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REGULARIZED MODIFIED LOG-SOBOLEV INEQUALITIES AND COMPARISON OF MARKOV CHAINS

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In this work we develop a comparison procedure for the modified log-Sobolev inequality (MLSI) constants of two reversible Markov chains on a finite state space. Efficient comparison of the MLSI Dirichlet forms is a well-known obstacle in the theory of Markov chains. We approach this problem by introducing a *regularized* MLSI constant, which, under some assumptions, has the same order of magnitude as the usual MLSI constant yet is amenable for comparison and thus considerably simpler to estimate in certain cases. As an application of this general comparison procedure, we provide a sharp estimate of the MLSI constant of the switch chain on the set of simple bipartite regular graphs of size n with a fixed degree d. Our estimate implies that the total variation mixing time of the switch chain is of order $O_d(n \log n)$. The result is optimal up to a multiple depending on d and resolves a long-standing open problem. We expect that the MLSI comparison technique implemented in this paper will find further applications.

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RANDOM PARTITIONS UNDER THE PLANCHEREL–HURWITZ MEASURE, HIGH-GENUS HURWITZ NUMBERS AND MAPS

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We study the asymptotic behaviour of random integer partitions under a new probability law that we introduce, the Plancherel–Hurwitz measure. This distribution, which has a natural definition in terms of Young tableaux, is a deformation of the classical Plancherel measure, which appears naturally in the context of Hurwitz numbers, enumerating certain transposition factorisations in symmetric groups.

We study a regime in which the number of factors in the underlying factorisations grows linearly with the order of the group, and the corresponding topological objects, Hurwitz maps, are of high genus. We prove that the limiting behaviour exhibits a new, twofold, phenomenon: the first part becomes very large, while the rest of the partition has the standard Vershik–Kerov–Logan–Shepp limit shape. As a consequence, we obtain asymptotic estimates for unconnected Hurwitz numbers with linear Euler characteristic, which we use to study random Hurwitz maps in this regime. This result can also be interpreted as the return probability of the transposition random walk on the symmetric group after linearly many steps.

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THE DISCRETE GAUSSIAN MODEL, I. RENORMALISATION GROUP FLOW AT HIGH TEMPERATURE

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The Discrete Gaussian model is the lattice Gaussian free field conditioned to be integer-valued. In two dimensions and at sufficiently high temperature, we show that its macroscopic scaling limit on the torus is a multiple of the Gaussian free field. Our proof starts from a single renormalisation group step after which the integer-valued field becomes a smooth field, which we then analyse using the renormalisation group method.

This paper also provides the foundation for the construction of the scaling limit of the infinite-volume gradient Gibbs state of the Discrete Gaussian model in the companion paper. Moreover, we develop all estimates for general finite-range interaction with sharp dependence on the range. We expect these estimates to prepare for a future analysis of the spread-out version of the Discrete Gaussian model at its critical temperature.

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THE DISCRETE GAUSSIAN MODEL, II. INFINITE-VOLUME SCALING LIMIT AT HIGH TEMPERATURE

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The Discrete Gaussian model is the lattice Gaussian free field conditioned to be integer-valued. In two dimensions and at sufficiently high temperature, we show that the scaling limit of the infinite-volume gradient Gibbs state with zero mean is a multiple of the Gaussian free field.

This article is the second in a series on the Discrete Gaussian model, extending the methods of the first paper by the analysis of general external fields (rather than macroscopic test functions on the torus). As a byproduct, we also obtain a scaling limit for mesoscopic test functions on the torus.

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ENVIRONMENT SEEN FROM INFINITE GEODESICS IN LIOUVILLE QUANTUM GRAVITY

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First passage percolation (FPP) on $\mathbb{Z}^d$ or $\mathbb{R}^d$ is a canonical model of a random metric space where the standard Euclidean geometry is distorted by random noise. Of central interest is the length and the geometry of the geodesic, the shortest path between points. Since the latter, owing to its length minimization, traverses through atypically low values of the underlying noise variables, it is an important problem to quantify the disparity between the environment rooted at a point on the geodesic and the typical one. We investigate this in the context of γ-Liouville quantum gravity (LQG) (where $\gamma \in (0, 2)$ is a parameter)—a random Riemannian surface induced on the complex plane by the random metric tensor $e^{2\gamma h/d_\gamma}(dx^2 + dy^2)$, where h is the whole plane, properly centered, Gaussian Free Field (GFF), and d_γ is the associated dimension. We consider the unique infinite geodesic Γ from the origin, parametrized by the logarithm of its chemical length, and show that, for an almost sure realization of h, the distributions of the appropriately scaled field and the induced metric on a ball, rooted at a point "uniformly" sampled on Γ, converge to deterministic measures on the space of generalized functions and continuous metrics on the unit disk, respectively. Moreover, toward a better understanding of the limiting objects living on the unit disk, we show that they are singular with respect to their typical counterparts but become absolutely continuous away from the origin. Our arguments rely on unearthing a regeneration structure with fast decay of correlation in the geodesic owing to coalescence and the domain Markov property of the GFF. While there have been significant recent advances around this question for stochastic planar growth models in the Kardar–Parisi–Zhang universality class, the present work initiates this research program in the context of LQG.

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STRICT MONOTONICITY FOR FIRST PASSAGE PERCOLATION ON GRAPHS OF POLYNOMIAL GROWTH AND QUASI-TREES

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We prove a strict monotonicity theorem for first passage percolation on any Cayley graph of a virtually nilpotent group which is not isomorphic to the standard Cayley graph of $\mathbb{Z}$: given two distributions ν and $\tilde{\nu}$ with finite mean, if $\tilde{\nu}$ is strictly more variable than ν and ν is subcritical in an appropriate sense, then the expected passage times associated to ν exceed those of $\tilde{\nu}$ by an amount proportional to the graph distance. This generalizes a theorem of van den Berg and Kesten from 1993, which treats the standard Cayley graphs of $\mathbb{Z}^d, d \geq 2$. In fact, our theorem applies to any bounded-degree graph which either is of strict polynomial growth or is quasi-isometric to a tree and which satisfies a certain geometric condition we call "admitting detours." If a bounded degree graph does not admit detours, such a strict monotonicity theorem with respect to variability cannot hold.

Moreover, we show that for the same class of graphs, independent of whether the graph admits detours, any appropriately subcritical weight measure is "absolutely continuous with respect to the expected empirical measure of the geodesic." This implies a strict monotonicity theorem with respect to stochastic domination.

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ASYMPTOTIC FLUCTUATIONS IN SUPERCRITICAL CRUMP–MODE–JAGERS PROCESSES

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Consider a supercritical Crump–Mode–Jagers process $(\mathcal{Z}_t^\varphi)_{t\geq 0}$ counted with a random characteristic φ. Nerman's celebrated law of large numbers (*Z. Wahrsch. Verw. Gebiete* **57** (1981) 365–395) states that, under some mild assumptions, $e^{-\alpha t}\mathcal{Z}_t^\varphi$ converges almost surely as $t \to \infty$ to aW. Here, $\alpha > 0$ is the Malthusian parameter, a is a constant and W is the limit of Nerman's martingale, which is positive on the survival event. In this general situation, under additional (second moment) assumptions, we prove a central limit theorem for $(\mathcal{Z}_t^\varphi)_{t\geq 0}$. More precisely, we show that there exist a constant $k \in \mathbb{N}_0$ and a function $H(t)$, a finite random linear combination of functions of the form $t^j e^{\lambda t}$ with $\alpha/2 \leq \mathrm{Re}(\lambda) < \alpha$, such that $(\mathcal{Z}_t^\varphi - ae^{\alpha t}W - H(t))/\sqrt{t^k e^{\alpha t}}$ converges in distribution to a normal random variable with random variance. This result unifies and extends various central limit theorem-type results for specific branching processes.

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