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# LINEAR COVER TIME IS EXPONENTIALLY UNLIKELY

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Proving a 2009 conjecture of Itai Benjamini, we show:
*Theorem. For any $C$ there is an $\varepsilon > 0$ such that for any simple graph $G$ on $V$ of size $n$, and $X_0, \dots$ an ordinary random walk on $G$,*

$$\mathbb{P}(\{X_0, \dots, X_{Cn}\} = V) < e^{-\varepsilon n}.$$

A first ingredient in the proof of this is a similar statement for Markov chains in which all transition probabilities are sufficiently small relative to $C$.

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# UNIFORM SPANNING TREE IN TOPOLOGICAL POLYGONS, PARTITION FUNCTIONS FOR SLE(8), AND CORRELATIONS IN $c = -2$ LOGARITHMIC CFT

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We find explicit SLE(8) partition functions for the scaling limits of Peano curves in the uniform spanning tree (UST) in topological polygons with general boundary conditions. They are given in terms of Coulomb gas integral formulas, which can also be expressed in terms of determinants involving $a$-periods of a hyperelliptic Riemann surface. We also identify the crossing probabilities for the UST Peano curves as ratios of these partition functions.

The partition functions are interpreted as correlation functions in a logarithmic conformal field theory (log-CFT) of central charge $c = -2$. Indeed, it is clear from our results that this theory is not a minimal model and exhibits logarithmic phenomena—the limit functions have logarithmic asymptotic behavior, that we calculate explicitly. General fusion rules for them could also be inferred from the explicit formulas. The discovered algebraic structure matches the known Virasoro staggered module classification, so in this sense, we give a direct probabilistic construction for correlation functions in a log-CFT of central charge $-2$ describing the UST model.

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# REGULARIZATION BY NOISE FOR ROUGH DIFFERENTIAL EQUATIONS DRIVEN BY GAUSSIAN ROUGH PATHS

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We consider the rough differential equation with drift driven by a Gaussian geometric rough path. Under natural conditions on the rough path, namely nondeterminism, and uniform ellipticity conditions on the diffusion coefficient, we prove path-by-path well-posedness of the equation for poorly regular drifts. In the case of the fractional Brownian motion $B^H$ for $H > \frac{1}{4}$, we prove that the drift may be taken to be $\kappa > 0$ Hölder continuous and bounded for $\kappa > \frac{3}{2} - \frac{1}{2H}$. A flow transform of the equation and Malliavin calculus for Gaussian rough paths are used to achieve such a result.

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# CORRELATION DECAY FOR FINITE LATTICE GAUGE THEORIES AT WEAK COUPLING

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In the setting of lattice gauge theories with finite (possibly non-Abelian) gauge groups at weak coupling, we prove exponential decay of correlations for a wide class of gauge invariant functions, which in particular includes arbitrary functions of Wilson loop observables.

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# BROWNIAN MOTION WITH ASYMPTOTICALLY NORMAL REFLECTION IN UNBOUNDED DOMAINS: FROM TRANSIENCE TO STABILITY

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We quantify the asymptotic behaviour of multidimensional driftless diffusions in domains unbounded in a single direction with asymptotically normal reflections from the boundary. We identify the critical growth/contraction rates of the domain that separate stability, null recurrence and transience. In the stable case, we prove existence and uniqueness of the invariant distribution and establish the polynomial rate of decay of its tail. We also establish matching polynomial upper and lower bounds on the rate of convergence to stationarity in total variation. All exponents are explicit in the model parameters that determine the asymptotics of the growth rate of the domain, the interior covariance and the reflection vector field.

Proofs are probabilistic and use upper and lower tail bounds for additive functionals up to return times to compact sets for which we develop novel sub-/supermartingale criteria, applicable to general continuous semimartingales. Narrowing domains fall outside of the standard literature, in part because boundary local time can accumulate arbitrarily rapidly. Establishing Feller continuity (essential for characterizing stability) thus requires an extension of the usual approach.

Our recurrence/transience classification extends previous work on strictly normal reflections and expands the range of phenomena observed across all dimensions. For all recurrent cases, we provide quantitative information through upper and lower bounds on tails of return times to compact sets.

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# SOLUTIONS TO THE STOCHASTIC HEAT EQUATION WITH POLYNOMIALLY GROWING MULTIPLICATIVE NOISE DO NOT EXPLODE IN THE CRITICAL REGIME

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We investigate the finite time explosion of the stochastic heat equation $\frac{\partial u}{\partial t} = \Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x)$ in the critical setting where $\sigma$ grows like $\sigma(u) \approx C(1+|u|^\gamma)$ and $\gamma = \frac{3}{2}$. Mueller previously identified $\gamma = \frac{3}{2}$ as the critical growth rate for explosion and proved that solutions cannot explode in finite time if $\gamma < \frac{3}{2}$ and solutions will explode with positive probability if $\gamma > \frac{3}{2}$. This paper proves that explosion does not occur in the critical $\gamma = \frac{3}{2}$ setting.

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# TRACY-WIDOM LIMIT FOR FREE SUM OF RANDOM MATRICES

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We consider fluctuations of the largest eigenvalues of the random matrix model $A + UBU^*$ where $A$ and $B$ are $N \times N$ deterministic Hermitian (or symmetric) matrices and $U$ is a Haar-distributed unitary (or orthogonal) matrix. We prove that the largest eigenvalue weakly converges to the GUE (or GOE) Tracy–Widom distribution, under mild assumptions on $A$ and $B$ to guarantee that the density of states of the model decays as square root around the upper edge. Our proof is based on the comparison of the Green function along the Dyson Brownian motion starting from the matrix $A + UBU^*$ and ending at time $N^{-1/3+o(1)}$. As a byproduct of our proof, we also prove an optimal local law for the Dyson Brownian motion up to the constant time scale.

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# A DETERMINANTAL POINT PROCESS APPROACH TO SCALING AND LOCAL LIMITS OF RANDOM YOUNG TABLEAUX

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We obtain scaling and local limit results for large random Young tableaux of fixed shape $\lambda^0$ via the asymptotic analysis of a determinantal point process due to Gorin and Rahman (2019). More precisely, we prove:

- an explicit description of the limiting surface of a uniform random Young tableau of shape $\lambda^0$, based on solving a complex-valued polynomial equation,
- a simple criteria to determine if the limiting surface is continuous in the whole domain,
- and a local limit result in the bulk of a random Poissonized Young tableau of shape $\lambda^0$.

Our results have several consequences, for instance: they lead to explicit formulas for the limiting surface of $L$-shaped tableaux, generalizing the results of Pittel and Romik (2007) for rectangular shapes; they imply that the limiting surface for $L$-shaped tableaux is discontinuous for almost-every $L$-shape, and they give a new one-parameter family of infinite random Young tableaux, constructed from the so-called *random infinite bead process*.

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# A STOCHASTIC DIFFERENTIAL EQUATION FOR LOCAL TIMES OF SUPER-BROWNIAN MOTION

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We show that local times of super-Brownian motion, or of Brownian motion indexed by the Brownian tree, satisfy an explicit stochastic differential equation. Our proofs rely on both excursion theory for the Brownian snake and tools from the theory of superprocesses.

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