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TAP FREE ENERGY, SPIN GLASSES AND VARIATIONAL INFERENCE

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We consider the Sherrington–Kirkpatrick model of spin glasses with ferromagnetically biased couplings. For a specific choice of the couplings mean, the resulting Gibbs measure is equivalent to the Bayesian posterior for a high-dimensional estimation problem known as "$\mathbb{Z}_2$ synchronization." Statistical physics suggests to compute the expectation with respect to this Gibbs measure (the posterior mean in the synchronization problem), by minimizing the so-called Thouless–Anderson–Palmer (TAP) free energy, instead of the mean field (MF) free energy. We prove that this identification is correct, provided the ferromagnetic bias is larger than a constant (i.e., the noise level is small enough in synchronization). Namely, we prove that the scaled ℓ_2 distance between any low energy local minimizers of the TAP free energy and the mean of the Gibbs measure vanishes in the large size limit. Our proof technique is based on upper bounding the expected number of critical points of the TAP free energy using the Kac–Rice formula.

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LOCAL LAWS AND RIGIDITY FOR COULOMB GASES AT ANY TEMPERATURE

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We study Coulomb gases in any dimension $\mathsf{d} \geq 2$ and in a broad temperature regime. We prove local laws on the energy, separation and number of points down to the microscopic scale. These yield the existence of limiting point processes after extraction, generalizing the Ginibre point process for arbitrary temperature and dimension. The local laws come together with a quantitative expansion of the free energy with a new explicit error rate in the case of a uniform background density. These estimates have explicit temperature dependence, allowing to treat regimes of very large or very small temperature, and exhibit a new minimal lengthscale for rigidity and screening, depending on the temperature. They apply as well to energy minimizers (formally zero temperature). The method is based on a bootstrap on scales and reveals the additivity of the energy modulo surface terms, via the introduction of subadditive and superadditive approximate energies.

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2D ANISOTROPIC KPZ AT STATIONARITY: SCALING, TIGHTNESS AND NONTRIVIALITY

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In this work we focus on the two-dimensional anisotropic KPZ (aKPZ) equation, which is formally given by

$$\partial_t h = \frac{\nu}{2}\Delta h + \lambda\big((\partial_1 h)^2 - (\partial_2 h)^2\big) + \nu^{\frac{1}{2}}\xi,$$

where ξ denotes a noise which is white in both space and time, and λ and ν are positive constants. Due to the wild oscillations of the noise and the quadratic nonlinearity, the previous equation is classically ill posed. It is not possible to linearise it via the Cole–Hopf transformation and the pathwise techniques for singular SPDEs (the theory of regularity structures by M. Hairer or the paracontrolled distributions approach of M. Gubinelli, P. Imkeller, N. Perkowski) are not applicable. In the present work we consider a regularised version of aKPZ which preserves its invariant measure. We prove the existence of subsequential limits once the regularisation is removed, provided λ and ν are suitably renormalised. Moreover, we show that, in the regime in which ν is constant and the coupling constant λ converges to 0 as the inverse of the square root logarithm, any limit differs from the solution to the linear equation obtained by simply dropping the nonlinearity in aKPZ.

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FREENESS OVER THE DIAGONAL FOR LARGE RANDOM MATRICES

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We prove that independent families of permutation invariant random matrices are asymptotically free with amalgamation over the diagonal, both in expectation and in probability, under a uniform boundedness assumption on the operator norm. We can relax the operator norm assumption to an estimate on sums associated to graphs of matrices, further extending the range of applications (e.g., to Wigner matrices with exploding moments and the sparse regime of the Erdős–Rényi model). The result still holds even if the matrices are multiplied entrywise by random variables satisfying a certain growth condition (e.g., as in the case of matrices with a variance profile and percolation models). Our analysis relies on a modified method of moments based on graph observables.

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THE OVERLAP GAP PROPERTY AND APPROXIMATE MESSAGE PASSING ALGORITHMS FOR p-SPIN MODELS

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We consider the algorithmic problem of finding a near ground state (near optimal solution) of a p-spin model. We show that for a class of algorithms broadly defined as Approximate Message Passing (AMP), the presence of the Overlap Gap Property (OGP), appropriately defined, is a barrier. We conjecture that, when $p \geq 4$, the model does indeed exhibit OGP (and prove it for the space of binary solutions). Assuming the validity of this conjecture, as an implication the AMP fails to find near ground states in these models, per our result. We extend our result to the problem of finding pure states by means of Thouless, Anderson and Palmer (TAP) based iterations which is yet another example of AMP type algorithms. We show that such iterations fail to find pure states approximately, subject to the conjecture that the space of pure states exhibits the OGP, appropriately stated, when $p \geq 4$.

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DISTRIBUTION OF THE RANDOM WALK CONDITIONED ON SURVIVAL AMONG QUENCHED BERNOULLI OBSTACLES

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Place an obstacle with probability $1-p$ independently at each vertex of $\mathbb{Z}^d$, and consider a simple symmetric random walk that is killed upon hitting one of the obstacles. For $d \geq 2$ and p strictly above the critical threshold for site percolation, we condition on the environment such that the origin is contained in an infinite connected component free of obstacles. It has previously been shown that, with high probability, the random walk conditioned on survival up to time n will be localized in a ball of volume, asymptotically, $d \log_{1/p} n$. In this work we prove that this ball is free of obstacles, and we derive the limiting one-time distributions of the random walk conditioned on survival. Our proof is based on obstacle modifications and estimates on how such modifications affect the probability of the obstacle configurations as well as their associated Dirichlet eigenvalues which is of independent interest.

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SURVIVAL AND EXTINCTION OF EPIDEMICS ON RANDOM GRAPHS WITH GENERAL DEGREE

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In this paper we establish the necessary and sufficient criterion for the contact process on Galton–Watson trees (resp., random graphs) to exhibit the phase of extinction (resp., short survival). We prove that the survival threshold λ_1 for a Galton–Watson tree is strictly positive if and only if its offspring distribution ξ has an exponential tail, that is, $\mathbb{E}e^{c\xi} < \infty$ for some $c > 0$, settling a conjecture by Huang and Durrett (2018). On the random graph with degree distribution μ, we show that if μ has an exponential tail, then for small enough λ the contact process with the all-infected initial condition survives for $n^{1+o(1)}$-time whp (short survival), while for large enough λ it runs over $e^{\Theta(n)}$-time whp (long survival). When μ is subexponential, we prove that the contact process whp displays long survival for any fixed $\lambda > 0$.

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GRAVITATIONAL ALLOCATION FOR UNIFORM POINTS ON THE SPHERE

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Given a collection $\mathcal{L}$ of n points on a sphere $\mathbf{S}_n^2$ of surface area n, a fair allocation is a partition of the sphere into n cells each of area 1, and each associated with a distinct point of $\mathcal{L}$. We show that if the n points are chosen uniformly at random and the partition is defined by considering a "gravitational" potential defined by the n points, then the expected distance between a point on the sphere and the associated point of $\mathcal{L}$ is $O(\sqrt{\log n})$. We use our result to define a matching between two collections of n independent and uniform points on the sphere and prove that the expected distance between a pair of matched points is $O(\sqrt{\log n})$, which is optimal by a result of Ajtai, Komlós and Tusnády. Furthermore, we prove that the expected number of maxima for the gravitational potential is $\Theta(n/\log n)$. We also study gravitational allocation on the sphere to the zero set $\mathcal{L}$ of a particular Gaussian polynomial, and we quantify the repulsion between the points of $\mathcal{L}$ by proving that the expected distance between a point on the sphere and the associated point of $\mathcal{L}$ is $O(1)$.

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SCALING LIMIT OF DYNAMICAL PERCOLATION ON CRITICAL ERDŐS–RÉNYI RANDOM GRAPHS

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Consider a critical Erdős–Rényi random graph: n is the number of vertices, each one of the $\binom{n}{2}$ possible edges is kept in the graph independently from the others with probability $n^{-1} + \lambda n^{-4/3}$, λ being a fixed real number. When n goes to infinity, Addario-Berry, Broutin and Goldschmidt (*Probab. Theory Related Fields* **152** (2012) 367–406) have shown that the collection of connected components, viewed as suitably normalized measured compact metric spaces, converges in distribution to a continuous limit $\mathcal{G}_\lambda$ made of random real graphs. In this paper, we consider notably the dynamical percolation on critical Erdős–Rényi random graphs. To each pair of vertices is attached a Poisson process of intensity $n^{-1/3}$, and every time it rings, one resamples the corresponding edge. Under this process, the collection of connected components undergoes coalescence and fragmentation. We prove that this process converges in distribution, as n goes to infinity, toward a fragmentation-coalescence process on the continuous limit $\mathcal{G}_\lambda$. We also prove convergence of discrete coalescence and fragmentation processes and provide general Feller-type properties associated to fragmentation and coalescence.

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ON THE UNIQUENESS OF GLOBAL MULTIPLE SLES

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This article focuses on the characterization of global multiple Schramm–Loewner evolutions (SLE). The chordal SLE describes the scaling limit of a single interface in various critical lattice models with Dobrushin boundary conditions, and similarly, global multiple SLEs describe scaling limits of collections of interfaces in critical lattice models with alternating boundary conditions. In this article, we give a minimal amount of characterizing properties for the global multiple SLEs: we prove that there exists a unique probability measure on collections of pairwise disjoint continuous simple curves with a certain conditional law property. As a consequence, we obtain the convergence of multiple interfaces in the critical Ising, FK-Ising and percolation models.

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THE 2D-DIRECTED SPANNING FOREST CONVERGES TO THE BROWNIAN WEB

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The two-dimensional *directed spanning forest* (DSF) introduced by Baccelli and Bordenave is a planar directed forest whose vertex set is given by a homogeneous Poisson point process $\mathcal{N}$ on $\mathbb{R}^2$. If the DSF has direction $-e_y$, the ancestor $h(\mathbf{u})$ of a vertex $\mathbf{u} \in \mathcal{N}$ is the nearest Poisson point (in the L_2 distance) having strictly larger y-coordinate. This construction induces complex geometrical dependencies. In this paper, we show that the collection of DSF paths, properly scaled, converges in distribution to the Brownian web (BW). This verifies a conjecture made by Baccelli and Bordenave in 2007 (*Ann. Appl. Probab.* **17** (2007) 305–359).

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FRACTAL GEOMETRY OF AIRY$_2$ PROCESSES COUPLED VIA THE AIRY SHEET

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In last passage percolation models lying in the Kardar–Parisi–Zhang universality class, maximizing paths that travel over distances of order n accrue energy that fluctuates on scale $n^{1/3}$; and these paths deviate from the linear interpolation of their endpoints on scale $n^{2/3}$. These maximizing paths and their energies may be viewed via a coordinate system that respects these scalings. What emerges by doing so is a system indexed by $x, y \in \mathbb{R}$ and $s, t \in \mathbb{R}$ with $s < t$ of unit order quantities $W_n(x, s; y, t)$ specifying the scaled energy of the maximizing path that moves in scaled coordinates between (x, s) and (y, t). The space-time Airy sheet is, after a parabolic adjustment, the putative distributional limit W_∞ of this system as $n \to \infty$. The Airy sheet has recently been constructed in (Dauvergne, Ortmann and Virág (2020)) as such a limit of Brownian last passage percolation. In this article, we initiate the study of fractal geometry in the Airy sheet. We prove that the scaled energy difference profile given by $\mathbb{R} \to \mathbb{R} : z \to W_\infty(1, 0; z, 1) - W_\infty(-1, 0; z, 1)$ is a nondecreasing process that is constant in a random neighbourhood of almost every $z \in \mathbb{R}$; and that the exceptional set of $z \in \mathbb{R}$ that violate this condition almost surely has Hausdorff dimension one-half. Points of violation correspond to special behaviour for scaled maximizing paths, and we prove the result by investigating this behaviour, making use of two inputs from recent studies of scaled Brownian LPP; namely, Brownian regularity of profiles, and estimates on the rarity of pairs of disjoint scaled maximizing paths that begin and end close to each other.

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MODIFIED LOG-SOBOLEV INEQUALITIES FOR STRONGLY LOG-CONCAVE DISTRIBUTIONS

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We show that the modified log-Sobolev constant for a natural Markov chain which converges to an r-homogeneous strongly log-concave distribution is at least $1/r$. Applications include a sharp mixing time bound for the bases-exchange walk for matroids, and a concentration bound for Lipschitz functions over these distributions.

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