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A PHASE TRANSITION FOR REPEATED AVERAGES

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Let $x_1, \dots, x_n$ be a fixed sequence of real numbers. At each stage, pick two indices I and J uniformly at random, and replace x_I, x_J by $(x_I + x_J)/2$, $(x_I + x_J)/2$. Clearly, all the coordinates converge to $(x_1 + \cdots + x_n)/n$. We determine the rate of convergence, establishing a sharp "cutoff" transition answering a question of Jean Bourgain.

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HIDDEN INVARIANCE OF LAST PASSAGE PERCOLATION AND DIRECTED POLYMERS

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Last passage percolation and directed polymer models on $\mathbb{Z}^2$ are invariant under translation and certain reflections. When these models have an integrable structure coming from either the RSK correspondence or the geometric RSK correspondence (e.g., geometric last passage percolation or the log-gamma polymer), we show that these basic invariances can be combined with a decoupling property to yield a rich new set of symmetries. Among other results, we prove shift and rearrangement invariance statements for last passage times, geodesic locations, disjointness probabilities, polymer partition functions and quenched polymer measures. We also use our framework to find "scrambled" versions of the classical RSK correspondence and to find an RSK correspondence for moon polyominoes. The results extend to limiting models, including the KPZ equation and the Airy sheet.

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SHANNON'S THEOREM FOR LOCALLY COMPACT GROUPS

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We establish the ray and strip approximation criteria for the identification of the Poisson boundary of random walks on locally compact groups. This settles a conjecture from the 1990s by Kaimanovich, who formulated and proved the criterion for discrete groups. The key result is the proof of a version of the Shannon–McMillan–Breiman theorem for locally compact groups. We provide several applications to locally compact groups of isometries of nonpositively curved spaces, as well as Diestel–Leader graphs and horocylic products.

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UNIVERSALITY OF THE GEODESIC TREE IN LAST PASSAGE PERCOLATION

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In this paper, we consider the geodesic tree in exponential last passage percolation. We show that for a large class of initial conditions around the origin, the line-to-point geodesic that terminates in a cylinder located around the point (N, N), and whose width and length are $o(N^{2/3})$ and $o(N)$, respectively, agrees in the cylinder, with the stationary geodesic sharing the same end-point. In the case of the point-to-point model where the geodesic starts from the origin, we consider width $\delta N^{2/3}$, length up to $\delta^{3/2}N/(\log(\delta^{-1}))^3$, and provide lower and upper bounds for the probability that the geodesics agree in that cylinder.

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LARGE N LIMIT OF THE $O(N)$ LINEAR SIGMA MODEL VIA STOCHASTIC QUANTIZATION

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This article studies large N limits of a coupled system of N interacting Φ^4 equations posed over $\mathbb{T}^d$ for $d = 2$, known as the $O(N)$ linear sigma model. Uniform in N bounds on the dynamics are established, allowing us to show convergence to a mean-field singular SPDE, also proved to be globally well posed. Moreover, we show tightness of the invariant measures in the large N limit. For large enough mass, they converge to the (massive) Gaussian free field, the unique invariant measure of the mean-field dynamics, at a rate of order $1/\sqrt{N}$ with respect to the Wasserstein distance. We also consider fluctuations and obtain tightness results for certain $O(N)$ invariant observables, along with an exact description of the limiting correlations.

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UNIVERSALITY OF CUTOFF FOR GRAPHS WITH AN ADDED RANDOM MATCHING

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We establish universality of cutoff for simple random walk on a class of random graphs defined as follows. Given a finite graph $G = (V, E)$ with $|V|$ even we define a random graph $G^* = (V, E \cup E')$ obtained by picking E' to be the (unordered) pairs of a random perfect matching of V. We show that for a sequence of such graphs G_n of diverging sizes and of uniformly bounded degree, if the minimal size of a connected component of G_n is at least 3 for all n, then the random walk on G_n^* exhibits cutoff w.h.p. This provides a simple generic operation of adding some randomness to a given graph, which results in cutoff.

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ALMOST-SURE EXPONENTIAL MIXING OF PASSIVE SCALARS BY THE STOCHASTIC NAVIER–STOKES EQUATIONS

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We deduce almost-sure exponentially fast mixing of passive scalars advected by solutions of the stochastically-forced 2D Navier–Stokes equations and 3D hyper-viscous Navier–Stokes equations in $\mathbb{T}^d$ subjected to nondegenerate H^σ-regular noise for any σ sufficiently large. That is, for all $s > 0$ there is a deterministic exponential decay rate such that all mean-zero H^s passive scalars decay in H^{-s} at this same rate with probability one. This is equivalent to what is known as *quenched correlation decay* for the Lagrangian flow in the dynamical systems literature. This is a follow-up to our previous work, which establishes a positive Lyapunov exponent for the Lagrangian flow—in general, almost-sure exponential mixing is much stronger than this. Our methods also apply to velocity fields evolving according to finite-dimensional models, for example, Galerkin truncations of Navier–Stokes or the Stokes equations with very degenerate forcing. For all $0 \le k < \infty$, this exhibits many examples of $C_t^k C_x^\infty$ random velocity fields that are almost-sure exponentially fast mixers.

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CUTOFF FOR RANDOM LIFTS OF WEIGHTED GRAPHS

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We prove the cutoff phenomenon for the random walk on random n-lifts of finite weighted graphs, even when the random walk on the base graph $\mathcal{G}$ of the lift is not reversible. The mixing time is w.h.p. $t_{\mathrm{mix}} = h^{-1} \log n$, where h is a constant associated to $\mathcal{G}$, namely the entropy of its universal cover. Moreover, this mixing time is the smallest possible among all n-lifts of $\mathcal{G}$. In the particular case where the base graph is a vertex with $d/2$ loops, d even, we obtain a cutoff for a d-regular random graph, as did Lubetzky and Sly in (*Duke Math. J.* **153** (2010) 475–510) (with a slightly different distribution on d-regular graphs, but the mixing time is the same).

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SCALING LIMIT OF THE SUBDIFFUSIVE RANDOM WALK ON A GALTON–WATSON TREE IN RANDOM ENVIRONMENT

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We consider a random walk on a Galton–Watson tree in random environment, in the subdiffusive case. We prove the convergence of the renormalised height function of the walk towards the continuous-time height process of a spectrally positive strictly stable Lévy process, jointly with the convergence of the renormalised trace of the walk towards the real tree coded by the latter continuous-time height process.

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THE SUPREMA OF INFINITELY DIVISIBLE PROCESSES

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In this paper we complete the full characterization of the expected suprema of infinitely divisible processes. In particular, we remove the technical assumption called $H(C_0, \delta)$ condition and settle positively the conjecture posed by M. Talagrand.

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