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AN INVARIANCE PRINCIPLE FOR THE 1D KPZ EQUATION

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Consider a discrete one-dimensional random surface whose height at a point grows as a function of the heights at neighboring points, plus an independent random noise. Assuming that this function is equivariant under constant shifts, symmetric in its arguments, and at least six times continuously differentiable in a neighborhood of the origin, we show that, as the variance of the noise goes to zero, any such process converges to the Cole–Hopf solution of the 1D KPZ equation under a suitable scaling of space and time. This proves an invariance principle for the 1D KPZ equation in the spirit of Donsker's invariance principle for Brownian motion.

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BIASED RANDOM WALK ON DYNAMICAL PERCOLATION

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We study biased random walks on dynamical percolation on $\mathbb{Z}^d$. We establish a law of large numbers and an invariance principle for the random walk using regeneration times. Moreover, we verify that the Einstein relation holds, and we investigate the speed of the walk as a function of the bias. While for $d = 1$ the speed is increasing, we show that, in general, this fails in dimension $d \geq 2$. As our main result, we establish two regimes of parameters, separated by an explicit critical curve such that the speed is either eventually strictly increasing or eventually strictly decreasing. This is in sharp contrast to the biased random walk on a static supercritical percolation cluster where the speed is known to be eventually zero.

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THE NUMBER OF ENDS IN THE UNIFORM SPANNING TREE FOR RECURRENT UNIMODULAR RANDOM GRAPHS

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We prove that if a unimodular random rooted graph is recurrent, the number of ends of its uniform spanning tree is almost surely equal to the number of ends of the graph. Together with previous results in the transient case, this completely resolves the problem of the number of ends of wired uniform spanning forest components in unimodular random rooted graphs and confirms a conjecture of Aldous and Lyons (2006).

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A BRANCHING PARTICLE SYSTEM AS A MODEL OF SEMIPUSHED FRONTS

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We consider a system of particles performing a one-dimensional dyadic branching Brownian motion with space-dependent branching rate, negative drift $-\mu$ and killed upon reaching 0, starting with N particles. More precisely, particles branch at rate $\rho/2$ in the interval $[0, 1]$, for some $\rho > 1$, and at rate $1/2$ in $(1, +\infty)$. The drift $\mu(\rho)$ is chosen in such a way that, heuristically, the system is critical in some sense: the number of particles stays roughly constant before it eventually dies out. This particle system can be seen as an analytically tractable model for fluctuating fronts, describing the internal mechanisms driving the invasion of a habitat by a cooperating population. Recent studies from Birzu, Hallatschek and Korolev suggest the existence of three classes of fluctuating fronts: pulled, semipushed and fully pushed fronts. Here we rigorously verify and make precise this classification and focus on the semipushed regime. This complements previous results from Berestycki, Berestycki and Schweinsberg for the case $\rho = 1$.

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INTERNAL DLA ON MATED-CRT MAPS

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We prove a shape theorem for internal diffusion limited aggregation on mated-CRT maps, a family of random planar maps which approximate Liouville quantum gravity (LQG) surfaces. The limit is a LQG harmonic ball, which we constructed in a companion paper. We also prove an analogous result for the divisible sandpile.

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RANDOM TREES HAVE HEIGHT $O(\sqrt{n})$

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We obtain new nonasymptotic tail bounds for the height of uniformly random trees with a given degree sequence, simply generated trees and conditioned Bienaymé trees (the family trees of branching processes) in the process settling three conjectures of Janson (*Probab. Surv.* **9** (2012) 103–252) and answering several other questions from the literature. Moreover, we define a partial ordering on degree sequences and show that it induces a stochastic ordering on the heights of uniformly random trees with given degree sequences. The latter result can also be used to show that sub-binary random trees are stochastically the tallest trees with a given number of vertices and leaves (and thus that random binary trees are the stochastically tallest random homeomorphically irreducible trees (*Acta Math.* **101** (1959) 141–162) with a given number of vertices).

Our proofs are based in part on the Foata–Fuchs bijection between trees and sequences (*J. Combin. Theory* **8** (1970) 361–375), which can be recast to provide a line-breaking construction of random trees with given vertex degrees (*Electron. Commun. Probab.* **28** (2023) 1–13).

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ON PERFECTLY FRIENDLY BISECTIONS OF RANDOM GRAPHS

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We prove that there exists a constant $\gamma_{\text{crit}} \approx 0.17566$ such that if $G \sim \mathbb{G}(n, 1/2)$, then for any $\varepsilon > 0$ with high probability G has a equipartition such that each vertex has $(\gamma_{\text{crit}} - \varepsilon)\sqrt{n}$ more neighbors in its own part than in the other part and with high probability no such partition exists for a separation of $(\gamma_{\text{crit}} + \varepsilon)\sqrt{n}$. The proof involves a number of tools ranging from isoperimetric results on vertex-transitive sets of graphs coming from Boolean functions, switchings, enumeration of graphs with a given degree sequence, and the second moment method. Our results substantially strengthen recent work of Ferber, Kwan, Narayanan, and the last two authors on a conjecture of Füredi from 1988 and, in particular, prove the existence of fully-friendly bisections in $\mathbb{G}(n, 1/2)$.

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POISSON–VORONOI PERCOLATION IN THE HYPERBOLIC PLANE WITH SMALL INTENSITIES

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We consider percolation on the Voronoi tessellation generated by a homogeneous Poisson point process on the hyperbolic plane. We show that the critical probability for the existence of an infinite cluster is asymptotically equal to $\pi\lambda/3$ as $\lambda \to 0$. This answers a question of Benjamini and Schramm (*J. Amer. Math. Soc.* **14** (2001) 487–507).

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