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## Articles

- An invariance principle for the 1D KPZ equation  
ARKA ADHIKARI AND SOURAV CHATTERJEE 2019
- Biased random walk on dynamical percolation  
SEBASTIAN ANDRES, NINA GANTERT, DOMINIK SCHMID AND PERLA SOUSI 2051
- The number of ends in the uniform spanning tree for recurrent unimodular random graphs  
DIEDERIK VAN ENGELENBURG AND TOM HUTCHCROFT 2079
- A branching particle system as a model of semipushed fronts . . . . . JULIE TOURNIAIRE 2104
- Internal DLA on mated-CRT maps . . . . . AHMED BOU-RABEE AND EWAIN GWYNNE 2173
- Random trees have height  $O(\sqrt{n})$   
LOUIGI ADDARIO-BERRY AND SERTE DONDERWINKEL 2238
- On perfectly friendly bisections of random graphs  
DOR MINZER, ASHWIN SAH AND MEHTAAB SAWHNEY 2281
- Poisson–Voronoi percolation in the hyperbolic plane with small intensities  
BENJAMIN HANSEN AND TOBIAS MÜLLER 2342

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# AN INVARIANCE PRINCIPLE FOR THE 1D KPZ EQUATION

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Consider a discrete one-dimensional random surface whose height at a point grows as a function of the heights at neighboring points, plus an independent random noise. Assuming that this function is equivariant under constant shifts, symmetric in its arguments, and at least six times continuously differentiable in a neighborhood of the origin, we show that, as the variance of the noise goes to zero, any such process converges to the Cole–Hopf solution of the 1D KPZ equation under a suitable scaling of space and time. This proves an invariance principle for the 1D KPZ equation in the spirit of Donsker’s invariance principle for Brownian motion.

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## BIASED RANDOM WALK ON DYNAMICAL PERCOLATION

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We study biased random walks on dynamical percolation on  $\mathbb{Z}^d$ . We establish a law of large numbers and an invariance principle for the random walk using regeneration times. Moreover, we verify that the Einstein relation holds, and we investigate the speed of the walk as a function of the bias. While for  $d = 1$  the speed is increasing, we show that, in general, this fails in dimension  $d \geq 2$ . As our main result, we establish two regimes of parameters, separated by an explicit critical curve such that the speed is either eventually strictly increasing or eventually strictly decreasing. This is in sharp contrast to the biased random walk on a static supercritical percolation cluster where the speed is known to be eventually zero.

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# THE NUMBER OF ENDS IN THE UNIFORM SPANNING TREE FOR RECURRENT UNIMODULAR RANDOM GRAPHS

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We prove that if a unimodular random rooted graph is recurrent, the number of ends of its uniform spanning tree is almost surely equal to the number of ends of the graph. Together with previous results in the transient case, this completely resolves the problem of the number of ends of wired uniform spanning forest components in unimodular random rooted graphs and confirms a conjecture of Aldous and Lyons (2006).

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# A BRANCHING PARTICLE SYSTEM AS A MODEL OF SEMIPUSHED FRONTS

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We consider a system of particles performing a one-dimensional dyadic branching Brownian motion with space-dependent branching rate, negative drift  $-\mu$  and killed upon reaching 0, starting with  $N$  particles. More precisely, particles branch at rate  $\rho/2$  in the interval  $[0, 1]$ , for some  $\rho > 1$ , and at rate  $1/2$  in  $(1, +\infty)$ . The drift  $\mu(\rho)$  is chosen in such a way that, heuristically, the system is critical in some sense: the number of particles stays roughly constant before it eventually dies out. This particle system can be seen as an analytically tractable model for fluctuating fronts, describing the internal mechanisms driving the invasion of a habitat by a cooperating population. Recent studies from Birzu, Hallatschek and Korolev suggest the existence of three classes of fluctuating fronts: pulled, semipushed and fully pushed fronts. Here we rigorously verify and make precise this classification and focus on the semipushed regime. This complements previous results from Berestycki, Berestycki and Schweinsberg for the case  $\rho = 1$ .

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## INTERNAL DLA ON MATED-CRT MAPS

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We prove a shape theorem for internal diffusion limited aggregation on mated-CRT maps, a family of random planar maps which approximate Liouville quantum gravity (LQG) surfaces. The limit is a LQG harmonic ball, which we constructed in a companion paper. We also prove an analogous result for the divisible sandpile.

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# RANDOM TREES HAVE HEIGHT $O(\sqrt{n})$

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We obtain new nonasymptotic tail bounds for the height of uniformly random trees with a given degree sequence, simply generated trees and conditioned Bienaymé trees (the family trees of branching processes) in the process settling three conjectures of Janson (*Probab. Surv.* **9** (2012) 103–252) and answering several other questions from the literature. Moreover, we define a partial ordering on degree sequences and show that it induces a stochastic ordering on the heights of uniformly random trees with given degree sequences. The latter result can also be used to show that sub-binary random trees are stochastically the tallest trees with a given number of vertices and leaves (and thus that random binary trees are the stochastically tallest random homeomorphically irreducible trees (*Acta Math.* **101** (1959) 141–162) with a given number of vertices).

Our proofs are based in part on the Foata–Fuchs bijection between trees and sequences (*J. Combin. Theory* **8** (1970) 361–375), which can be recast to provide a line-breaking construction of random trees with given vertex degrees (*Electron. Commun. Probab.* **28** (2023) 1–13).

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# ON PERFECTLY FRIENDLY BISECTIONS OF RANDOM GRAPHS

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We prove that there exists a constant  $\gamma_{\text{crit}} \approx 0.17566$  such that if  $G \sim \mathbb{G}(n, 1/2)$ , then for any  $\varepsilon > 0$  with high probability  $G$  has a equipartition such that each vertex has  $(\gamma_{\text{crit}} - \varepsilon)\sqrt{n}$  more neighbors in its own part than in the other part and with high probability no such partition exists for a separation of  $(\gamma_{\text{crit}} + \varepsilon)\sqrt{n}$ . The proof involves a number of tools ranging from isoperimetric results on vertex-transitive sets of graphs coming from Boolean functions, switchings, enumeration of graphs with a given degree sequence, and the second moment method. Our results substantially strengthen recent work of Ferber, Kwan, Narayanan, and the last two authors on a conjecture of Füredi from 1988 and, in particular, prove the existence of fully-friendly bisections in  $\mathbb{G}(n, 1/2)$ .

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# POISSON–VORONOI PERCOLATION IN THE HYPERBOLIC PLANE WITH SMALL INTENSITIES

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We consider percolation on the Voronoi tessellation generated by a homogeneous Poisson point process on the hyperbolic plane. We show that the critical probability for the existence of an infinite cluster is asymptotically equal to  $\pi\lambda/3$  as  $\lambda \rightarrow 0$ . This answers a question of Benjamini and Schramm (*J. Amer. Math. Soc.* **14** (2001) 487–507).

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VOLUME 52

2024

Articles

ADAMCZAK, RADOSŁAW, PIVOVAROV, PETER AND SIMANJUNTAK, PAUL. Limit theorems for the volumes of small codimensional random sections of $\ell_p^n$ -balls .....	93–126
ADDARIO-BERRY, LOUIGI AND DONDERWINKEL, SERTE. Random trees have height $O(\sqrt{n})$ .....	2238–2280
ADHIKARI, ARKA AND CHATTERJEE, SOURAV. An invariance principle for the 1D KPZ equation .....	2019–2050
ANDRES, SEBASTIAN, GANTERT, NINA, SCHMID, DOMINIK AND SOUSI, PERLA. Biased random walk on dynamical percolation .....	2051–2078
ASSELAH, AMINE, FORIEN, NICOLAS AND GAUDILLIÈRE, ALEXANDRE. The critical density for activated random walks is always less than 1 .....	1607–1649
BARNES, CLAYTON, MYTNIK, LEONID AND SUN, ZHENYAO. On the coming down from infinity of coalescing Brownian motions .....	67–92
BASU, RIDDHIPRATIM, BHATIA, MANAN AND GANGULY, SHIRSHENDU. En- vironment seen from infinite geodesics in Liouville Quantum Gravity ....	1399–1486
BAUERSCHMIDT, ROLAND, PARK, JIWOON AND RODRIGUEZ, PIERRE- FRANÇOIS. The Discrete Gaussian model, I. Renormalisation group flow at high temperature .....	1253–1359
BAUERSCHMIDT, ROLAND, PARK, JIWOON AND RODRIGUEZ, PIERRE- FRANÇOIS. The discrete Gaussian model, II. Infinite-volume scaling limit at high temperature .....	1360–1398
BELIAEV, DMITRY, MCAULEY, MICHAEL AND MUIRHEAD, STEPHEN. A central limit theorem for the number of excursion set components of Gaus- sian fields .....	882–922
BERESTYCKI, JULIEN, KIM, YUJIN H., LUBETZKY, EYAL, MALLEIN, BASTIEN AND ZEITOUNI, OFER. The extremal point process of branching Brownian motion in $\mathbb{R}^d$ .....	955–982
BÉTHENCOURT, LOÏC. Fractional diffusion limit for a kinetic Fokker–Planck equation with diffusive boundary conditions in the half-line .....	1713–1757
BHATIA, MANAN, GANGULY, SHIRSHENDU AND BASU, RIDDHIPRATIM. En- vironment seen from infinite geodesics in Liouville Quantum Gravity ....	1399–1486
BISKUP, MAREK AND LOUIDOR, OREN. A limit law for the most favorite point of simplerandom walk on a regular tree .....	502–544
BODINEAU, THIERRY, GALLAGHER, ISABELLE, SAINT-RAYMOND, LAURE AND SIMONELLA, SERGIO. Long-time derivation at equilibrium of the fluctuating Boltzmann equation .....	217–295
BOU-RABEE, AHMED AND GWYNNE, EWAIN. Internal DLA on mated-CRT maps .....	2173–2237
BURDZY, KRZYSZTOF AND ENGLÄNDER, JÁNOS. The spine of the Fleming– Viot process driven by Brownian motion .....	983–1015
BUSANI, OFER. Diffusive scaling limit of the Busemann process in last passage percolation .....	1650–1712
BUSANI, OFER, SEPPÄLÄINEN, TIMO AND SORENSEN, EVAN. The stationary horizon and semi-infinite geodesics in the directed landscape .....	1–66
CELENTANO, MICHAEL. Sudakov–Fernique post-AMP, and a new proof of the local convexity of the TAP free energy .....	923–954

CHAPUY, GUILLAUME, LOUF, BAPTISTE AND WALSH, HARRIET. Random partitions under the Plancherel–Hurwitz measure, high-genus Hurwitz numbers and maps .....	1225–1252
CHATTERJEE, SOURAV AND ADHIKARI, ARKA. An invariance principle for the 1D KPZ equation .....	2019–2050
CHEN, XIN AND WANG, JIAN. Two-sided heat kernel estimates for Schrödinger operators with unbounded potentials .....	1016–1047
CHEN, YU-TING. Delta-Bose gas from the viewpoint of the two-dimensional stochastic heat equation .....	127–187
DAREIOTIS, KONSTANTINOS AND GERENCSÉR, MÁTÉ. Path-by-path regularisation through multiplicative noise in rough, Young, and ordinary differential equations .....	1864–1902
DEMBO, AMIR AND OKADA, IZUMI. Capacity of the range of random walk: The law of the iterated logarithm .....	1954–1991
DOMINGUEZ, TOMAS AND MOURRAT, JEAN-CHRISTOPHE. Mutual information for the sparse stochastic block model .....	434–501
DONDERWINKEL, SERTE AND ADDARIO-BERRY, LOUIGI. Random trees have height $O(\sqrt{n})$ .....	2238–2280
ENGLÄNDER, JÁNOS AND BURDZY, KRZYSZTOF. The spine of the Fleming–Viot process driven by Brownian motion .....	983–1015
FEHRMAN, BENJAMIN. Stochastic homogenization with space-time ergodic divergence-free drift .....	350–380
FORIEN, NICOLAS, GAUDILLIÈRE, ALEXANDRE AND ASSELAH, AMINE. The critical density for activated random walks is always less than 1 .....	1607–1649
GALLAGHER, ISABELLE, SAINT-RAYMOND, LAURE, SIMONELLA, SERGIO AND BODINEAU, THIERRY. Long-time derivation at equilibrium of the fluctuating Boltzmann equation .....	217–295
GANGULY, SHIRSHENDU, BASU, RIDDHIPRATIM AND BHATIA, MANAN. Environment seen from infinite geodesics in Liouville Quantum Gravity ....	1399–1486
GANTERT, NINA, SCHMID, DOMINIK, SOUSI, PERLA AND ANDRES, SEBASTIAN. Biased random walk on dynamical percolation .....	2051–2078
GAUDILLIÈRE, ALEXANDRE, ASSELAH, AMINE AND FORIEN, NICOLAS. The critical density for activated random walks is always less than 1 .....	1607–1649
GERENCSÉR, MÁTÉ AND DAREIOTIS, KONSTANTINOS. Path-by-path regularisation through multiplicative noise in rough, Young, and ordinary differential equations .....	1864–1902
GESS, BENJAMIN AND TSATSOULIS, PAVLOS. Lyapunov exponents and synchronisation by noise for systems of SPDEs .....	1903–1953
GONZÁLEZ CASANOVA, ADRIÁN, TÓBIÁS, ANDRÁS AND VALESIN, DANIEL. Scaling limit of an adaptive contact process .....	296–349
GORIN, VADIM AND HUANG, JIAOYANG. Dynamical loop equation .....	1758–1863
GORSKI, CHRISTIAN. Strict monotonicity for first passage percolation on graphs of polynomial growth and quasi-trees .....	1487–1537
GWYNNE, EWAIN AND BOU-RABEE, AHMED. Internal DLA on mated-CRT maps .....	2173–2237
GWYNNE, EWAIN AND SUNG, JINWOO. The Minkowski content measure for the Liouville quantum gravity metric .....	658–712

HANSEN, BENJAMIN AND MÜLLER, TOBIAS. Poisson–Voronoi percolation in the hyperbolic plane with small intensities . . . . .	2342–2405
HAO, ZIMO, ZHANG, XICHENG, ZHU, RONGCHAN AND ZHU, XIANGCHAN. Singular kinetic equations and applications . . . . .	576–657
HARRIS, SIMON, JOHNSTON, SAMUEL G. G. AND PARDO, JUAN CARLOS. Universality classes for the coalescent structure of heavy-tailed Galton–Watson trees . . . . .	387–433
HERRY, RONAN, MALICET, DOMINIQUE AND POLY, GUILLAUME. Superconvergence phenomenon in Wiener chaoses . . . . .	1162–1200
HUANG, JIAOYANG AND GORIN, VADIM. Dynamical loop equation . . . . .	1758–1863
HUTCHCROFT, TOM AND VAN ENGELENBURG, DIEDERIK. The number of ends in the uniform spanning tree for recurrent unimodular random graphs . . . . .	2079–2103
IKSANOV, ALEXANDER, KOLESKO, KONRAD AND MEINERS, MATTHIAS. Asymptotic fluctuations in supercritical Crump–Mode–Jagers processes . . . . .	1538–1606
JOHNSTON, SAMUEL G. G., PARDO, JUAN CARLOS AND HARRIS, SIMON. Universality classes for the coalescent structure of heavy-tailed Galton–Watson trees . . . . .	387–433
KIM, YUJIN H., LUBETZKY, EYAL, MALLEIN, BASTIEN, ZEITOUNI, OFER AND BERESTYCKI, JULIEN. The extremal point process of branching Brownian motion in $\mathbb{R}^d$ . . . . .	955–982
KOLESKO, KONRAD, MEINERS, MATTHIAS AND IKSANOV, ALEXANDER. Asymptotic fluctuations in supercritical Crump–Mode–Jagers processes . . . . .	1538–1606
LE GALL, JEAN-FRANÇOIS. The Markov property of local times of Brownian motion indexed by the Brownian tree . . . . .	188–216
LIPNOWSKI, MICHAEL AND WRIGHT, ALEX. Towards optimal spectral gaps in large genus . . . . .	545–575
LOUF, BAPTISTE, WALSH, HARRIET AND CHAPUY, GUILLAUME. Random partitions under the Plancherel–Hurwitz measure, high-genus Hurwitz numbers and maps . . . . .	1225–1252
LOUDOR, OREN AND BISKUP, MAREK. A limit law for the most favorite point of simplerandom walk on a regular tree . . . . .	502–544
LUBETZKY, EYAL, MALLEIN, BASTIEN, ZEITOUNI, OFER, BERESTYCKI, JULIEN AND KIM, YUJIN H. The extremal point process of branching Brownian motion in $\mathbb{R}^d$ . . . . .	955–982
MAINI, LEONARDO AND NOURDIN, IVAN. Spectral central limit theorem for additive functionals of isotropic and stationary Gaussian fields . . . . .	737–763
MALICET, DOMINIQUE, POLY, GUILLAUME AND HERRY, RONAN. Superconvergence phenomenon in Wiener chaoses . . . . .	1162–1200
MALLEIN, BASTIEN, ZEITOUNI, OFER, BERESTYCKI, JULIEN, KIM, YUJIN H. AND LUBETZKY, EYAL. The extremal point process of branching Brownian motion in $\mathbb{R}^d$ . . . . .	955–982
MCAULEY, MICHAEL, MUIRHEAD, STEPHEN AND BELIAEV, DMITRY. A central limit theorem for the number of excursion set components of Gaussian fields . . . . .	882–922
MEINERS, MATTHIAS, IKSANOV, ALEXANDER AND KOLESKO, KONRAD. Asymptotic fluctuations in supercritical Crump–Mode–Jagers processes . . . . .	1538–1606
MINZER, DOR, SAH, ASHWIN AND SAWHNEY, MEHTAAB. On perfectly friendly bisections of random graphs . . . . .	2281–2341

MOURRAT, JEAN-CHRISTOPHE AND DOMINGUEZ, TOMAS. Mutual information for the sparse stochastic block model .....	434–501
MUIRHEAD, STEPHEN. Percolation of strongly correlated Gaussian fields II. Sharpness of the phase transition .....	838–881
MUIRHEAD, STEPHEN, BELIAEV, DMITRY AND MCAULEY, MICHAEL. A central limit theorem for the number of excursion set components of Gaussian fields .....	882–922
MÜLLER, TOBIAS AND HANSEN, BENJAMIN. Poisson–Voronoi percolation in the hyperbolic plane with small intensities .....	2342–2405
MYTNIK, LEONID, SUN, ZHENYAO AND BARNES, CLAYTON. On the coming down from infinity of coalescing Brownian motions .....	67–92
NARAYANAN, HARIHARAN AND SHEFFIELD, SCOTT. Large deviations for random hives and the spectrum of the sum of two random matrices .....	1093–1152
NOURDIN, IVAN AND MAINI, LEONARDO. Spectral central limit theorem for additive functionals of isotropic and stationary Gaussian fields .....	737–763
OKADA, IZUMI AND DEMBO, AMIR. Capacity of the range of random walk: The law of the iterated logarithm .....	1954–1991
PARDO, JUAN CARLOS, HARRIS, SIMON AND JOHNSTON, SAMUEL G. G. Universality classes for the coalescent structure of heavy-tailed Galton–Watson trees .....	387–433
PARK, JIWOON, RODRIGUEZ, PIERRE-FRANÇOIS AND BAUERSCHMIDT, ROLAND. The Discrete Gaussian model, I. Renormalisation group flow at high temperature .....	1253–1359
PARK, JIWOON, RODRIGUEZ, PIERRE-FRANÇOIS AND BAUERSCHMIDT, ROLAND. The discrete Gaussian model, II. Infinite-volume scaling limit at high temperature .....	1360–1398
PIVOVAROV, PETER, SIMANJUNTAK, PAUL AND ADAMCZAK, RADOSŁAW. Limit theorems for the volumes of small codimensional random sections of $\ell_p^n$ -balls .....	93–126
POLY, GUILLAUME, HERRY, RONAN AND MALICET, DOMINIQUE. Superconvergence phenomenon in Wiener chaoses .....	1162–1200
RODRIGUEZ, PIERRE-FRANÇOIS, BAUERSCHMIDT, ROLAND AND PARK, JIWOON. The Discrete Gaussian model, I. Renormalisation group flow at high temperature .....	1253–1359
RODRIGUEZ, PIERRE-FRANÇOIS, BAUERSCHMIDT, ROLAND AND PARK, JIWOON. The discrete Gaussian model, II. Infinite-volume scaling limit at high temperature .....	1360–1398
RUDELSON, MARK. A large deviation inequality for the rank of a random matrix .....	1992–2018
SAH, ASHWIN, SAWHNEY, MEHTAAB AND MINZER, DOR. On perfectly friendly bisections of random graphs .....	2281–2341
SAINT-RAYMOND, LAURE, SIMONELLA, SERGIO, BODINEAU, THIERRY AND GALLAGHER, ISABELLE. Long-time derivation at equilibrium of the fluctuating Boltzmann equation .....	217–295
SALEZ, JUSTIN. Spectral gap and curvature of monotone Markov chains .....	1153–1161
SAWHNEY, MEHTAAB, MINZER, DOR AND SAH, ASHWIN. On perfectly friendly bisections of random graphs .....	2281–2341
SCHMID, DOMINIK, SOUSI, PERLA, ANDRES, SEBASTIAN AND GANTERT, NINA. Biased random walk on dynamical percolation .....	2051–2078

SEPPÄLÄINEN, TIMO, SORENSEN, EVAN AND BUSANI, OFER. The stationary horizon and semi-infinite geodesics in the directed landscape . . . . .	1–66
SETHURAMAN, SUNDER AND XUE, JIANFEI. Condensation, boundary conditions, and effects of slow sites in zero-range systems . . . . .	1048–1092
SHEFFIELD, SCOTT AND NARAYANAN, HARIHARAN. Large deviations for random hives and the spectrum of the sum of two random matrices . . . . .	1093–1152
SIMANJUNTAK, PAUL, ADAMCZAK, RADOSŁAW AND PIVOVAROV, PETER. Limit theorems for the volumes of small codimensional random sections of $\ell_p^n$ -balls . . . . .	93–126
SIMONELLA, SERGIO, BODINEAU, THIERRY, GALLAGHER, ISABELLE AND SAINT-RAYMOND, LAURE. Long-time derivation at equilibrium of the fluctuating Boltzmann equation . . . . .	217–295
SORENSEN, EVAN, BUSANI, OFER AND SEPPÄLÄINEN, TIMO. The stationary horizon and semi-infinite geodesics in the directed landscape . . . . .	1–66
SOUSI, PERLA, ANDRES, SEBASTIAN, GANTERT, NINA AND SCHMID, DOMINIK. Biased random walk on dynamical percolation . . . . .	2051–2078
SUN, ZHENYAO, BARNES, CLAYTON AND MYTNIK, LEONID. On the coming down from infinity of coalescing Brownian motions . . . . .	67–92
SUNG, JINWOO AND GWYNNE, EWAIN. The Minkowski content measure for the Liouville quantum gravity metric . . . . .	658–712
TIKHOMIROV, KONSTANTIN AND YOUSSEF, PIERRE. Regularized modified log-Sobolev inequalities and comparison of Markov chains . . . . .	1201–1224
TÓBIÁS, ANDRÁS, VALESIN, DANIEL AND GONZÁLEZ CASANOVA, ADRIÁN. Scaling limit of an adaptive contact process . . . . .	296–349
TOURNAIRE, JULIE. A branching particle system as a model of semipushed fronts . . . . .	2104–2172
TSATSOU LIS, PAVLOS AND GESS, BENJAMIN. Lyapunov exponents and synchronisation by noise for systems of SPDEs . . . . .	1903–1953
VALESIN, DANIEL, GONZÁLEZ CASANOVA, ADRIÁN AND TÓBIÁS, ANDRÁS. Scaling limit of an adaptive contact process . . . . .	296–349
VAN ENGELENBURG, DIEDERIK AND HUTCHCROFT, TOM. The number of ends in the uniform spanning tree for recurrent unimodular random graphs . . . . .	2079–2103
WALSH, HARRIET, CHAPUY, GUILLAUME AND LOUF, BAPTISTE. Random partitions under the Plancherel–Hurwitz measure, high-genus Hurwitz numbers and maps . . . . .	1225–1252
WANG, JIAN AND CHEN, XIN. Two-sided heat kernel estimates for Schrödinger operators with unbounded potentials . . . . .	1016–1047
WRIGHT, ALEX AND LIPNOWSKI, MICHAEL. Towards optimal spectral gaps in large genus . . . . .	545–575
XU, CHANGJI, YANG, FAN, YAU, HORNG-TZER AND YIN, JUN. Bulk universality and quantum unique ergodicity for random band matrices in high dimensions . . . . .	765–837
XUE, JIANFEI AND SETHURAMAN, SUNDER. Condensation, boundary conditions, and effects of slow sites in zero-range systems . . . . .	1048–1092
YANG, FAN, YAU, HORNG-TZER, YIN, JUN AND XU, CHANGJI. Bulk universality and quantum unique ergodicity for random band matrices in high dimensions . . . . .	765–837



YAU, HORNG-TZER, YIN, JUN, XU, CHANGJI AND YANG, FAN. Bulk universality and quantum unique ergodicity for random band matrices in high dimensions . . . . .	765–837
YIN, JUN, XU, CHANGJI, YANG, FAN AND YAU, HORNG-TZER. Bulk universality and quantum unique ergodicity for random band matrices in high dimensions . . . . .	765–837
YOUSSEF, PIERRE AND TIKHOMIROV, KONSTANTIN. Regularized modified log-Sobolev inequalities and comparison of Markov chains . . . . .	1201–1224
ZEITOUNI, OFER, BERESTYCKI, JULIEN, KIM, YUJIN H., LUBETZKY, EYAL AND MALLEIN, BASTIEN. The extremal point process of branching Brownian motion in $\mathbb{R}^d$ . . . . .	955–982
ZHANG, LINGFU. Cutoff profile of the Metropolis biased card shuffling . . . . .	713–736
ZHANG, XICHENG, ZHU, RONGCHAN, ZHU, XIANGCHAN AND HAO, ZIMO. Singular kinetic equations and applications . . . . .	576–657
ZHU, RONGCHAN, ZHU, XIANGCHAN, HAO, ZIMO AND ZHANG, XICHENG. Singular kinetic equations and applications . . . . .	576–657
ZHU, XIANGCHAN, HAO, ZIMO, ZHANG, XICHENG AND ZHU, RONGCHAN. Singular kinetic equations and applications . . . . .	576–657

### Erratum

DUMINIL-COPIN, HUGO, RIVERA, ALEJANDRO, RODRIGUEZ, PIERRE-FRANÇOIS AND VANNEUVILLE, HUGO. “Existence of an unbounded nodal hypersurface for smooth Gaussian fields in dimension $d \geq 3$ ” . . . . .	381–385
--	---------

# The Annals of Probability

## Future Issues

### Articles

- Linear cover time is exponentially unlikely  
QUENTIN DUBROFF AND JEFF KAHN
- Uniform spanning tree in topological polygons, partition functions for SLE(8), and correlations in  $c = -2$  logarithmic CFT  
MINGCHANG LIU, EVELIINA PELTOLA AND HAO WU
- Regularization by noise for rough differential equations driven by Gaussian rough paths  
RÉMI CATELLIER AND ROMAIN DUBOSCQ
- Correlation decay for finite lattice gauge theories at weak coupling  
ARKA ADHIKARI AND SKY CAO
- Brownian motion with asymptotically normal reflection in unbounded domains: From transience to stability  
MIHA BREŠAR, ALEKSANDAR MIJATOVIĆ AND ANDREW WADE
- Solutions to the stochastic heat equation with polynomially growing multiplicative noise do not explode in the critical regime  
MICHAEL SALINS
- Tracy-Widom limit for free sum of random matrices  
HONG CHANG JI AND JAEWHI PARK
- A determinantal point process approach to scaling and local limits of random Young tableaux  
JACOPO BORGA, CÉDRIC BOUTILLIER, VALENTIN FÉRAY AND PIERRE-LOÏC MÉLIOT
- A stochastic differential equation for local times of super-Brownian motion  
JEAN-FRANÇOIS LE GALL AND EDWIN PERKINS
- Branching random walks on relatively hyperbolic groups  
MATTHIEU DUSSAULE, LONGMIN WANG AND WENYUAN YANG
- The three-dimensional stochastic Zakharov system  
SEBASTIAN HERR, MICHAEL RÖCKNER, MARTIN SPITZ AND DENG ZHANG
- Central limit theorem for Rényi divergence of infinite order  
SERGEY G. BOBKOV AND FRIEDRICH GÖTZE
- Macroscopic loops in the loop  $O(n)$  model via the XOR trick  
NICHOLAS CRAWFORD, ALEXANDER GLAZMAN, MATAN HAREL AND RON PELED
- On the tightness of the maximum of branching Brownian motion in random environment  
JIŘÍ ČERNÝ, ALEXANDER DREWITZ AND PASCAL OSWALD
- On the influence of edges in first-passage percolation on  $\mathbb{Z}^d$   
BARBARA DEMBIN, DOR ELBOIM AND RON PELED
- Fluctuations of partition functions of directed polymers in weak disorder beyond the  $L^2$ -phase  
STEFAN JUNK
- Approximation method to metastability: An application to nonreversible, two-dimensional Ising and Potts models without external fields  
SEONWOO KIM AND INSUK SEO
- The central limit theorem on nilpotent Lie groups  
TIMOTHÉE BÉNARD AND EMMANUEL BREUILLARD
- Degenerate processes killed at the boundary of a domain  
MICHEL BENAÏM, NICOLAS CHAMPAGNAT, WILLIAM OÇAÏRAIN AND DENIS VILLEMONAIS
- The contact process on dynamic regular graphs: Subcritical phase and monotonicity  
BRUNO SCHAPIRA AND DANIEL VALESIN
- Boundary touching probability and nested-path exponent for nonsimple CLE  
MORRIS ANG, XIN SUN, PU YU AND ZIJIE ZHUANG
- Second order fractional mean-field SDEs with singular kernels and measure initial data  
MICHAEL RÖCKNER, ZIMO HAO AND XICHENG ZHANG
- A stochastic analysis of subcritical Euclidean fermionic field theories  
FRANCESCO CARLO DE VECCHI, LUCA FRESTA AND MASSIMILIANO GUBINELLI
- Directed spatial permutations on asymmetric tori  
TYLER HELMUTH AND ALAN HAMMOND
- Exact phase transitions for stochastic block models and reconstruction on trees  
ELCHANAN MOSSEL, ALLAN SLY AND YOUNGTAK SOHN
- Universality of directed polymers in the intermediate disorder regime  
JULIAN RANSFORD
- Finite range interlacements and couplings  
HUGO DUMINIL-COPIN, SUBHAJIT GOSWAMI, PIERRE-FRANÇOIS RODRIGUEZ, FRANCO SEVERO AND AUGUSTO TEIXEIRA
- Liouville conformal field theory and the quantum zipper  
MORRIS ANG

Moderate deviations for the capacity of the random walk range in dimension four  
ARKA ADHIKARI AND IZUMI OKADA

Recurrence and transience of multidimensional elephant random walks  
SHUO QIN

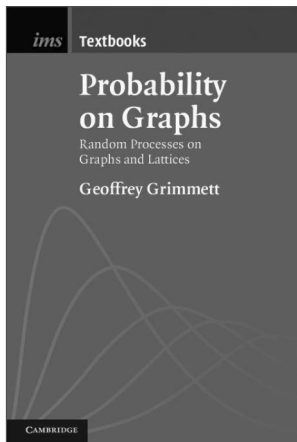
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