data in many fields. To model such data suitably, we consider symmetric scalar and multivariate α-stable linear processes generated by infinite variance innovation sequence. We use a Whittle likelihood type estimating function in the empirical likelihood ratio function and derive the asymptotic distribution of the empirical likelihood ratio statistic for α-stable linear processes. With the empirical likelihood statistic approach, the theory of estimation and testing for second order stationary processes is nicely extended to heavy-tailed data analyses, not straightforward, and applicable to a lot of financial statistical analyses.

Keywords: confidence region; empirical likelihood ratio; heavy tail; normalized power transfer function; self-normalized periodogram; symmetric α-stable process; Whittle likelihood

References

[1] Akashi, F., Liu, Y. and Taniguchi, M. (2014). Supplement to “An empirical likelihood approach for symmetric α-stable processes.” DOI:10.3150/14-BEJ636SUPP.
[2] Bhansali, R.J. (1980). Autoregressive and window estimates of the inverse correlation function. *Biometrika* **67** 551–566. MR0601091
[3] Brillinger, D.R. (2001). *Time Series*. *Classics in Applied Mathematics* **36**. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM). Data analysis and theory, Reprint of the 1981 edition. MR1853554
[4] Brockwell, P.J. and Davis, R.A. (1991). *Time Series*: *Theory and Methods*, 2nd ed. *Springer Series in Statistics*. New York: Springer. MR1093459
[5] Cleveland, W.S. (1972). The inverse autocorrelations of a time series and their applications. *Technometrics* **14** 277–293.
[6] Davis, R. and Resnick, S. (1985). Limit theory for moving averages of random variables with regularly varying tail probabilities. *Ann*. *Probab.* **13** 179–195. MR0770636
[7] Davis, R. and Resnick, S. (1985). More limit theory for the sample correlation function of moving averages. *Stochastic Process*. *Appl*. **20** 257–279. MR0808161
[8] Davis, R. and Resnick, S. (1986). Limit theory for the sample covariance and correlation functions of moving averages. *Ann*. *Statist*. **14** 533–558. MR0840513
[9] Drees, H., de Haan, L. and Resnick, S. (2000). How to make a Hill plot. *Ann*. *Statist*. **28** 254–274. MR1762911

[10] Fama, E.F. (1965). The behavior of stock-market prices. *J. Bus*. **38** 34–105.
[11] Hall, P. (1982). On some simple estimates of an exponent of regular variation. *J. R. Stat. Soc. Ser. B Stat. Methodol*. **44** 37–42. MR0655370
[12] Hannan, E.J. (1970). *Multiple Time Series*. New York: Wiley. MR0279952
[13] Hsing, T. (1991). On tail index estimation using dependent data. *Ann. Statist*. **19** 1547–1569. MR1126337
[14] Klüppelberg, C. and Mikosch, T. (1993). Spectral estimates and stable processes. *Stochastic Process. Appl*. **47** 323–344. MR1239844
[15] Klüppelberg, C. and Mikosch, T. (1994). Some limit theory for the self-normalised periodogram of stable processes. *Scand. J. Stat*. **21** 485–491. MR1310091
[16] Klüppelberg, C. and Mikosch, T. (1996). The integrated periodogram for stable processes. *Ann. Statist*. **24** 1855–1879. MR1421152
[17] Mandelbrot, B.B. (1963). New methods in statistical economics. *J. Polit. Econ*. **71** 421–440.
[18] Mikosch, T., Gadrich, T., Klüppelberg, C. and Adler, R.J. (1995). Parameter estimation for ARMA models with infinite variance innovations. *Ann. Statist*. **23** 305–326. MR1331670
[19] Mikosch, T., Resnick, S. and Samorodnitsky, G. (2000). The maximum of the periodogram for a heavy-tailed sequence. *Ann. Probab.* **28** 885–908. MR1782277
[20] Monti, A.C. (1997). Empirical likelihood confidence regions in time series models. *Biometrika* **84** 395–405. MR1467055
[21] Nolan, J.P. (2015). *Stable Distributions – Models for Heavy Tailed Data*. Boston: Birkhäuser. To appear.
[22] Ogata, H. and Taniguchi, M. (2010). An empirical likelihood approach for non-Gaussian vector stationary processes and its application to minimum contrast estimation. *Aust. N. Z. J. Stat*. **52** 451–468. MR2791530
[23] Owen, A.B. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika* **75** 237–249. MR0946049
[24] Petrov, V.V. (1975). *Sums of Independent Random Variables*. New York: Springer. MR0388499
[25] Resnick, S. and Stărică, C. (1998). Tail index estimation for dependent data. *Ann. Appl. Probab.* **8** 1156–1183. MR1661160
[26] Resnick, S.I. and Stărică, C. (1996). Asymptotic behavior of Hill's estimator for autoregressive data. *Stoch. Models* **13** 703–723.
[27] Rosiński, J. and Woyczyński, W.A. (1987). Multilinear forms in Pareto-like random variables and product random measures. *Colloq. Math*. **51** 303–313. MR0891300
[28] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes. Stochastic Modeling. Stochastic Models with Infinite Variance*. New York: Chapman and Hall. MR1280932

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Maximum likelihood estimators uniformly minimize distribution variance among distribution unbiased estimators in exponential families

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We employ a parameter-free distribution estimation framework where estimators are random distributions and utilize the Kullback–Leibler (KL) divergence as a loss function. Wu and Vos [*J. Statist. Plann. Inference* **142** (2012) 1525–1536] show that when an estimator obtained from an i.i.d. sample is viewed as a random distribution, the KL risk of the estimator decomposes in a fashion parallel to the mean squared error decomposition when the estimator is a real-valued random variable. In this paper, we explore how conditional versions of distribution expectation ($E^{\dagger}$) can be defined so that a distribution version of the Rao–Blackwell theorem holds. We define distributional expectation and variance ($V^{\dagger}$) that also provide a decomposition of KL risk in exponential and mixture families. For exponential families, we show that the maximum likelihood estimator (viewed as a random distribution) is distribution unbiased and is the unique uniformly minimum distribution variance unbiased (UMV†U) estimator. Furthermore, we show that the MLE is robust against model specification in that if the true distribution does not belong to the exponential family, the MLE is UMV†U for the KL projection of the true distribution onto the exponential families provided these two distribution have the same expectation for the canonical statistic. To allow for estimators taking values outside of the exponential family, we include results for KL projection and define an extended projection to accommodate the non-existence of the MLE for families having discrete sample space. Illustrative examples are provided.

Keywords: distribution unbiasedness; extended KL projection; Kullback–Leibler loss; MVUE; Pythagorean relationship; Rao–Blackwell

References

[1] Amari, S.-i. (1990). *Differential-Geometrical Methods in Statistics*. New York: Springer.
[2] Blackwell, D. (1947). Conditional expectation and unbiased sequential estimation. *Ann. Math. Statist.* **18** 105–110. MR0019903
[3] Brown, L.D. (1986). Fundamentals of statistical exponential families with applications in statistical decision theory. In *Institute of Mathematical Statistics Lecture Notes – Monograph Series* **9**. Hayward, CA: IMS. MR0882001
[4] Čencov, N.N. (1982). *Statistical Decision Rules and Optimal Inference*. *Translations of Mathematical Monographs* **53**. Providence, RI: Amer. Math. Soc. Translation from the Russian edited by Lev J. Leifman. MR0645898

1350-7265

[5] Chow, M.S. and Fong, D.K.H. (1992). Simultaneous estimation of the Hardy–Weinberg proportions. *Canad. J. Statist.* **20** 291–296. MR1190573

[6] Csiszár, I. (1975). I-divergence geometry of probability distributions and minimization problems. *Ann. Probab.* **3** 146–158. MR0365798

[7] Darmois, G. (1935). Sur les lois de probabilité à estimation exhausitve. *C. R. Math. Acad. Sci. Paris* **200** 1265–1266.

[8] Denny, J.L. (1972). Sufficient statistics and discrete exponential families. *Ann. Math. Statist.* **43** 1320–1322. MR0339366

[9] Kass, R.E. and Vos, P.W. (1997). *Geometrical Foundations of Asymptotic Inference*. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. New York: Wiley. MR1461540

[10] Koopman, B.O. (1936). On distributions admitting a sufficient statistic. *Trans. Amer. Math. Soc.* **39** 399–409. MR1501854

[11] Lehmann, E.L. (1983). *Theory of Point Estimation*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. New York: Wiley. MR0702834

[12] Pitman, E.J.G. (1936). Sufficient statistics and intrinsic accuracy. *Math. Proc. Cambridge Philos. Soc.* **32** 567–579.

[13] Wu, Q. and Vos, P. (2012). Decomposition of Kullback–Leibler risk and unbiasedness for parameter-free estimators. *J. Statist. Plann. Inference* **142** 1525–1536. MR2891504

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Poisson convergence on the free Poisson algebra

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Based on recent findings by Bourguin and Peccati, we give a *fourth moment* type condition for an element of a free Poisson chaos of arbitrary order to converge to a free (centered) Poisson distribution. We also show that free Poisson chaos of order strictly greater than one do not contain any non-zero free Poisson random variables. We are also able to give a sufficient and necessary condition for an element of the first free Poisson chaos to have a free Poisson distribution. Finally, depending on the parity of the considered free Poisson chaos, we provide a general counterexample to the naive universality of the semicircular Wigner chaos established by Deya and Nourdin as well as a transfer principle between the Wigner and the free Poisson chaos.

Keywords: chaos structure; combinatorics of free Poisson random measures; contractions; diagram formulae; fourth moment theorem; free Poisson distribution; free probability; multiplication formula

References

[1] Bernhart, F.R. (1999). Catalan, Motzkin, and Riordan numbers. *Discrete Math*. **204** 73–112. MR1691863

[2] Bourguin, S. and Peccati, G. (2014). Semicircular limits on the free Poisson chaos: Counterexamples to a transfer principle. *J. Funct. Anal*. **267** 963–997. MR3217054

[3] Deya, A. and Nourdin, I. (2014). Invariance principles for homogeneous sums of free random variables. *Bernoulli* **20** 586–603. MR3178510

[4] Kemp, T., Nourdin, I., Peccati, G. and Speicher, R. (2012). Wigner chaos and the fourth moment. *Ann. Probab*. **40** 1577–1635. MR2978133

[5] Nica, A. and Speicher, R. (2006). *Lectures on the Combinatorics of Free Probability*. *London Mathematical Society Lecture Note Series* **335**. Cambridge: Cambridge Univ. Press. MR2266879

[6] Nourdin, I. and Peccati, G. (2009). Noncentral convergence of multiple integrals. *Ann. Probab*. **37** 1412–1426. MR2546749

[7] Nourdin, I. and Peccati, G. (2009). Stein's method on Wiener chaos. *Probab. Theory Related Fields* **145** 75–118. MR2520122

[8] Nourdin, I. and Peccati, G. (2012). *Normal Approximations with Malliavin Calculus*. *From Stein's Method to Universality*. *Cambridge Tracts in Mathematics* **192**. Cambridge: Cambridge Univ. Press. MR2962301

[9] Nourdin, I. and Peccati, G. (2013). Poisson approximations on the free Wigner chaos. *Ann. Probab*. **41** 2709–2723. MR3112929

[10] Nualart, D. and Peccati, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab*. **33** 177–193. MR2118863

[11] Peccati, G. (2012). The Chen–Stein method for Poisson functionals. Available at arXiv:1112.5051.

1350-7265

[12] Peccati, G. and Taqqu, M.S. (2011). *Wiener Chaos*: *Moments*, *Cumulants and Diagrams*. *A survey With Computer Implementation*. Milan: Springer. MR2791919
[13] Peccati, G. and Thäle, C. (2013). Gamma limits and U-statistics on the Poisson space. *ALEA Lat*. *Am*. *J*. *Probab*. *Math*. *Stat*. **10** 525–560. MR3083936

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Rates of convergence for multivariate normal approximation with applications to dense graphs and doubly indexed permutation statistics

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We provide a new general theorem for multivariate normal approximation on convex sets. The theorem is formulated in terms of a multivariate extension of Stein couplings. We apply the results to a homogeneity test in dense random graphs and to prove multivariate asymptotic normality for certain doubly indexed permutation statistics.

Keywords: dense graph limits; multivariate normal approximation; non-smooth metrics; permutation statistics; random graphs; Stein's method

References

[1] Aldous, D.J. (1981). Representations for partially exchangeable arrays of random variables. *J. Multivariate Anal.* **11** 581–598. MR0637937
[2] Ball, K. (1993). The reverse isoperimetric problem for Gaussian measure. *Discrete Comput. Geom.* **10** 411–420. MR1243336
[3] Barbour, A.D. (1990). Stein's method for diffusion approximations. *Probab. Theory Related Fields* **84** 297–322. MR1035659
[4] Barbour, A.D., Karoński, M. and Ruciński, A. (1989). A central limit theorem for decomposable random variables with applications to random graphs. *J. Combin. Theory Ser. B* **47** 125–145. MR1047781
[5] Bentkus, V. (2003). On the dependence of the Berry–Esseen bound on dimension. *J. Statist. Plann. Inference* **113** 385–402. MR1965117
[6] Bhattacharya, R.N. and Holmes, S. (2010). An exposition of Götze's estimation of the rate of convergence in the multivariate central limit theorem. Available at arXiv:1003.4254.
[7] Bollobás, B. and Riordan, O. (2009). Metrics for sparse graphs. In *Surveys in Combinatorics* 2009. *London Mathematical Society Lecture Note Series* **365** 211–287. Cambridge: Cambridge Univ. Press. MR2588543
[8] Borgs, C., Chayes, J.T., Lovász, L., Sós, V.T. and Vesztergombi, K. (2008). Convergent sequences of dense graphs I. Subgraph frequencies, metric properties and testing. *Adv. Math.* **219** 1801–1851. MR2455626
[9] Borgs, C., Chayes, J.T., Lovász, L., Sós, V.T. and Vesztergombi, K. (2012). Convergent sequences of dense graphs II. Multiway cuts and statistical physics. *Ann. of Math.* (2) **176** 151–219. MR2925382

1350-7265

[10] Chatterjee, S. and Meckes, E. (2008). Multivariate normal approximation using exchangeable pairs. *ALEA Lat. Am. J. Probab. Math. Stat.* **4** 257–283. MR2453473
[11] Chen, L.H.Y. and Röllin, A. (2010). Stein couplings for normal approximation. Preprint. Available at arXiv:1003.6039.
[12] Chung, F.R.K., Graham, R.L. and Wilson, R.M. (1989). Quasi-random graphs. *Combinatorica* **9** 345–362. MR1054011
[13] Diaconis, P. (1988). *Group Representations in Probability and Statistics*. *Institute of Mathematical Statistics Lecture Notes – Monograph Series* **11**. Hayward, CA: IMS. MR0964069
[14] Diaconis, P. and Janson, S. (2008). Graph limits and exchangeable random graphs. *Rend. Mat. Appl.* (7) **28** 33–61. MR2463439
[15] Fang, X. (2012). Multivariate, combinatorial and discretized normal approximations by Stein's methods. Ph.D. thesis, National Univ. Singapore.
[16] Fulman, J. (2004). Stein's method and non-reversible Markov chains. In *Stein's Method: Expository Lectures and Applications*. *Institute of Mathematical Statistics Lecture Notes – Monograph Series* **46** 69–77. Beachwood, OH: IMS. MR2118603
[17] Goldstein, L. and Rinott, Y. (1996). Multivariate normal approximations by Stein's method and size bias couplings. *J. Appl. Probab.* **33** 1–17. MR1371949
[18] Götze, F. (1991). On the rate of convergence in the multivariate CLT. *Ann. Probab.* **19** 724–739. MR1106283
[19] Hardy, M. (2006). Combinatorics of partial derivatives. *Electron. J. Combin.* **13** 1–13 (electronic). MR2200529
[20] Janson, S. and Nowicki, K. (1991). The asymptotic distributions of generalized U-statistics with applications to random graphs. *Probab. Theory Related Fields* **90** 341–375. MR1133371
[21] Lovász, L. and Szegedy, B. (2011). Finitely forcible graphons. *J. Combin. Theory Ser. B* **101** 269–301. MR2802882
[22] Pao, H., Coppersmith, G.A. and Priebe, C.E. (2011). Statistical inference on random graphs: Comparative power analyses via Monte Carlo. *J. Comput. Graph. Statist.* **20** 395–416. MR2847801
[23] Raič, M. (2004). A multivariate CLT for decomposable random vectors with finite second moments. *J. Theoret. Probab.* **17** 573–603. MR2091552
[24] Reinert, G. and Röllin, A. (2009). Multivariate normal approximation with Stein's method of exchangeable pairs under a general linearity condition. *Ann. Probab.* **37** 2150–2173. MR2573554
[25] Rinott, Y. and Rotar, V. (1996). A multivariate CLT for local dependence with $n^{-1/2}\log n$ rate and applications to multivariate graph related statistics. *J. Multivariate Anal.* **56** 333–350. MR1379533
[26] Zhao, L., Bai, Z., Chao, C.-C. and Liang, W.-Q. (1997). Error bound in a central limit theorem of double-indexed permutation statistics. *Ann. Statist.* **25** 2210–2227. MR1474091

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Integrability conditions for space–time stochastic integrals: Theory and applications

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We derive explicit integrability conditions for stochastic integrals taken over time and space driven by a random measure. Our main tool is a canonical decomposition of a random measure which extends the results from the purely temporal case. We show that the characteristics of this decomposition can be chosen as predictable strict random measures, and we compute the characteristics of the stochastic integral process. We apply our conditions to a variety of examples, in particular to ambit processes, which represent a rich model class.

Keywords: ambit process; continuous-time moving average; integrability conditions; Lévy basis; martingale measure; predictable characteristics; random measure; stochastic integration; stochastic partial differential equation; supCARMA; supCOGARCH; supOU; Volterra process

References

[1] Albeverio, S., Wu, J.-L. and Zhang, T.-S. (1998). Parabolic SPDEs driven by Poisson white noise. *Stochastic Process. Appl.* **74** 21–36. MR1624076

[2] Andresen, A., Benth, F.E., Koekebakker, S. and Zakamulin, V. (2014). The CARMA interest rate model. *Int. J. Theor. Appl. Finance* **17** 1–27. MR3198712

[3] Applebaum, D. and Wu, J.-L. (2000). Stochastic partial differential equations driven by Lévy space–time white noise. *Random Oper. Stoch. Equ.* **8** 245–259. MR1796675

[4] Barndorff-Nielsen, O.E. (2000). Superposition of Ornstein–Uhlenbeck type processes. *Theory Probab. Appl.* **45** 175–194.

[5] Barndorff-Nielsen, O.E., Benth, F.E. and Veraart, A.E.D. (2011). Ambit processes and stochastic partial differential equations. In *Advanced Mathematical Methods for Finance* (G. di Nunno and B. Øksendal, eds.) 35–74. Berlin: Springer. MR2752540

[6] Barndorff-Nielsen, O.E., Benth, F.E. and Veraart, A.E.D. (2013). Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. *Bernoulli* **19** 803–845. MR3079297

[7] Barndorff-Nielsen, O.E. and Schmiegel, J. (2007). Ambit processes: With applications to turbulence and tumour growth. In *Stochastic Analysis and Applications* (F.E. Benth, ed.). *Abel Symp.* **2** 93–124. Berlin: Springer. MR2397785

[8] Barndorff-Nielsen, O.E. and Shephard, N. (2001). Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **63** 167–241. MR1841412

[9] Barndorff-Nielsen, O.E. and Stelzer, R. (2011). Multivariate supOU processes. *Ann. Appl. Probab.* **21** 140–182. MR2759198

[10] Barndorff-Nielsen, O.E. and Veraart, A.E.D. (2012). Stochastic volatility of volatility and variance risk premia. *J. Financ. Econom.* **11** 1–46.

1350-7265

[11] Basse-O'Connor, A., Graversen, S.-E. and Pedersen, J. (2012). Multiparameter processes with stationary increments: Spectral representation and integration. *Electron. J. Probab.* **17** 1–21. MR2968681
[12] Basse-O'Connor, A., Graversen, S.-E. and Pedersen, J. (2014). Stochastic integration on the real line. *Theory Probab. Appl.* **58** 193–215.
[13] Behme, A., Chong, C. and Klüppelberg, C. (2013). Superposition of COGARCH processes. Preprint. Available at arXiv:1305.2296 [math.PR].
[14] Bichteler, K. (2002). *Stochastic Integration with Jumps*. *Encyclopedia of Mathematics and Its Applications* **89**. Cambridge: Cambridge Univ. Press. MR1906715
[15] Bichteler, K. and Jacod, J. (1983). Random measures and stochastic integration. In *Theory and Application of Random Fields* (*Bangalore*, 1982). *Lecture Notes in Control and Inform. Sci.* **49** 1–18. Berlin: Springer. MR0799929
[16] Billingsley, P. (1995). *Probability and Measure*, 3rd ed. *Wiley Series in Probability and Mathematical Statistics*. New York: Wiley. MR1324786
[17] Cambanis, S., Nolan, J.P. and Rosiński, J. (1990). On the oscillation of infinitely divisible and some other processes. *Stochastic Process. Appl.* **35** 87–97. MR1062585
[18] Chang, D.K. and Rao, M.M. (1983). Bimeasures and sampling theorems for weakly harmonizable processes. *Stoch. Anal. Appl.* **1** 21–55. MR0700356
[19] Cherny, A. and Shiryaev, A. (2005). On stochastic integrals up to infinity and predictable criteria for integrability. In *Séminaire de Probabilités XXXVIII*. *Lecture Notes in Math.* **1857** 165–185. Berlin: Springer. MR2126973
[20] Dalang, R.C. (1999). Extending the martingale measure stochastic integral with applications to spatially homogeneous S.P.D.E.'s. *Electron. J. Probab.* **4** 1–29 (electronic). MR1684157
[21] Doob, J.L. (1953). *Stochastic Processes*. New York: Wiley. MR0058896
[22] Émery, M. (1979). Une topologie sur l'espace des semimartingales. In *Séminaire de Probabilités XIII* (C. Dellacherie, P.A. Meyer and M. Weil, eds.) 260–280. Berlin: Springer.
[23] Fasen, V. (2009). Extremes of Lévy driven mixed MA processes with convolution equivalent distributions. *Extremes* **12** 265–296. MR2533953
[24] Fasen, V. and Klüppelberg, C. (2007). Extremes of supOU processes. In *Stochastic Analysis and Applications* (F.E. Benth, G. di Nunno, B. Øksendal, T. Lindstrøm and T. Zhang, eds.). *Abel Symp.* **2** 340–359 Heidelberg: Springer.
[25] Fuchs, F. and Stelzer, R. (2013). Mixing conditions for multivariate infinitely divisible processes with an application to mixed moving averages and the supOU stochastic volatility model. *ESAIM Probab. Stat.* **17** 455–471. MR3070886
[26] Itô, K. (1944). Stochastic integral. *Proc. Imp. Acad. Tokyo* **20** 519–524. MR0014633
[27] Jacod, J., Klüppelberg, C. and Müller, G. (2012). Functional relationships between price and volatility jumps and their consequences for discretely observed data. *J. Appl. Probab.* **49** 901–914. MR3058978
[28] Jacod, J. and Shiryaev, A.N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Berlin: Springer. MR1943877
[29] Jensen, E.BV., Jónsdóttir, KÝ., Schmiegel, J. and Barndorff-Nielsen, O.E. (2007). Spatio-temporal modelling: With a view to biological growth. In *Statistical Methods for Spatio-Temporal Systems* (B. Finkenstädt, L. Held and V. Isham, eds.) 47–76. Boca Raton, FL: Chapman and Hall.
[30] Jónsdóttir, K.Ý., Rønn-Nielsen, A., Mouridsen, K. and Jensen, E.B.V. (2013). Lévy-based modelling in brain imaging. *Scand. J. Stat.* **40** 511–529. MR3091695
[31] Kallenberg, O. (1986). *Random Measures*, 4th ed. Berlin: Akademie-Verlag. MR0854102
[32] Klüppelberg, C., Lindner, A. and Maller, R. (2004). A continuous-time GARCH process driven by a Lévy process: Stationarity and second-order behaviour. *J. Appl. Probab.* **41** 601–622. MR2074811

[33] Kurtz, T.G. and Protter, P.E. (1996). Weak convergence of stochastic integrals and differential equations. II. Infinite-dimensional case. In *Probabilistic Models for Nonlinear Partial Differential Equations* (*Montecatini Terme*, 1995). *Lecture Notes in Math*. **1627** 197–285. Berlin: Springer. MR1431303
[34] Kwapień, S. and Woyczyński, W.A. (1992). *Random Series and Stochastic Integrals*: *Single and Multiple*. *Probability and Its Applications*. Boston: Birkhäuser. MR1167198
[35] Lebedev, V.A. (1995). Behavior of random measures under a change of filtration. *Theory Probab. Appl*. **40** 645–652.
[36] Marcus, M.B. and Rosiński, J. (2005). Continuity and boundedness of infinitely divisible processes: A Poisson point process approach. *J. Theoret. Probab.* **18** 109–160. MR2132274
[37] Marquardt, T. and James, L.F. (2007). Generating long memory models based on CARMA processes. Technical report, Technische Univ. München.
[38] Mémin, J. (1980). Espaces de semi martingales et changement de probabilité. *Z. Wahrsch. Verw. Gebiete* **52** 9–39. MR0568256
[39] Métivier, M. and Pellaumail, J. (1977). Mesures stochastiques à valeurs dans des espaces L_0. *Z. Wahrsch. Verw. Gebiete* **40** 101–114. MR0471080
[40] Moser, M. and Stelzer, R. (2013). Functional regular variation of Lévy-driven multivariate mixed moving average processes. *Extremes* **16** 351–382. MR3101877
[41] Mueller, C. (1998). The heat equation with Lévy noise. *Stochastic Process. Appl.* **74** 67–82. MR1624088
[42] Mytnik, L. (2002). Stochastic partial differential equation driven by stable noise. *Probab. Theory Related Fields* **123** 157–201. MR1900321
[43] Nagasawa, M. and Tanaka, H. (1999). Stochastic differential equations of pure-jumps in relativistic quantum theory. *Chaos Solitons Fractals* **10** 1265–1280. MR1697664
[44] Pedersen, J. (2003). The Lévy–Itô decomposition of an independently scattered random measure. Technical report, MaPhySto, Univ. Aarhus.
[45] Pipiras, V. and Taqqu, M.S. (2000). Integration questions related to fractional Brownian motion. *Probab. Theory Related Fields* **118** 251–291. MR1790083
[46] Rajput, B.S. and Rosiński, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields* **82** 451–487. MR1001524
[47] Rosiński, J. (1989). On path properties of certain infinitely divisible processes. *Stochastic Process. Appl.* **33** 73–87. MR1027109
[48] Saint Loubert Bié, E. (1998). Étude d'une EDPS conduite par un bruit poissonnien. *Probab. Theory Related Fields* **111** 287–321. MR1633586
[49] Sato, K.-I. (2004). Stochastic integrals in additive processes and application to semi-Lévy processes. *Osaka J. Math.* **41** 211–236. MR2040073
[50] Todorov, V. and Tauchen, G. (2006). Simulation methods for Lévy-driven continuous-time autoregressive moving average (CARMA) stochastic volatility models. *J. Bus. Econom. Statist.* **24** 455–469. MR2328465
[51] Walsh, J.B. (1986). An introduction to stochastic partial differential equations. In *École d'Été de Probabilités de Saint-Flour XIV* – 1984. *Lecture Notes in Math.* **1180** 265–439. Berlin: Springer. MR0876085

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Optimal method in multiple regression with structural changes

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In this paper, we consider an estimation problem of the regression coefficients in multiple regression models with several unknown change-points. Under some realistic assumptions, we propose a class of estimators which includes as a special cases shrinkage estimators (SEs) as well as the unrestricted estimator (UE) and the restricted estimator (RE). We also derive a more general condition for the SEs to dominate the UE. To this end, we generalize some identities for the evaluation of the bias and risk functions of shrinkage-type estimators. As illustrative example, our method is applied to the "gross domestic product" data set of 10 countries whose USA, Canada, UK, France and Germany. The simulation results corroborate our theoretical findings.

Keywords: ADB; ADR; change-points; multiple regression; pre-test estimators; restricted estimator; shrinkage estimators; unrestricted estimator

References

[1] Bai, J. and Perron, P. (2003). Computation and analysis of multiple structural change models. *J. Appl. Econometr.* **18** 1–22.
[2] Baranchick, A. (1964). Multiple regression and estimation of the mean of a multivariate normal distribution. Technical Report No. 51, Dept. Statistics, Stanford Univ.
[3] Braun, J.V. and Muller, H.G. (1998). Statistical methods for DNA sequence segmentation. *Statist. Sci.* **13** 142–162.
[4] Broemeling, L.D. and Tsurumi, H. (1987). *Econometrics and Structural Change*. *Statistics*: *Textbooks and Monographs* **74**. New York: Dekker, Inc. MR0922263
[5] Fu, Y.-X. and Curnow, R.N. (1990). Locating a changed segment in a sequence of Bernoulli variables. *Biometrika* **77** 295–304. MR1064801
[6] Fu, Y.-X. and Curnow, R.N. (1990). Maximum likelihood estimation of multiple change points. *Biometrika* **77** 563–573. MR1087847
[7] Hossain, S., Doksum, K.A. and Ahmed, S.E. (2009). Positive shrinkage, improved pretest and absolute penalty estimators in partially linear models. *Linear Algebra Appl.* **430** 2749–2761. MR2509855
[8] James, W. and Stein, C. (1961). Estimation with quadratic loss. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob.* **I** 361–379. Berkeley, CA: Univ. California Press. MR0133191
[9] Judge, G.G. and Bock, M.E. (1978). *The Statistical Implications of Pre-Test and Stein-Rule Estimators in Econometrics*. Amsterdam: North-Holland. MR0483199
[10] Judge, G.G. and Mittelhammer, R.C. (2004). A semiparametric basis for combining estimation problems under quadratic loss. *J. Amer. Statist. Assoc.* **99** 479–487. MR2062833
[11] Lombard, F. (1986). The change-point problem for angular data: A nonparametric approach. *Technometrics* **28** 391–397.

1350-7265

[12] McLeish, D.L. (1977). On the invariance principle for nonstationary mixingales. *Ann. Probab.* **5** 616–621. MR0445583

[13] Nkurunziza, S. (2011). Shrinkage strategy in stratified random sample subject to measurement error. *Statist. Probab. Lett.* **81** 317–325. MR2764300

[14] Nkurunziza, S. (2012). The risk of pretest and shrinkage estimators. *Statistics* **46** 305–312. MR2929155

[15] Nkurunziza, S. and Ahmed, S.E. (2010). Shrinkage drift parameter estimation for multi-factor Ornstein–Uhlenbeck processes. *Appl. Stoch. Models Bus. Ind.* **26** 103–124. MR2722886

[16] Perron, P. and Qu, Z. (2006). Estimating restricted structural change models. *J. Econometrics* **134** 373–399. MR2328414

[17] Perron, P. and Yabu, T. (2009). Testing for shifts in trend with an integrated or stationary noise component. *J. Bus. Econom. Statist.* **27** 369–396. MR2554242

[18] Saleh, A.K.Md.E. (2006). *Theory of Preliminary Test and Stein-Type Estimation with Applications*. *Wiley Series in Probability and Statistics*. Hoboken, NJ: Wiley. MR2218139

[19] Tan, Z. (2014). Improved minimax estimation of a multivariate normal mean under heteroscedasticity. *Bernoulli*. To appear.

[20] Zeileis, A., Kleiber, C., Krämer, W. and Hornik, K. (2003). Testing and dating of structural changes in practice. *Comput. Statist. Data Anal.* **44** 109–123. MR2019790

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Weak noise and non-hyperbolic unstable fixed points: Sharp estimates on transit and exit times

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We consider certain one dimensional ordinary stochastic differential equations driven by additive Brownian motion of variance ε^2. When $\varepsilon = 0$ such equations have an unstable non-hyperbolic fixed point and the drift near such a point has a power law behavior. For $\varepsilon > 0$ small, the fixed point property disappears, but it is replaced by a random escape or transit time which diverges as $\varepsilon \searrow 0$. We show that this random time, under suitable (easily guessed) rescaling, converges to a limit random variable that essentially depends only on the power exponent associated to the fixed point. Such random variables, or laws, have therefore a universal character and they arise of course in a variety of contexts. We then obtain quantitative sharp estimates, notably tail properties, on these universal laws.

Keywords: martingale theory; Schrödinger equation; stochastic differential equations; unstable non-hyperbolic fixed points; WKB analysis

References

[1] Arecchi, F.T., Politi, A. and Ulivi, L. (1982). Stochastic time description of transitions in unstable and multistable systems. *Il Nuovo Cimento* **71B** 119–154.
[2] Bakhtin, Y. (2008). Exit asymptotics for small diffusion about an unstable equilibrium. *Stochastic Process. Appl.* **118** 839–851. MR2411523
[3] Bender, C.M. and Orszag, S.A. (1978). *Advanced Mathematical Methods for Scientists and Engineers. International Series in Pure and Applied Mathematics*. New York: McGraw-Hill. MR0538168
[4] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge: Cambridge Univ. Press. MR0898871
[5] Cáceres, M.O., Fuentes, M.A. and Budde, C.E. (1997). Stochastic escape processes from a non-symmetric potential normal form. II. The marginal case. *J. Phys. A* **30** 2287–2296. MR1457379
[6] Carinci, G. (2013). Random hysteresis loops. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 307–339. MR3088372
[7] Coddington, E.A. and Levinson, N. (1955). *Theory of Ordinary Differential Equations*. New York: McGraw-Hill. MR0069338
[8] Colet, P., De Pasquale, F., Caceres, M.O. and San Miguel, M. (1990). Theory for relaxation at a subcritical pitchfork bifurcation. *Phys. Rev. A* (3) **41** 1901–1911.
[9] Conway, J.B. (1978). *Functions of One Complex Variable*, 2nd ed. *Graduate Texts in Mathematics* **11**. New York: Springer. MR0503901
[10] Costin, O. (2009). *Asymptotics and Borel Summability. Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics* **141**. Boca Raton, FL: CRC Press. MR2474083

[11] Flajolet, P., Gourdon, X. and Dumas, P. (1995). Mellin transforms and asymptotics: Harmonic sums. *Theoret. Comput. Sci.* **144** 3–58. Special volume on mathematical analysis of algorithms. MR1337752
[12] Freidlin, M.I. and Wentzell, A.D. (1998). *Random Perturbations of Dynamical Systems*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **260**. New York: Springer. Translated from the 1979 Russian original by Joseph Szücs. MR1652127
[13] Giacomin, G. and Merle, M. Scaling limits of transit and exit times from non hyperbolic fixed points. Preprint.
[14] Groisman, P. and Rossi, J.D. (2007). Explosion time in stochastic differential equations with small diffusion. *Electron. J. Differential Equations* **2007** No. 140, 9. MR2349968
[15] Horsthemke, W. and Lefever, R. (1984). *Noise-Induced Transitions: Theory and Applications in Physics, Chemistry, and Biology*. *Springer Series in Synergetics* **15**. Berlin: Springer. MR0724433
[16] Karatzas, I. and Shreve, S.E. (1988). *Brownian Motion and Stochastic Calculus*. *Graduate Texts in Mathematics* **113**. New York: Springer. MR0917065
[17] Le Gall, J.-F. (2013). *Mouvement Brownien, Martingales et Calcul Stochastique*. *Mathématiques & Applications* (*Berlin*) [*Mathematics & Applications*] **71**. Heidelberg: Springer. MR3184878
[18] Lindner, B., García-Ojalvo, J. and Schimansky-Geier, L. (2004). Effects of noise in excitable systems. *Phys. Rep.* **392** 321–424.
[19] Lindner, B., Longtin, A. and Bulsara, A. (2003). Analytic expressions for rate and CV of a type I neuron driven by white Gaussian noise. *Neural Comput.* **15** 1760–1787.
[20] McGill, P. (1981). A direct proof of the Ray–Knight theorem. In *Seminar on Probability, XV* (*Univ. Strasbourg, Strasbourg*, 1979/1980) (*French*). *Lecture Notes in Math.* **850** 206–209. Berlin: Springer. MR0622564
[21] Nakagawa, K. (2007). Application of Tauberian theorem to the exponential decay of the tail probability of a random variable. *IEEE Trans. Inform. Theory* **53** 3239–3249. MR2417689
[22] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Berlin: Springer. MR1725357
[23] Sibuya, Y. (1975). *Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient*. *North-Holland Mathematics Studies* **18**. New York: North-Holland. MR0486867
[24] Sigeti, D. and Horsthemke, W. (1989). Pseudo-regular oscillations induced by external noise. *J. Stat. Phys.* **54** 1217–1222. MR0993147
[25] Titchmarsh, E.C. (1962). *Eigenfunction Expansions Associated with Second-Order Differential Equations. Part i*, 2nd ed. Oxford: Clarendon Press. MR0176151

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Adaptive-treed bandits

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We describe a novel algorithm for noisy global optimisation and continuum-armed bandits, with good convergence properties over any continuous reward function having finitely many polynomial maxima. Over such functions, our algorithm achieves square-root regret in bandits, and inverse-square-root error in optimisation, without prior information.

Our algorithm works by reducing these problems to tree-armed bandits, and we also provide new results in this setting. We show it is possible to adaptively combine multiple trees so as to minimise the regret, and also give near-matching lower bounds on the regret in terms of the zooming dimension.

Keywords: bandits on taxonomies; continuum-armed bandits; noisy global optimisation; tree-armed bandits; zooming dimension

References

[1] Agrawal, R. (1995). The continuum-armed bandit problem. *SIAM J. Control Optim.* **33** 1926–1951. MR1358102

[2] Auer, P., Cesa-Bianchi, N. and Fischer, P. (2002). Finite-time analysis of the multiarmed bandit problem. *Mach. Learn.* **47** 235–256.

[3] Auer, P., Ortner, R. and Szepesvári, C. (2007). Improved rates for the stochastic continuum-armed bandit problem. In *Learning Theory. Lecture Notes in Computer Science* **4539** 454–468. Berlin: Springer. MR2397605

[4] Bubeck, S. (2010). Jeux de bandits et fondations du clustering. Ph.D. thesis, Univ. Lille 1.

[5] Bubeck, S. and Cesa-Bianchi, N. (2012). Regret analysis of stochastic and nonstochastic multi-armed bandit problems. *Found. Trends Mach. Learn.* **5** 1–122.

[6] Bubeck, S., Munos, R. and Stoltz, G. (2009). Pure exploration in multi-armed bandits problems. In *Algorithmic Learning Theory. Lecture Notes in Computer Science* **5809** 23–37. Berlin: Springer. MR2564216

[7] Bubeck, S., Munos, R., Stoltz, G. and Szepesvári, C. (2011). $\mathcal{X}$-armed bandits. *J. Mach. Learn. Res.* **12** 1655–1695. MR2813150

[8] Bubeck, S., Stoltz, G. and Yu, J. (2011). Lipschitz bandits without the Lipschitz constant. In *Algorithmic Learning Theory* **22** 144–158. New York: Springer.

[9] Bull, A.D. (2014). Supplement to "Adaptive-treed bandits." DOI:10.3150/14-BEJ644SUPP.

[10] Cope, E.W. (2009). Regret and convergence bounds for a class of continuum-armed bandit problems. *IEEE Trans. Automat. Control* **54** 1243–1253. MR2532613

[11] Frazier, P., Powell, W. and Dayanik, S. (2009). The knowledge-gradient policy for correlated normal beliefs. *INFORMS J. Comput.* **21** 599–613. MR2588343

[12] Gelly, S., Kocsis, L., Schoenauer, M., Sebag, M., Silver, D., Szepesvári, C. and Teytaud, O. (2012). The grand challenge of computer Go: Monte Carlo tree search and extensions. *Comm. ACM* **55** 106–113.

[13] Huang, D., Allen, T.T., Notz, W.I. and Miller, R.A. (2006). Sequential kriging optimization using multiple-fidelity evaluations. *Struct. Multidiscip. Optim.* **32** 369–382.

[14] Kiefer, J. and Wolfowitz, J. (1952). Stochastic estimation of the maximum of a regression function. *Ann. Math. Stat.* **23** 462–466. MR0050243
[15] Kleinberg, R., Slivkins, A. and Upfal, E. (2008). Multi-armed bandits in metric spaces. In *STOC'08* 681–690. New York: ACM. MR2582691
[16] Kleinberg, R.D. (2005). Nearly tight bounds for the continuum-armed bandit problem. In *Advances in Neural Information Processing Systems* **17** 697–704. Cambridge, MA: MIT Press.
[17] Müller, H.-G. (1985). Kernel estimators of zeros and of location and size of extrema of regression functions. *Scand. J. Stat.* **12** 221–232. MR0817940
[18] Munos, R. (2011). Optimistic optimization of a deterministic function without the knowledge of its smoothness. In *Advances in Neural Information Processing Systems* **24** 783–791. Cambridge, MA: MIT Press.
[19] Parsopoulos, K.E. and Vrahatis, M.N. (2002). Recent approaches to global optimization problems through particle swarm optimization. *Nat. Comput.* **1** 235–306. MR1999724
[20] Slivkins, A. (2011). Multi-armed bandits on implicit metric spaces. In *Advances in Neural Information Processing Systems* **24** 1602–1610. Cambridge, MA: MIT Press.
[21] Srinivas, N., Krause, A., Kakade, S.M. and Seeger, M. (2010). Gaussian process optimization in the bandit setting: No regret and experimental design. In *Proceedings of the* 27*th International Conference on Machine Learning* (*ICML*-10).
[22] Valko, M., Carpentier, A. and Munos, R. (2013). Stochastic simultaneous optimistic optimization. In *Proceedings of the* 30*th International Conference on Machine Learning* (*ICML*-13) 19–27.
[23] Yu, J.Y. and Mannor, S. (2011). Unimodal bandits. In *Proceedings of the* 28*th International Conference on Machine Learning* (*ICML*-11).

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Geometric median and robust estimation in Banach spaces

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In many real-world applications, collected data are contaminated by noise with heavy-tailed distribution and might contain outliers of large magnitude. In this situation, it is necessary to apply methods which produce reliable outcomes even if the input contains corrupted measurements. We describe a general method which allows one to obtain estimators with tight concentration around the true parameter of interest taking values in a Banach space. Suggested construction relies on the fact that the geometric median of a collection of independent "weakly concentrated" estimators satisfies a much stronger deviation bound than each individual element in the collection. Our approach is illustrated through several examples, including sparse linear regression and low-rank matrix recovery problems.

Keywords: distributed computing; heavy-tailed noise; large deviations; linear models; low-rank matrix estimation; principal component analysis; robust estimation

References

[1] Alon, N., Matias, Y. and Szegedy, M. (1996). The space complexity of approximating the frequency moments. In *Proceedings of the Twenty-Eighth Annual ACM Symposium on the Theory of Computing* (*Philadelphia*, *PA*, 1996) 20–29. New York: ACM. MR1427494

[2] Audibert, J.-Y. and Catoni, O. (2011). Robust linear least squares regression. *Ann*. *Statist*. **39** 2766–2794. MR2906886

[3] Bickel, P.J. and Levina, E. (2008). Regularized estimation of large covariance matrices. *Ann*. *Statist*. **36** 199–227. MR2387969

[4] Bickel, P.J. and Levina, E. (2008). Covariance regularization by thresholding. *Ann*. *Statist*. **36** 2577–2604. MR2485008

[5] Bickel, P.J., Ritov, Y. and Tsybakov, A.B. (2009). Simultaneous analysis of lasso and Dantzig selector. *Ann*. *Statist*. **37** 1705–1732. MR2533469

[6] Bose, P., Maheshwari, A. and Morin, P. (2003). Fast approximations for sums of distances, clustering and the Fermat–Weber problem. *Comput*. *Geom*. **24** 135–146. MR1947896

[7] Bubeck, S., Cesa-Bianchi, N. and Lugosi, G. (2013). Bandits with heavy tail. *IEEE Trans*. *Inform*. *Theory* **59** 7711–7717. MR3124669

[8] Bühlmann, P. and van de Geer, S. (2011). *Statistics for High-Dimensional Data*. *Methods*, *Theory and Applications*. *Springer Series in Statistics*. Heidelberg: Springer. MR2807761

[9] Candès, E.J., Li, X., Ma, Y. and Wright, J. (2011). Robust principal component analysis? *J*. *ACM* **58** Art. 11, 37. MR2811000

[10] Candès, E.J. and Plan, Y. (2011). Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements. *IEEE Trans*. *Inform*. *Theory* **57** 2342–2359. MR2809094

[11] Candès, E.J. and Recht, B. (2009). Exact matrix completion via convex optimization. *Found. Comput. Math.* **9** 717–772. MR2565240

[12] Candès, E.J., Romberg, J.K. and Tao, T. (2006). Stable signal recovery from incomplete and inaccurate measurements. *Comm. Pure Appl. Math.* **59** 1207–1223. MR2230846

[13] Cardot, H., Cénac, P. and Zitt, P.-A. (2013). Efficient and fast estimation of the geometric median in Hilbert spaces with an averaged stochastic gradient algorithm. *Bernoulli* **19** 18–43. MR3019484

[14] Catoni, O. (2012). Challenging the empirical mean and empirical variance: A deviation study. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 1148–1185. MR3052407

[15] Chandrasekaran, R. and Tamir, A. (1990). Algebraic optimization: The Fermat–Weber location problem. *Math. Program.* **46** 219–224. MR1047376

[16] Davis, C. and Kahan, W.M. (1970). The rotation of eigenvectors by a perturbation. III. *SIAM J. Numer. Anal.* **7** 1–46. MR0264450

[17] Haldane, J.B.S. (1948). Note on the median of a multivariate distribution. *Biometrika* **35** 414–417.

[18] Hsu, D. and Sabato, S. (2013). Loss minimization and parameter estimation with heavy tails. Preprint. Available at arXiv:1307.1827.

[19] Huber, P.J. and Ronchetti, E.M. (2009). *Robust Statistics*, 2nd ed. *Wiley Series in Probability and Statistics*. Hoboken, NJ: Wiley. MR2488795

[20] Hubert, M., Rousseeuw, P.J. and Van Aelst, S. (2008). High-breakdown robust multivariate methods. *Statist. Sci.* **23** 92–119. MR2431867

[21] Ioffe, A.D. and Tikhomirov, V.M. (1974). *Theory of Extremal Problems*. Moscow: Nauka.

[22] Kemperman, J.H.B. (1987). The median of a finite measure on a Banach space. In *Statistical Data Analysis Based on the L_1-Norm and Related Methods* (*Neuchâtel*, 1987) 217–230. Amsterdam: North-Holland. MR0949228

[23] Koltchinskii, V. (2011). *Oracle Inequalities in Empirical Risk Minimization and Sparse Recovery Problems*. *Lecture Notes in Math.* **2033**. *Lectures from the 38th Probability Summer School held in Saint-Flour 2008. École d'Été de Probabilités de Saint-Flour*. Heidelberg: Springer. MR2829871

[24] Koltchinskii, V., Lounici, K. and Tsybakov, A.B. (2011). Nuclear-norm penalization and optimal rates for noisy low-rank matrix completion. *Ann. Statist.* **39** 2302–2329. MR2906869

[25] Kuhn, H.W. (1973). A note on Fermat's problem. *Math. Program.* **4** 98–107. MR0316102

[26] Lambert-Lacroix, S. and Zwald, L. (2011). Robust regression through the Huber's criterion and adaptive lasso penalty. *Electron. J. Stat.* **5** 1015–1053. MR2836768

[27] Ledoit, O. and Wolf, M. (2012). Nonlinear shrinkage estimation of large-dimensional covariance matrices. *Ann. Statist.* **40** 1024–1060. MR2985942

[28] Lerasle, M. and Oliveira, R.I. (2011). Robust empirical mean estimators. Preprint. Available at arXiv:1112.3914.

[29] Lounici, K. (2014). High-dimensional covariance matrix estimation with missing observations. *Bernoulli* **20** 1029–1058. MR3217437

[30] Minsker, S. (2013). Geometric median and robust estimation in Banach spaces. Preprint. Available at http://sminsker.wordpress.com/publications/.

[31] Negahban, S. and Wainwright, M.J. (2011). Estimation of (near) low-rank matrices with noise and high-dimensional scaling. *Ann. Statist.* **39** 1069–1097. MR2816348

[32] Negahban, S. and Wainwright, M.J. (2012). Restricted strong convexity and weighted matrix completion: Optimal bounds with noise. *J. Mach. Learn. Res.* **13** 1665–1697. MR2930649

[33] Nemirovski, A. (2000). Topics in non-parametric statistics. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1998). *Lecture Notes in Math.* **1738** 85–277. Berlin: Springer. MR1775640

[34] Nemirovski, A. and Yudin, D. (1983). *Problem Complexity and Method Efficiency in Optimization*. New York: Wiley.

[35] Nguyen, N.H. and Tran, T.D. (2013). Robust Lasso with missing and grossly corrupted observations. *IEEE Trans. Inform. Theory* **59** 2036–2058. MR3043781

[36] Ostresh, L.M. Jr. (1978). On the convergence of a class of iterative methods for solving the Weber location problem. *Oper. Res.* **26** 597–609. MR0496728

[37] Overton, M.L. (1983). A quadratically convergent method for minimizing a sum of Euclidean norms. *Math. Program.* **27** 34–63. MR0712109

[38] Pearson, K. (1901). On lines and planes of closest fit to systems of points in space. *Lond. Edinb. Dubl. Phil. Mag. J. Sci.* **2** 559–572.

[39] Recht, B., Fazel, M. and Parrilo, P.A. (2010). Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Rev.* **52** 471–501. MR2680543

[40] Rohde, A. and Tsybakov, A.B. (2011). Estimation of high-dimensional low-rank matrices. *Ann. Statist.* **39** 887–930. MR2816342

[41] Small, C. (1990). A survey of multidimensional medians. *Internat. Statist. Rev.* **58** 263–277.

[42] Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **58** 267–288. MR1379242

[43] van der Vaart, A.W. and Wellner, J.A. (1996). *Weak Convergence and Empirical Processes*. *Springer Series in Statistics*. New York: Springer. MR1385671

[44] Vardi, Y. and Zhang, C.-H. (2000). The multivariate L_1-median and associated data depth. *Proc. Natl. Acad. Sci. USA* **97** 1423–1426 (electronic). MR1740461

[45] Weber, A. (1929). *Uber Den Standort der Industrien* (*Alfred Weber's Theory of the Location of Industries*). Chicago, IL: Univ. Chicago Press.

[46] Weiszfeld, E. (1937). Sur un problème de minimum dans l'espace. *Tohoku Math. J.* (2) **42** 274–280.

[47] Wright, J. and Ma, Y. (2010). Dense error correction via ℓ_1-minimization. *IEEE Trans. Inform. Theory* **56** 3540–3560. MR2799012

[48] Zhang, T. and Lerman, G. (2014). A novel M-estimator for robust PCA. *J. Mach. Learn. Res.* **15** 749–808.

[49] Zwald, L. and Blanchard, G. (2006). On the convergence of eigenspaces in kernel principal component analysis. In *Advances in Neural Information Processing Systems* **18** 1649–1656. Cambridge, MA: MIT Press.

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Density convergence in the Breuer–Major theorem for Gaussian stationary sequences

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Consider a Gaussian stationary sequence with unit variance $X = \{X_k; k \in \mathbb{N} \cup \{0\}\}$. Assume that the central limit theorem holds for a weighted sum of the form $V_n = n^{-1/2} \sum_{k=0}^{n-1} f(X_k)$, where f designates a finite sum of Hermite polynomials. Then we prove that the uniform convergence of the density of V_n towards the standard Gaussian density also holds true, under a mild additional assumption involving the causal representation of X.

Keywords: Breuer–Major theorem; density convergence; Gaussian stationary sequences; Malliavin calculus; moving average representation

References

[1] Arcones, M.A. (1994). Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. *Ann. Probab.* **22** 2242–2274. MR1331224
[2] Beran, J. (1994). *Statistics for Long-Memory Processes*. *Monographs on Statistics and Applied Probability* **61**. New York: Chapman and Hall. MR1304490
[3] Breuer, P. and Major, P. (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13** 425–441. MR0716933
[4] Brockwell, P.J. and Davis, R.A. (1991). *Time Series*: *Theory and Methods*, 2nd ed. *Springer Series in Statistics*. New York: Springer. MR1093459
[5] Carbery, A. and Wright, J. (2001). Distributional and L^q norm inequalities for polynomials over convex bodies in $\mathbb{R}^n$. *Math. Res. Lett.* **8** 233–248. MR1839474
[6] Chambers, D. and Slud, E. (1989). Central limit theorems for nonlinear functionals of stationary Gaussian processes. *Probab. Theory Related Fields* **80** 323–346. MR0976529
[7] Giraitis, L. and Surgailis, D. (1985). CLT and other limit theorems for functionals of Gaussian processes. *Z. Wahrsch. Verw. Gebiete* **70** 191–212. MR0799146
[8] Granger, C.W.J. and Joyeux, R. (1980). An introduction to long-memory time series models and fractional differencing. *J. Time Series Anal.* **1** 15–29. MR0605572
[9] Gubner, J.A. (2005). Theorems and fallacies in the theory of long-range-dependent processes. *IEEE Trans. Inform. Theory* **51** 1234–1239. MR2237996
[10] Hosking, J.R.M. (1981). Fractional differencing. *Biometrika* **68** 165–176. MR0614953
[11] Hu, Y., Lu, F. and Nualart, D. (2014). Convergence of densities of some functionals of Gaussian processes. *J. Funct. Anal.* **266** 814–875. MR3132731

1350-7265

[12] Nourdin, I. and Nualart, D. (2013). Fisher information and the fourth moment theorem. Preprint. Available at arXiv:1312.5841.

[13] Nourdin, I. and Peccati, G. (2009). Stein's method on Wiener chaos. *Probab. Theory Related Fields* **145** 75–118. MR2520122

[14] Nourdin, I. and Peccati, G. (2012). *Normal Approximations with Malliavin Calculus*: *From Stein's Method to Universality*. *Cambridge Tracts in Mathematics* **192**. Cambridge: Cambridge Univ. Press. MR2962301

[15] Nourdin, I., Peccati, G. and Podolskij, M. (2011). Quantitative Breuer–Major theorems. *Stochastic Process*. *Appl*. **121** 793–812. MR2770907

[16] Nourdin, I. and Poly, G. (2013). Convergence in total variation on Wiener chaos. *Stochastic Process*. *Appl*. **123** 651–674. MR3003367

[17] Nualart, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Berlin: Springer. MR2200233

[18] Nualart, D. and Peccati, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann*. *Probab*. **33** 177–193. MR2118863

[19] Palma, W. (2007). *Long-Memory Time Series*: *Theory and Methods*. *Wiley Series in Probability and Statistics*. Hoboken, NJ: Wiley. MR2297359

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Large deviations for 2-D stochastic Navier–Stokes equations driven by multiplicative *Lévy* noises

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In this paper, we establish a large deviation principle for two-dimensional stochastic Navier–Stokes equations driven by multiplicative *Lévy* noises. The weak convergence method introduced by Budhiraja, Dupuis and Maroulas [*Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2011) 725–747] plays a key role.

Keywords: Brownian motions; large deviations; Poisson random measures; Skorohod representation; stochastic Navier–Stokes equations; tightness

References

[1] Aldous, D. (1978). Stopping times and tightness. *Ann. Probab.* **6** 335–340. MR0474446
[2] Bensoussan, A. and Temam, R. (1973). Équations stochastiques du type Navier–Stokes. *J. Funct. Anal.* **13** 195–222. MR0348841
[3] Bessaih, H. and Millet, A. (2009). Large deviation principle and inviscid shell models. *Electron. J. Probab.* **14** 2551–2579. MR2570011
[4] Boué, M. and Dupuis, P. (1998). A variational representation for certain functionals of Brownian motion. *Ann. Probab.* **26** 1641–1659. MR1675051
[5] Brzeźniak, Z., Liu, W. and Zhu, J. (2014). Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise. *Nonlinear Anal. Real World Appl.* **17** 283–310. MR3158475
[6] Budhiraja, A., Chen, J. and Dupuis, P. (2013). Large deviations for stochastic partial differential equations driven by a Poisson random measure. *Stochastic Process. Appl.* **123** 523–560. MR3003362
[7] Budhiraja, A. and Dupuis, P. (2000). A variational representation for positive functionals of infinite dimensional Brownian motion. *Probab. Math. Statist.* **20** 39–61. MR1785237
[8] Budhiraja, A., Dupuis, P. and Maroulas, V. (2008). Large deviations for infinite dimensional stochastic dynamical systems. *Ann. Probab.* **36** 1390–1420. MR2435853
[9] Budhiraja, A., Dupuis, P. and Maroulas, V. (2011). Variational representations for continuous time processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** 725–747. MR2841073
[10] Cardon-Weber, C. (1999). Large deviations for a Burgers'-type SPDE. *Stochastic Process. Appl.* **84** 53–70. MR1720097
[11] Cerrai, S. and Röckner, M. (2004). Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. *Ann. Probab.* **32** 1100–1139. MR2044675
[12] Chenal, F. and Millet, A. (1997). Uniform large deviations for parabolic SPDEs and applications. *Stochastic Process. Appl.* **72** 161–186. MR1486551

[13] Chow, P.L. (1992). Large deviation problem for some parabolic Itô equations. *Comm. Pure Appl. Math.* **45** 97–120. MR1135925
[14] Chueshov, I. and Millet, A. (2010). Stochastic 2D hydrodynamical type systems: Well posedness and large deviations. *Appl. Math. Optim.* **61** 379–420. MR2609596
[15] Da Prato, G. and Zabczyk, J. (1992). *Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and Its Applications* **44**. Cambridge: Cambridge Univ. Press. MR1207136
[16] Duan, J. and Millet, A. (2009). Large deviations for the Boussinesq equations under random influences. *Stochastic Process. Appl.* **119** 2052–2081. MR2519356
[17] Flandoli, F. (1994). Dissipativity and invariant measures for stochastic Navier–Stokes equations. *NoDEA Nonlinear Differential Equations Appl.* **1** 403–423. MR1300150
[18] Flandoli, F. and Gatarek, D. (1995). Martingale and stationary solutions for stochastic Navier–Stokes equations. *Probab. Theory Related Fields* **102** 367–391. MR1339739
[19] Hairer, M. and Mattingly, J.C. (2006). Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing. *Ann. of Math.* (2) **164** 993–1032. MR2259251
[20] Ikeda, N. and Watanabe, S. (1981). *Stochastic Differential Equations and Diffusion Processes. North-Holland Mathematical Library* **24**. Amsterdam: North-Holland. MR0637061
[21] Jakubowski, A. (1986). On the Skorokhod topology. *Ann. Inst. Henri Poincaré Probab. Stat.* **22** 263–285. MR0871083
[22] Lions, J.-L. (1969). *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*. Paris: Dunod. MR0259693
[23] Liu, W. (2010). Large deviations for stochastic evolution equations with small multiplicative noise. *Appl. Math. Optim.* **61** 27–56. MR2575313
[24] Manna, U., Sritharan, S.S. and Sundar, P. (2009). Large deviations for the stochastic shell model of turbulence. *NoDEA Nonlinear Differential Equations Appl.* **16** 493–521. MR2525514
[25] Mikulevicius, R. and Rozovskii, B.L. (2005). Global L_2-solutions of stochastic Navier–Stokes equations. *Ann. Probab.* **33** 137–176. MR2118862
[26] Ren, J. and Zhang, X. (2005). Freidlin–Wentzell's large deviations for homeomorphism flows of non-Lipschitz SDEs. *Bull. Sci. Math.* **129** 643–655. MR2166732
[27] Ren, J. and Zhang, X. (2005). Schilder theorem for the Brownian motion on the diffeomorphism group of the circle. *J. Funct. Anal.* **224** 107–133. MR2139106
[28] Röckner, M. and Zhang, T. (2007). Stochastic evolution equations of jump type: Existence, uniqueness and large deviation principles. *Potential Anal.* **26** 255–279. MR2286037
[29] Röckner, M., Zhang, T. and Zhang, X. (2010). Large deviations for stochastic tamed 3D Navier–Stokes equations. *Appl. Math. Optim.* **61** 267–285. MR2585144
[30] Sowers, R.B. (1992). Large deviations for a reaction–diffusion equation with non-Gaussian perturbations. *Ann. Probab.* **20** 504–537. MR1143433
[31] Sritharan, S.S. and Sundar, P. (2006). Large deviations for the two-dimensional Navier–Stokes equations with multiplicative noise. *Stochastic Process. Appl.* **116** 1636–1659. MR2269220
[32] Święch, A. and Zabczyk, J. (2011). Large deviations for stochastic PDE with Lévy noise. *J. Funct. Anal.* **260** 674–723. MR2737394
[33] Temam, R. (1979). *Navier–Stokes Equations: Theory and Numerical Analysis*. Revised ed. *Studies in Mathematics and Its Applications* **2**. Amsterdam: North-Holland. With an appendix by F. Thomasset. MR0603444
[34] Temam, R. (1983). *Navier–Stokes Equations and Nonlinear Functional Analysis. CBMS-NSF Regional Conference Series in Applied Mathematics* **41**. Philadelphia, PA: SIAM. MR0764933
[35] Višik, M.I. and Fursikov, A.V. (1988). *Mathematical Problems of Statistical Hydromechanics. Mathematics and Its Applications* **9**. Dordrecht: Kluver. Translated from 1980 Russian original *Matematicheskie Zadachi Statisticheskoi Gidromekhaniki*. Moscow: Nauka. MR0591678

[36] Wang, W. and Duan, J. (2009). Reductions and deviations for stochastic partial differential equations under fast dynamical boundary conditions. *Stoch. Anal. Appl.* **27** 431–459. MR2523176

[37] Xu, T. and Zhang, T. (2009). Large deviation principles for 2-D stochastic Navier–Stokes equations driven by Lévy processes. *J. Funct. Anal.* **257** 1519–1545. MR2541279

[38] Yang, X., Zhai, J. and Zhang, T. (2014). Large deviations for SPDEs of jump type. Available at arXiv:1211.0466.

[39] Zhang, T.S. (2000). On the small time asymptotics of diffusion processes on Hilbert spaces. *Ann. Probab.* **28** 537–557. MR1782266

[40] Zhang, X. (2010). Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. *J. Funct. Anal.* **258** 1361–1425. MR2565842

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Estimation of integrated volatility of volatility with applications to goodness-of-fit testing

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In this paper, we are concerned with nonparametric inference on the volatility of volatility process in stochastic volatility models. We construct several estimators for its integrated version in a high-frequency setting, all based on increments of spot volatility estimators. Some of those are positive by construction, others are bias corrected in order to attain the optimal rate $n^{-1/4}$. Associated central limit theorems are proven which can be widely used in practice, as they are the key to essentially all tools in model validation for stochastic volatility models. As an illustration we give a brief idea on a goodness-of-fit test in order to check for a certain parametric form of volatility of volatility.

Keywords: central limit theorem; goodness-of-fit testing; high-frequency observations; model validation; stable convergence; stochastic volatility model

References

[1] Aït-Sahalia, Y. and Kimmel, R. (2007). Maximum likelihood estimation of stochastic volatility models. *J. Financial Economics* **134** 507–551.
[2] Alvarez, A., Panloup, F., Pontier, M. and Savy, N. (2012). Estimation of the instantaneous volatility. *Stat. Inference Stoch. Process.* **15** 27–59. MR2892587
[3] Bandi, F. and Renò, R. (2008). Nonparametric stochastic volatility. Technical report.
[4] Barndorff-Nielsen, O. and Veraart, A. (2009). Stochastic volatility of volatility in continuous time. Technical report.
[5] Barndorff-Nielsen, O.E., Graversen, S.E., Jacod, J., Podolskij, M. and Shephard, N. (2006). A central limit theorem for realised power and bipower variations of continuous semimartingales. In *From Stochastic Calculus to Mathematical Finance* 33–68. Berlin: Springer. MR2233534
[6] Barndorff-Nielsen, O.E., Hansen, P.R., Lunde, A. and Shephard, N. (2011). Multivariate realised kernels: Consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading. *J. Econometrics* **162** 149–169. MR2795610
[7] Bollerslev, T. and Zhou, H. (2002). Estimating stochastic volatility diffusion using conditional moments of integrated volatility. *J. Econometrics* **109** 33–65. MR1899692
[8] Chernov, M. and Ghysels, E. (2000). Estimation of stochastic volatility models for the purpose of option pricing. In *Computational Finance* 1999 (Y. Abu-Mostafa, B. LeBaron, A. Lo and A. Weigend, eds.) 567–581. Cambridge: MIT Press.
[9] Comte, F., Genon-Catalot, V. and Rozenholc, Y. (2010). Nonparametric estimation for a stochastic volatility model. *Finance Stoch.* **14** 49–80. MR2563205

[10] Dette, H. and Podolskij, M. (2008). Testing the parametric form of the volatility in continuous time diffusion models – A stochastic process approach. *J. Econometrics* **143** 56–73. MR2384433
[11] Dette, H., Podolskij, M. and Vetter, M. (2006). Estimation of integrated volatility in continuous-time financial models with applications to goodness-of-fit testing. *Scand. J. Stat.* **33** 259–278. MR2279642
[12] Genon-Catalot, V., Jeantheau, T. and Laredo, C. (1999). Parameter estimation for discretely observed stochastic volatility models. *Bernoulli* **5** 855–872. MR1715442
[13] Gloter, A. (2007). Efficient estimation of drift parameters in stochastic volatility models. *Finance Stoch.* **11** 495–519. MR2335831
[14] Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bonds and currency options. *Rev. Financial Studies* **6** 327–343.
[15] Hoffmann, M. (2002). Rate of convergence for parametric estimation in a stochastic volatility model. *Stochastic Process. Appl.* **97** 147–170. MR1870964
[16] Jacod, J. (1997). On continuous conditional Gaussian martingales and stable convergence in law. In *Séminaire de Probabilités XXXI. Lecture Notes in Math.* **1655** 232–246. Berlin: Springer. MR1478732
[17] Jacod, J. (2008). Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Process. Appl.* **118** 517–559. MR2394762
[18] Jacod, J. and Protter, P. (2012). *Discretization of Processes. Stochastic Modelling and Applied Probability* **67**. Heidelberg: Springer. MR2859096
[19] Jacod, J. and Rosenbaum, M. (2013). Quarticity and other functionals of volatility: Efficient estimation. *Ann. Statist.* **41** 1462–1484. MR3113818
[20] Jacod, J. and Shiryaev, A.N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Berlin: Springer. MR1943877
[21] Jones, C.S. (2003). The dynamics of stochastic volatility: Evidence from underlying and options markets. *J. Econometrics* **116** 181–224. Frontiers of financial econometrics and financial engineering. MR2002525
[22] Mancini, C. (2009). Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps. *Scand. J. Stat.* **36** 270–296. MR2528985
[23] Podolskij, M. and Vetter, M. (2010). Understanding limit theorems for semimartingales: A short survey. *Stat. Neerl.* **64** 329–351. MR2683464
[24] Renò, R. (2006). Nonparametric estimation of stochastic volatility models. *Econom. Lett.* **90** 390–395. MR2212176
[25] Vetter, M. (2012). Estimation of correlation for continuous semimartingales. *Scand. J. Stat.* **39** 757–771. MR3000847
[26] Vetter, M. (2014). Supplement to "Estimation of integrated volatility of volatility with applications to goodness-of-fit testing." DOI:10.3150/14-BEJ648SUPP.
[27] Vetter, M. and Dette, H. (2012). Model checks for the volatility under microstructure noise. *Bernoulli* **18** 1421–1447. MR2995803
[28] Wang, C.D. and Mykland, P.A. (2014). The estimation of leverage effect with high-frequency data. *J. Amer. Statist. Assoc.* **109** 197–215. MR3180557

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Probabilistic proof of product formulas for Bessel functions

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We write, for geometric index values, a probabilistic proof of the product formula for spherical Bessel functions. Though our proof looks elementary in the real variable setting, it has the merit to carry over without any further effort to Bessel-type hypergeometric functions of one matrix argument, thereby avoid complicated arguments from differential geometry. Moreover, the representative probability distribution involved in the last setting is shown to be closely related to the symmetrization of upper-left corners of Haar-distributed orthogonal matrices. Analysis of this probability distribution is then performed and in case it is absolutely continuous with respect to Lebesgue measure on the space of real symmetric matrices, we derive an invariance-property of its density. As a by-product, Weyl integration formula leads to the product formula for Bessel-type hypergeometric functions of two matrix arguments.

Keywords: conditional independence; hypergeometric functions; matrix-variate normal distribution; product formula

References

[1] Biane, P., Bougerol, P. and O'Connell, N. (2009). Continuous crystal and Duistermaat–Heckman measure for Coxeter groups. *Adv. Math.* **221** 1522–1583. MR2522427
[2] Chikuse, Y. (2003). *Statistics on Special Manifolds*. *Lecture Notes in Statistics* **174**. New York: Springer. MR1960435
[3] Chybiryakov, O., Demni, N., Gallardo, L., Rösler, M., Voit, M. and Yor, M. (2008). *Harmonic and Stochastic Analysis of Dunkl Processes* (P. Graczyk, M. Rösler and M. Yor, eds.). *Travaux en Cours* **71**. Paris: Hermann.
[4] Collins, B. (2003). Intégrales matricielles et probabilités non commutatives. Ph.D. thesis, Paris VI.
[5] Dunkl, C.F. and Xu, Y. (2001). *Orthogonal Polynomials of Several Variables*. *Encyclopedia of Mathematics and Its Applications* **81**. Cambridge: Cambridge Univ. Press. MR1827871
[6] Faraut, J. (2008). *Analysis on Lie Groups*. *An Introduction*. *Cambridge Studies in Advanced Mathematics* **110**. Cambridge: Cambridge Univ. Press. MR2426516
[7] Faraut, J. and Korányi, A. (1994). *Analysis on Symmetric Cones*. *Oxford Mathematical Monographs*. New York: The Clarendon Press, Oxford Univ. Press. MR1446489
[8] Federer, H. (1969). *Geometric Measure Theory*. *Die Grundlehren der Mathematischen Wissenschaften* **153**. New York: Springer. MR0257325
[9] Herz, C.S. (1955). Bessel functions of matrix argument. *Ann. of Math.* (2) **61** 474–523. MR0069960
[10] Muirhead, R.J. (1982). *Aspects of Multivariate Statistical Theory*. *Wiley Series in Probability and Mathematical Statistics*. New York: Wiley. MR0652932

1350-7265

[11] Ragozin, D.L. (1973/74). Rotation invariant measure algebras on Euclidean space. *Indiana Univ. Math. J.* **23** 1139–1154. MR0338688

[12] Revuz, D. (1975). *Markov Chains*. *North-Holland Mathematical Library* **11**. Amsterdam: North-Holland. MR0415773

[13] Revuz, D. (1997). *Probabilités*, *Editeurs des Sciences et des Arts*. Paris: Hermann.

[14] Sato, K.-i. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge: Cambridge Univ. Press. Translated from the 1990 Japanese original. Revised by the author. MR1739520

[15] Watson, G.N. (1995). *A Treatise on the Theory of Bessel Functions*. *Cambridge Mathematical Library*. Cambridge: Cambridge Univ. Press. Reprint of the second (1944) edition. MR1349110

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On ADF goodness-of-fit tests for perturbed dynamical systems

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We consider the problem of construction of goodness-of-fit tests for diffusion processes with a *small noise*. The basic hypothesis is composite parametric and our goal is to obtain asymptotically distribution-free tests. We propose two solutions. The first one is based on a change of time, and the second test is obtained using a linear transformation of the "natural" statistics.

Keywords: asymptotically distribution free test; Cramér–von Mises tests; diffusion processes; goodness of fit test; perturbed dynamical systems

References

[1] Dachian, S. and Kutoyants, Y.A. (2008). On the goodness-of-fit tests for some continuous time processes. In *Statistical Models and Methods for Biomedical and Technical Systems* (F. Vonta, M. Nikulin, N. Limnios and C. Huber-Carol, eds.). *Stat. Ind. Technol.* 385–403. Boston, MA: Birkhäuser. MR2462767

[2] Darling, D.A. (1955). The Cramér–Smirnov test in the parametric case. *Ann. Math. Statist.* **26** 1–20. MR0067439

[3] Freidlin, M.I. and Wentzell, A.D. (1998). *Random Perturbations of Dynamical Systems*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **260**. New York: Springer. Translated from the 1979 Russian original by Joseph Szücs. MR1652127

[4] Hitsuda, M. (1968). Representation of Gaussian processes equivalent to Wiener process. *Osaka J. Math.* **5** 299–312. MR0243614

[5] Iacus, S.M. and Kutoyants, Yu.A. (2001). Semiparametric hypotheses testing for dynamical systems with small noise. *Math. Methods Statist.* **10** 105–120. MR1841810

[6] Khmaladze, È.V. (1981). A martingale approach in the theory of goodness-of-fit tests. *Theory Probab. Appl.* **26** 240–257.

[7] Kleptsyna, M. and Kutoyants, Y.A. (2014). On asymptotically distribution free tests with parametric hypothesis for ergodic diffusion processes. *Stat. Inference Stoch. Process.* To appear. Available at arXiv:1305.3382.

[8] Kutoyants, Yu. (1994). *Identification of Dynamical Systems with Small Noise*. *Mathematics and Its Applications* **300**. Dordrecht: Kluwer Academic. MR1332492

[9] Kutoyants, Y.A. (2011). Goodness-of-fit tests for perturbed dynamical systems. *J. Statist. Plann. Inference* **141** 1655–1666. MR2763197

[10] Kutoyants, Yu.A. (2014). On ADF goodness-of-fit tests for stochastic processes. In *New Perspectives on Stochastic Modeling and Data Analysis* (J. Bozeman, V. Girardin and C. Skiadas, eds.). To appear.

[11] Kutoyants, Yu.A. (2014). On score-function processes and goodness of fit tests for stochastic processes. Available at arXiv:1403.7715.

[12] Kutoyants, Y.A. (2014). On asymptotic distribution of parameter free tests for ergodic diffusion processes. *Stat*. *Inference Stoch*. *Process*. **17** 139–161. MR3219526
[13] Kutoyants, Y.A. and Zhou, L. (2014). On approximation of the backward stochastic differential equation. *J*. *Statist*. *Plann*. *Inference* **150** 111–123. MR3206723
[14] Liptser, R. and Shiryaev, A. (2001). *Statistics of Random Processes*. *Vols*. *I*, *II*, 2nd ed. Berlin: Springer.
[15] Maglaperidze, N.O., Tsigroshvili, Z.P. and van Pul, M. (1998). Goodness-of-fit tests for parametric hypotheses on the distribution of point processes. *Math*. *Methods Statist*. **7** 60–77. MR1626572
[16] Shepp, L.A. (1966). Radon–Nikodým derivatives of Gaussian measures. *Ann*. *Math*. *Statist*. **37** 321–354. MR0190999
[17] Yoshida, N. (1993). Asymptotic expansion of Bayes estimators for small diffusions. *Probab*. *Theory Related Fields* **95** 429–450. MR1217445
[18] Yoshida, N. (1996). Asymptotic expansions for perturbed systems on Wiener space: Maximum likelihood estimators. *J*. *Multivariate Anal*. **57** 1–36. MR1392575

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A new class of large claim size distributions: Definition, properties, and ruin theory

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We investigate a new natural class $\mathcal{J}$ of probability distributions modeling large claim sizes, motivated by the 'principle of one big jump'. Though significantly more general than the (sub-)class of subexponential distributions $\mathcal{S}$, many important and desirable structural properties can still be derived. We establish relations to many other important large claim distribution classes (such as $\mathcal{D}$, $\mathcal{S}$, $\mathcal{L}$, $\mathcal{K}$, $\mathcal{OS}$ and $\mathcal{OL}$), discuss the stability of $\mathcal{J}$ under tail-equivalence, convolution, convolution roots, random sums and mixture, and then apply these results to derive a partial analogue of the famous Pakes–Veraverbeke–Embrechts theorem from ruin theory for $\mathcal{J}$. Finally, we discuss the (weak) tail-equivalence of infinitely-divisible distributions in $\mathcal{J}$ with their Lévy measure.

Keywords: heavy-tailed; random walks; ruin theory; subexponential

References

[1] Asmussen, S., Foss, S. and Korshunov, D. (2003). Asymptotics for sums of random variables with local subexponential behaviour. *J. Theoret. Probab.* **16** 489–518. MR1982040

[2] Beck, S. (2014). New classes of large claim size distributions: Introduction, properties and applications. Ph.D. dissertation, TU Berlin. Available at http://opus4.kobv.de/opus4-tuberlin/frontdoor/index/index/docId/5364.

[3] Chistyakov, V.P. (1964). A theorem on sums of independent positive random variables and its applications to branching random processes. *Theory Probab. Appl.* **9** 640–648. MR0170394

[4] Chover, J., Ney, P. and Wainger, S. (1973). Degeneracy properties of subcritical branching processes. *Ann. Probab.* **1** 663–673. MR0348852

[5] Chover, J., Ney, P. and Wainger, S. (1973). Functions of probability measures. *J. Anal. Math.* **26** 255–302. MR0348393

[6] Embrechts, P. and Goldie, C.M. (1980). On closure and factorization properties of subexponential and related distributions. *J. Austral. Math. Soc. Ser. A* **29** 243–256. MR0566289

[7] Embrechts, P. and Goldie, C.M. (1982). On convolution tails. *Stochastic Process. Appl.* **13** 263–278. MR0671036

[8] Embrechts, P., Goldie, C.M. and Veraverbeke, N. (1979). Subexponentiality and infinite divisibility. *Z. Wahrsch. Verw. Gebiete* **49** 335–347. MR0547833

[9] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events. For Insurance and Finance. Applications of Mathematics* (*New York*) **33**. Berlin: Springer. MR1458613

[10] Embrechts, P. and Veraverbeke, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Math. Econom.* **1** 55–72. MR0652832

[11] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**. New York: Wiley.

1350-7265

[12] Foss, S., Korshunov, D. and Zachary, S. (2011). *An Introduction to Heavy-Tailed and Subexponential Distributions*. *Springer Series in Operations Research and Financial Engineering*. New York: Springer. MR2810144

[13] Geluk, J. (2009). Some closure properties for subexponential distributions. *Statist*. *Probab*. *Lett*. **79** 1108–1111. MR2510776

[14] Klüppelberg, C. (1990). Asymptotic ordering of distribution functions. *Semigroup Forum* **40** 77–92. MR1014226

[15] Korshunov, D. (1997). On distribution tail of the maximum of a random walk. *Stochastic Process*. *Appl*. **72** 97–103. MR1483613

[16] Leslie, J.R. (1989). On the nonclosure under convolution of the subexponential family. *J*. *Appl*. *Probab*. **26** 58–66. MR0981251

[17] Lin, J. and Wang, Y. (2012). New examples of heavy-tailed O-subexponential distributions and related closure properties. *Statist*. *Probab*. *Lett*. **82** 427–432. MR2887455

[18] Shimura, T. and Watanabe, T. (2005). Infinite divisibility and generalized subexponentiality. *Bernoulli* **11** 445–469. MR2146890

[19] Su, Z., Su, C., Hu, Z. and Liu, J. (2009). On domination problem of non-negative distributions. *Front*. *Math*. *China* **4** 681–696. MR2563649

[20] Wang, Y.B., Cheng, F.Y. and Yang, Y. (2005). Dominant relations on some subclasses of heavy-tailed distributions and their applications. *Chinese J*. *Appl*. *Probab*. *Statist*. **21** 21–30. MR2193514

[21] Watanabe, T. (2008). Convolution equivalence and distributions of random sums. *Probab*. *Theory Related Fields* **142** 367–397. MR2438696

[22] Xu, H., Cui, Z., Scheutzow, M. and Wang, Y. (2014). Distributions obeying the principle of a single big jump: New examples. Preprint. Available at arXiv:1406.2754.

[23] Yakymiv, A.L. (1997). Some properties of subexponential distributions. *Math*. *Notes* **62** 116–121. MR1619933

[24] Yang, Y. and Wang, K. (2011). Estimates for the tail probability of the supremum of a random walk with independent increments. *Chin*. *Ann*. *Math*. *Ser*. *B* **32** 847–856. MR2852306

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Size-biased permutation of a finite sequence with independent and identically distributed terms

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This paper focuses on the size-biased permutation of n independent and identically distributed (i.i.d.) positive random variables. This is a finite dimensional analogue of the size-biased permutation of ranked jumps of a subordinator studied in Perman–Pitman–Yor (PPY) [*Probab. Theory Related Fields* **92** (1992) 21–39], as well as a special form of *induced order statistics* [*Bull. Inst. Internat. Statist.* **45** (1973) 295–300; *Ann. Statist.* **2** (1974) 1034–1039]. This intersection grants us different tools for deriving distributional properties. Their comparisons lead to new results, as well as simpler proofs of existing ones. Our main contribution, Theorem 25 in Section 6, describes the asymptotic distribution of the last few terms in a finite i.i.d. size-biased permutation via a Poisson coupling with its few smallest order statistics.

Keywords: induced order statistics; Kingman paint box; Poisson–Dirichlet; size-biased permutation; subordinator

References

[1] Barouch, E. and Kaufman, G.M. (1976). Probabilistic modelling of oil and gas discovery. In *Energy: Mathematics and Models* 133–150. Philadelphia: SIAM.

[2] Bertoin, J. (1999). Subordinators: Examples and applications. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1997). *Lecture Notes in Math*. **1717** 1–91. Berlin: Springer. MR1746300

[3] Bhattacharya, P.K. (1974). Convergence of sample paths of normalized sums of induced order statistics. *Ann. Statist*. **2** 1034–1039. MR0386100

[4] Bickel, P.J., Nair, V.N. and Wang, P.C.C. (1992). Nonparametric inference under biased sampling from a finite population. *Ann. Statist*. **20** 853–878. MR1165596

[5] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation*. *Encyclopedia of Mathematics and Its Applications* **27**. Cambridge: Cambridge Univ. Press. MR0898871

[6] Chaumont, L. and Yor, M. (2003). *Exercises in Probability*. *Cambridge Series in Statistical and Probabilistic Mathematics* **13**. Cambridge: Cambridge Univ. Press. MR2016344

[7] Cheng, W., Dembczyński, K. and Hüllermeier, E. (2010). Label ranking methods based on the Plackett–Luce model. In *Proceedings of the 27th International Conference on Machine Learning (ICML-10)*, 215–222, Haifa, Israel, June 2010. Omnipress.

[8] David, H.A. (1973). Concomitants of order statistics. *Bull. Inst. Internat. Statist*. **45** 295–300. MR0373149

[9] David, H.A., O'Connell, M.J. and Yang, S.S. (1977). Distribution and expected value of the rank of a concomitant of an order statistic. *Ann. Statist*. **5** 216–223. MR0445673

1350-7265

[10] Davydov, Y. and Egorov, V. (2001). Functional CLT and LIL for induced order statistics. In *Asymptotic Methods in Probability and Statistics with Applications* (*St. Petersburg*, 1998). *Stat. Ind. Technol.* 333–349. Boston, MA: Birkhäuser. MR1890337
[11] de Haan, L. and Ferreira, A. (2006). *Extreme Value Theory*: *An Introduction*. *Springer Series in Operations Research and Financial Engineering*. New York: Springer. MR2234156
[12] Engen, S. (1978). *Stochastic Abundance Models with Emphasis on Biological Communities and Species Diversity*. London: Chapman & Hall. MR0515721
[13] Ewens, W.J. (1972). The sampling theory of selectively neutral alleles. *Theoret. Population Biology* **3** 87–112; erratum, ibid. 3 (1972), 240; erratum, ibid. 3 (1972), 376. MR0325177
[14] Gnedin, A., Haulk, C. and Pitman, J. (2010). Characterizations of exchangeable partitions and random discrete distributions by deletion properties. In *Probability and Mathematical Genetics*. *London Mathematical Society Lecture Note Series* **378** 264–298. Cambridge: Cambridge Univ. Press. MR2744243
[15] Gnedin, A.V. (1998). On convergence and extensions of size-biased permutations. *J. Appl. Probab.* **35** 642–650. MR1659532
[16] Goel, P.K. and Hall, P. (1994). On the average difference between concomitants and order statistics. *Ann. Probab.* **22** 126–144. MR1258869
[17] Gordon, L. (1983). Successive sampling in large finite populations. *Ann. Statist.* **11** 702–706. MR0696081
[18] Holst, L. (1973). Some limit theorems with applications in sampling theory. *Ann. Statist.* **1** 644–658. MR0365836
[19] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). New York: Springer. MR1876169
[20] Kingman, J.F.C. (1978). Random partitions in population genetics. *Proc. Roy. Soc. London Ser. A* **361** 1–20. MR0526801
[21] Kingman, J.F.C. (1978). The representation of partition structures. *J. Lond. Math. Soc.* (2) **18** 374–380. MR0509954
[22] Lukacs, E. (1955). A characterization of the Gamma distribution. *Ann. Math. Statist.* **26** 319–324. MR0069408
[23] McCloskey, J.W. (1965). A model for the distribution of individuals by species in an environment. Ph.D. thesis, Michigan State Univ.
[24] Nagaraja, H.N. and David, H.A. (1994). Distribution of the maximum of concomitants of selected order statistics. *Ann. Statist.* **22** 478–494. MR1272095
[25] Patil, G.P. and Taillie, C. (1977). Diversity as a concept and its implications for random communities. *Bull. Inst. Internat. Statist.* **47** 497–515. MR0617593
[26] Perman, M., Pitman, J. and Yor, M. (1992). Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields* **92** 21–39. MR1156448
[27] Pitman, J. (1996). Random discrete distributions invariant under size-biased permutation. *Adv. in Appl. Probab.* **28** 525–539. MR1387889
[28] Pitman, J. (2006). *Combinatorial Stochastic Processes*. *Lecture Notes in Math.* **1875**. Berlin: Springer. MR2245368
[29] Plackett, R.L. (1968). Random permutations. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **30** 517–534. MR0247695
[30] Rényi, A. (1953). On the theory of order statistics. *Acta Math. Acad. Sci. Hungar.* **4** 191–231. MR0061792
[31] Rosén, B. (1972). Asymptotic theory for successive sampling with varying probabilities without replacement. I, II. *Ann. Math. Statist.* **43** 373–397; ibid. 43 (1972), 748–776. MR0321223
[32] Sen, P.K. (1979). Invariance principles for the coupon collector's problem: A martingale approach. *Ann. Statist.* **7** 372–380. MR0520246

[33] Sturmfels, B. and Welker, V. (2012). Commutative algebra of statistical ranking. *J. Algebra* **361** 264–286. MR2921622
[34] Wellner, J.A. (1981). A Glivenko–Cantelli theorem for empirical measures of independent but non-identically distributed random variables. *Stochastic Process. Appl.* **11** 309–312. MR0622172
[35] Yang, S.S. (1977). General distribution theory of the concomitants of order statistics. *Ann. Statist.* **5** 996–1002. MR0501519

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Lévy processes and stochastic integrals in the sense of generalized convolutions

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In this paper, we present a comprehensive theory of generalized and weak generalized convolutions, illustrate it by a large number of examples, and discuss the related infinitely divisible distributions. We consider Lévy and additive process with respect to generalized and weak generalized convolutions as certain Markov processes, and then study stochastic integrals with respect to such processes. We introduce the representability property of weak generalized convolutions. Under this property and the related weak summability, a stochastic integral with respect to random measures related to such convolutions is constructed.

Keywords: Lévy process; scale mixture; stochastic integral; symmetric stable distribution; weakly stable distribution

References

[1] Bingham, N.H. (1984). On a theorem of Kłosowska about generalised convolutions. *Colloq. Math.* **48** 117–125. MR0750763
[2] Bohnenblust, F. (1940). An axiomatic characterization of L_p-spaces. *Duke Math. J.* **6** 627–640. MR0002701
[3] Cambanis, S., Keener, R. and Simons, G. (1983). On α-symmetric multivariate distributions. *J. Multivariate Anal.* **13** 213–233. MR0705548
[4] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications. Vol. II*. New York: Wiley. MR0210154
[5] Jarczyk, W. and Misiewicz, J.K. (2009). On weak generalized stability and (c,d)-pseudostable random variables via functional equations. *J. Theoret. Probab.* **22** 482–505. MR2501331
[6] Jasiulis, B.H. (2010). Limit property for regular and weak generalized convolution. *J. Theoret. Probab.* **23** 315–327. MR2591916
[7] Jasiulis, B.H. and Misiewicz, J.K. (2008). On the connections between weakly stable and pseudo-isotropic distributions. *Statist. Probab. Lett.* **78** 2751–2755. MR2465118
[8] Jasiulis-Gołdyn, B. and Kula, A. (2012). The Urbanik generalized convolutions in the non-commutative probability and a forgotten method of constructing generalized convolution. *Proc. Indian Acad. Sci. Math. Sci.* **122** 437–458. MR2972664

1350-7265

[9] Jasiulis-Gołdyn, B.H. (2014). On the random walk under the Kendall convolution. To appear.
[10] Jasiulis-Gołdyn, B.H. and Misiewicz, J.K. (2014). Weak Lévy–Khintchine representation for weak infinite divisibility. *Theory Probab. Appl.* To appear.
[11] Kallenberg, O. (1997). *Foundations of Modern Probability*. *Probability and Its Applications* (*New York*). New York: Springer. MR1464694
[12] Killmann, F. and von Collani, E. (2001). A note on the convolution of the uniform and related distributions and their use in quality control. *Econ*. *Qual*. *Control* **16** 17–41. MR1959535
[13] Kingman, J.F.C. (1963). Random walks with spherical symmetry. *Acta Math*. **109** 11–53. MR0149567
[14] Kozubowski, T.J. and Panorska, A.K. (1996). On moments and tail behavior of ν-stable random variables. *Statist*. *Probab*. *Lett*. **29** 307–315. MR1409326
[15] Kucharczak, J. and Urbanik, K. (1974). Quasi-stable functions. *Bull*. *Acad*. *Polon*. *Sci*. *Sér*. *Sci*. *Math*. *Astronom*. *Phys*. **22** 263–268. MR0343338
[16] Kucharczak, J. and Urbanik, K. (1986). Transformations preserving weak stability. *Bull*. *Pol*. *Acad*. *Sci*. *Math*. **34** 475–486. MR0874894
[17] Misiewicz, J.K. (1996). Infinite divisibility of sub-stable processes. II. Logarithm of probability measure. *J*. *Math*. *Sci*. **81** 2970–2979. MR1420898
[18] Misiewicz, J.K. (2006). Weak stability and generalized weak convolution for random vectors and stochastic processes. In *Dynamics & Stochastics*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **48** 109–118. Beachwood, OH: IMS. MR2306193
[19] Misiewicz, J.K. and Mazurkiewicz, G. (2005). On (c, p)-pseudostable random variables. *J*. *Theoret*. *Probab*. **18** 837–852. MR2289934
[20] Misiewicz, J.K., Oleszkiewicz, K. and Urbanik, K. (2005). Classes of measures closed under mixing and convolution. Weak stability. *Studia Math*. **167** 195–213. MR2131418
[21] Oleszkiewicz, K. (2003). On p-pseudostable random variables, Rosenthal spaces and l_p^n ball slicing. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math*. **1807** 188–210. Berlin: Springer. MR2083398
[22] Panorska, A.K. (1994). Rate of convergence in the central limit theorem for generalized convolutions. In *Approximation*, *Probability*, *and Related Fields* (*Santa Barbara*, *CA*, 1993) 379–393. New York: Plenum. MR1309282
[23] Panorska, A.K. (1999). Generalized convolutions on $\mathbf{R}$ with applications to financial modeling. *Math*. *Comput*. *Modelling* **29** 263–274. MR1704778
[24] Rajput, B.S. and Rosiński, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields* **82** 451–487. MR1001524
[25] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes*. *Stochastic Modeling*. New York: Chapman and Hall. MR1280932
[26] Sato, K.-i. (2013). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge: Cambridge Univ. Press. Translated from the 1990 Japanese original. Revised edition of the 1999 English translation. MR3185174
[27] Schoenberg, I.J. (1938). Metric spaces and completely monotonic functions. *Ann*. *Math*. **38** 811–841.
[28] Thu, N.V. (1994). Generalized independent increments processes. *Nagoya Math*. *J*. **133** 155–175. MR1266366
[29] Thu, N.V. (2009). A Kingman convolution approach to Bessel processes. *Probab*. *Math*. *Statist*. **29** 119–134. MR2553003
[30] Urbanik, K. (1963/1964). Generalized convolutions. *Studia Math*. **23** 217–245. MR0160267
[31] Urbanik, K. (1973). Generalized convolutions. II. *Studia Math*. **45** 57–70. MR0328991
[32] Urbanik, K. (1976). Remarks on B-stable probability distributions. *Bull*. *Acad*. *Polon*. *Sci*. *Sér*. *Sci*. *Math*. *Astronom*. *Phys*. **24** 783–787. MR0423472
[33] Urbanik, K. (1984). Generalized convolutions. III. *Studia Math*. **80** 167–189. MR0781333

[34] Urbanik, K. (1986). Generalized convolutions. IV. *Studia Math*. **83** 57–95. MR0829899
[35] Urbanik, K. (1988). Analytical methods in probability theory. In *Transactions of the Tenth Prague Conference on Information Theory*, *Statistical Decision Functions*, *Random Processes*, *Vol*. *A* (*Prague*, 1986) 151–163. Dordrecht: Reidel. MR1136270
[36] Vol'kovich, V.È. (1984). Multidimensional $\mathcal{B}$-stable distributions and realizations of generalized convolutions. In *Stability Problems for Stochastic Models* (*Moscow*, 1984) 40–54. Moscow: Vsesoyuz. Nauchno-Issled. Inst. Sistem. Issled. In Rusian. MR0859210
[37] Vol'kovich, V.È. (1985). On infinitely decomposable measures in algebras with stochastic convolution. In *Stability Problems of Stochastic Models* (*Moscow*, 1985) 15–24. Moscow: Vsesoyuz. Nauchno-Issled. Inst. Sistem. Issled. In Russian.
[38] Vol'kovich, V. (1992). On symmetric stochastic convolutions. *J*. *Theoret*. *Probab*. **5** 417–430. MR1176429
[39] Zolotarev, V.M. (1986). *One-Dimensional Stable Distributions*. *Translations of Mathematical Monographs* **65**. Providence, RI: Amer. Math. Soc. Translated from the Russian by H. H. McFaden, translation edited by Ben Silver. MR0854867

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On the infinite divisibility of inverse Beta distributions

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We show that all negative powers $\beta_{a,b}^{-s}$ of the Beta distribution are infinitely divisible. The case $b \le 1$ follows by complete monotonicity, the case $b > 1$, $s \ge 1$ by hyperbolically complete monotonicity and the case $b > 1$, $s < 1$ by a Lévy perpetuity argument involving the hypergeometric series. We also observe that $\beta_{a,b}^{-s}$ is self-decomposable if and only if $2a + b + s + bs \ge 1$, and that in this case it is not necessarily a generalized Gamma convolution. On the other hand, we prove that all negative powers of the Gamma distribution are generalized Gamma convolutions, answering to a recent question of L. Bondesson.

Keywords: Beta distribution; Gamma distribution; generalized Gamma convolution; hyperbolically complete monotonicity; hypergeometric series; Lévy perpetuity; self-decomposability; Stieltjes transform

References

[1] Anderson, G.D., Vamanamurthy, M.K. and Vuorinen, M. (2007). Generalized convexity and inequalities. *J. Math. Anal. Appl.* **335** 1294–1308. MR2346906
[2] Bertoin, J. (1996). *Lévy Processes. Cambridge Tracts in Mathematics* **121**. Cambridge: Cambridge Univ. Press. MR1406564
[3] Bertoin, J. and Yor, M. (2002). On the entire moments of self-similar Markov processes and exponential functionals of Lévy processes. *Ann. Fac. Sci. Toulouse Math.* (6) **11** 33–45. MR1986381
[4] Bertoin, J. and Yor, M. (2005). Exponential functionals of Lévy processes. *Probab. Surv.* **2** 191–212. MR2178044
[5] Bondesson, L. (1992). *Generalized Gamma Convolutions and Related Classes of Distributions and Densities. Lecture Notes in Statistics* **76**. New York: Springer. MR1224674
[6] Bondesson, L. (2014). A class of probability distributions that is closed with respect to addition as well as multiplication of independent random variables. *J. Theoret. Probab.* To appear. Available at DOI:10.1007/s10959-013-0523-y.
[7] Bosch, P. (2014). HCM property and the half-Cauchy distribution. Available at arXiv:1402.1059.
[8] Bosch, P. and Simon, T. (2013). On the self-decomposability of the Fréchet distribution. *Indag. Math.* (*N.S.*) **24** 626–636. MR3064566
[9] Bustoz, J. and Ismail, M.E.H. (1986). On gamma function inequalities. *Math. Comp.* **47** 659–667. MR0856710
[10] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953). *Higher Transcendental Functions Vol. I and II*. New York: McGraw-Hill.
[11] Gjessing, H.K. and Paulsen, J. (1997). Present value distributions with applications to ruin theory and stochastic equations. *Stochastic Process. Appl.* **71** 123–144. MR1480643

[12] James, L.F., Roynette, B. and Yor, M. (2008). Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surv.* **5** 346–415. MR2476736

[13] Janson, S. (2010). Moments of gamma type and the Brownian supremum process area. *Probab. Surv.* **7** 1–52. MR2645216

[14] Jedidi, W. and Simon, T. (2013). Further examples of GGC and HCM densities. *Bernoulli* **19** 1818–1838. MR3129035

[15] Klein, F. (1890). Ueber die Nullstellen der hypergeometrischen Reihe. *Math. Ann.* **37** 573–590. MR1510659

[16] Sato, K.-i. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge: Cambridge Univ. Press. Translated from the 1990 Japanese original. Revised by the author. MR1739520

[17] Schilling, R.L., Song, R. and Vondraček, Z. (2010). *Bernstein Functions*. *Theory and Applications*. *De Gruyter Studies in Mathematics* **37**. Berlin: de Gruyter. MR2598208

[18] Simon, T. (2014). Comparing Fréchet and positive stable laws. *Electron. J. Probab.* **19** 1–25. MR3164769

[19] Steutel, F.W. and Van Harn, K. (2003). *Infinite Divisibility of Probability Distributions on the Real Line*. New York: Dekker. MR2011862

[20] Van Vleck, E.B. (1902). A determination of the number of real and imaginary roots of the hypergeometric series. *Trans. Amer. Math. Soc.* **3** 110–131. MR1500590

[21] Vervaat, W. (1979). On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Adv. in Appl. Probab.* **11** 750–783. MR0544194

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Efficient pointwise estimation based on discrete data in ergodic nonparametric diffusions

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A truncated sequential procedure is constructed for estimating the drift coefficient at a given state point based on discrete data of ergodic diffusion process. A nonasymptotic upper bound is obtained for a pointwise absolute error risk. The optimal convergence rate and a sharp constant in the bounds are found for the asymptotic pointwise minimax risk. As a consequence, the efficiency is obtained of the proposed sequential procedure.

Keywords: discrete data; drift coefficient estimation; efficient procedure; ergodic diffusion process; minimax; nonparametric sequential estimation

References

[1] Arató, M. (1982). *Linear Stochastic Systems with Constant Coefficients*: *A Statistical Approach*. *Lecture Notes in Control and Information Sciences* **45**. Berlin: Springer. MR0791212
[2] Arato, M., Kolmogorov, A.N. and Sinai, Ya.G. (1962). On parameter estimation of a complex stationary Gaussian process. *Reports of the Acad*. *Sciences USSR* **146** 747–750.
[3] Bensoussan, A. (1992). *Stochastic Control of Partially Observable Systems*. Cambridge: Cambridge Univ. Press. MR1191160
[4] Borisov, V.Z. and Konev, V.V. (1977). Sequential estimation of parameters of discrete processes. *Autom*. *Remote Control* **10** 58–64.
[5] Comte, F., Genon-Catalot, V. and Rozenholc, Y. (2009). Nonparametric adaptive estimation for integrated diffusions. *Stochastic Process*. *Appl*. **119** 811–834. MR2499859
[6] Florens-Zmirou, D. (1993). On estimating the diffusion coefficient from discrete observations. *J. Appl*. *Probab*. **30** 790–804. MR1242012
[7] Galtchouk, L. and Pergamenshchikov, S. (2001). Sequential nonparametric adaptive estimation of the drift coefficient in diffusion processes. *Math*. *Methods Statist*. **10** 316–330. MR1867163
[8] Galtchouk, L. and Pergamenshchikov, S. (2005). Nonparametric sequential minimax estimation of the drift coefficient in diffusion processes. *Sequential Anal*. **24** 303–330. MR2187340
[9] Galtchouk, L. and Pergamenshchikov, S. (2006). Asymptotically efficient estimates for nonparametric regression models. *Statist*. *Probab. Lett*. **76** 852–860. MR2266100

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[10] Galtchouk, L. and Pergamenshchikov, S. (2006). Asymptotically efficient sequential kernel estimates of the drift coefficient in ergodic diffusion processes. *Stat*. *Inference Stoch*. *Process*. **9** 1–16. MR2224837

[11] Galtchouk, L. and Pergamenshchikov, S. (2007). Uniform concentration inequality for ergodic diffusion processes. *Stochastic Process*. *Appl*. **117** 830–839. MR2330721

[12] Galtchouk, L. and Pergamenshchikov, S. (2013). Uniform concentration inequality for ergodic diffusion processes observed at discrete times. *Stochastic Process*. *Appl*. **123** 91–109. MR2988111

[13] Galtchouk, L. and Pergamenshchikov, S. (2014). Geometric ergodicity for classes of homogeneous Markov chains. *Stochastic Process*. *Appl*. **124** 3362–3391. MR3231623

[14] Galtchouk, L.I. and Pergamenshchikov, S.M. (2011). Adaptive sequential estimation for ergodic diffusion processes in quadratic metric. *J*. *Nonparametr. Stat*. **23** 255–285. MR2801293

[15] Genon-Catalot, V., Laredo, C. and Picard, D. (1992). Nonparametric estimation of the diffusion coefficient by wavelets methods. *Scand*. *J*. *Stat*. **19** 317–335. MR1211787

[16] Gihman, I.I. and Skorohod, A.V. (1968). *Stokhasticheskie Differentsialnye Uravneniya*. Kiev: Izdat. "Naukova Dumka". MR0263172

[17] Gobet, E. (2002). LAN property for ergodic diffusions with discrete observations. *Ann*. *Inst*. *Henri Poincaré Probab. Stat*. **38** 711–737. MR1931584

[18] Gobet, E., Hoffmann, M. and Reiß, M. (2004). Nonparametric estimation of scalar diffusions based on low frequency data. *Ann*. *Statist*. **32** 2223–2253. MR2102509

[19] Hoffmann, M. (1999). Adaptive estimation in diffusion processes. *Stochastic Process*. *Appl*. **79** 135–163. MR1670522

[20] Jacod, J. (2000). Non-parametric kernel estimation of the coefficient of a diffusion. *Scand*. *J*. *Stat*. **27** 83–96. MR1774045

[21] Jacod, J. (2006). Parametric inference for discretely observed non-ergodic diffusions. *Bernoulli* **12** 383–401. MR2232724

[22] Kabanov, Y. and Pergamenshchikov, S. (2003). *Two-Scale Stochastic Systems*. *Applications of Mathematics* (*New York*) **49**. Berlin: Springer. MR1942740

[23] Karatzas, I. and Shreve, S.E. (1998). *Methods of Mathematical Finance*. *Applications of Mathematics* (*New York*) **39**. New York: Springer. MR1640352

[24] Kessler, M. (1997). Estimation of an ergodic diffusion from discrete observations. *Scand*. *J*. *Stat*. **24** 211–229. MR1455868

[25] Konev, V.V. and Pergamenshchikov, S.M. (1981). Sequential estimation of the parameters of random processes with continuous time. In *Math*. *Statistics and Appl*. (F. Tarasenko, ed.) 93–101. Tomsk: Publishing Hause of Tomsk State Univ.

[26] Konev, V.V. and Pergamenshchikov, S.M. (1992). On truncated sequential estimation of the parameters of diffusion processes. In *Methods of Economical Analysis* (S. Aivazian, ed.) 3–31. Moscow: Central Economical and Mathematical Institute of Russian Academy of Science.

[27] Kutoyants, Y.A. (2004). *Statistical Inference for Ergodic Diffusion Processes*. London: Springer. MR2144185

[28] Liptser, R.Sh. and Shiryaev, A.N. (1978). *Statistics of a Random Process II*. Berlin: Springer. MR0488267

[29] Novikov, A.A. (1972). Sequential estimation of the parameters of processes of diffusion type. *Mat*. *Zametki* **12** 627–638. MR0317493

[30] Yoshida, N. (1992). Estimation for diffusion processes from discrete observation. *J*. *Multivariate Anal*. **41** 220–242. MR1172898

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