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MAXIMUM OF THE GINZBURG–LANDAU FIELDS

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We study a two-dimensional massless field in a box with potential $V(\nabla\phi(\cdot))$ and zero boundary condition, where V is any symmetric and uniformly convex function. Naddaf–Spencer (*Comm. Math. Phys.* **183** (1997) 55–84) and Miller (*Comm. Math. Phys.* **308** (2011) 591–639) proved that the rescaled macroscopic averages of this field converge to a continuum Gaussian free field. In this paper, we prove that the distribution of local marginal $\phi(x)$, for any x in the bulk, has a Gaussian tail. We further characterize the leading order of the maximum and the dimension of high points of this field, thus generalizing the results of Bolthausen–Deuschel–Giacomin (*Ann. Probab.* **29** (2001) 1670–1692) and Daviaud (*Ann. Probab.* **34** (2006) 962–986) for the discrete Gaussian free field.

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THE RIEMANN ZETA FUNCTION AND GAUSSIAN MULTIPLICATIVE CHAOS: STATISTICS ON THE CRITICAL LINE

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We prove that if ω is uniformly distributed on $[0, 1]$, then as $T \to \infty$, $t \mapsto \zeta(i\omega T + it + 1/2)$ converges to a nontrivial random generalized function, which in turn is identified as a product of a very well-behaved random smooth function and a random generalized function known as a complex Gaussian multiplicative chaos distribution. This demonstrates a novel rigorous connection between probabilistic number theory and the theory of multiplicative chaos—the latter is known to be connected to various branches of modern probability theory and mathematical physics.

We also investigate the statistical behavior of the zeta function on the mesoscopic scale. We prove that if we let δ_T approach zero slowly enough as $T \to \infty$, then $t \mapsto \zeta(1/2 + i\delta_T t + i\omega T)$ is asymptotically a product of a divergent scalar quantity suggested by Selberg's central limit theorem and a strictly Gaussian multiplicative chaos. We also prove a similar result for the characteristic polynomial of a Haar distributed random unitary matrix, where the scalar quantity is slightly different but the multiplicative chaos part is identical. This says that up to scalar multiples, the zeta function and the characteristic polynomial of a Haar distributed random unitary matrix have an identical distribution on the mesoscopic scale.

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LOCALIZATION IN GAUSSIAN DISORDERED SYSTEMS AT LOW TEMPERATURE

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For a broad class of Gaussian disordered systems at low temperature, we show that the Gibbs measure is asymptotically localized in small neighborhoods of a small number of states. From a single argument, we obtain: (i) a version of "complete" path localization for directed polymers that is not available even for exactly solvable models, and (ii) a result about the exhaustiveness of Gibbs states in spin glasses not requiring the Ghirlanda–Guerra identities.

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SPECTRAL GAPS FOR REVERSIBLE MARKOV PROCESSES WITH CHAOTIC INVARIANT MEASURES: THE KAC PROCESS WITH HARD SPHERE COLLISIONS IN THREE DIMENSIONS

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We develop a method for producing estimates on the spectral gaps of reversible Markov jump processes with chaotic invariant measures, that is effective in the case of degenerate jump rates, and we apply it to prove the Kac conjecture for hard sphere collision in three dimensions.

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A COVARIANCE FORMULA FOR TOPOLOGICAL EVENTS OF SMOOTH GAUSSIAN FIELDS

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We derive a covariance formula for the class of 'topological events' of smooth Gaussian fields on manifolds; these are events that depend only on the topology of the level sets of the field, for example, (i) crossing events for level or excursion sets, (ii) events measurable with respect to the number of connected components of level or excursion sets of a given diffeomorphism class and (iii) persistence events. As an application of the covariance formula, we derive strong mixing bounds for topological events, as well as lower concentration inequalities for additive topological functionals (e.g., the number of connected components) of the level sets that satisfy a law of large numbers. The covariance formula also gives an alternate justification of the Harris criterion, which conjecturally describes the boundary of the percolation university class for level sets of stationary Gaussian fields. Our work is inspired by (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 1679–1711), in which a correlation inequality was derived for certain topological events on the plane, as well as by (*Asymptotic Methods in the Theory of Gaussian Processes and Fields* (1996) Amer. Math. Soc.), in which a similar covariance formula was established for finite-dimensional Gaussian vectors.

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COMPARISON THEOREM FOR SOME EXTREMAL EIGENVALUE STATISTICS

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We introduce a method for the comparison of some extremal eigenvalue statistics of random matrices. For example, it allows one to compare the maximal eigenvalue gap in the bulk of two generalized Wigner ensembles, provided that the first four moments of their matrix entries match. As an application, we extend results of Ben Arous–Bourgade and Feng–Wei that identify the limit of the maximal eigenvalue gap in the bulk of the GUE to all complex Hermitian generalized Wigner matrices.

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ON SINGULARITY OF ENERGY MEASURES FOR SYMMETRIC DIFFUSIONS WITH FULL OFF-DIAGONAL HEAT KERNEL ESTIMATES

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We show that, for a strongly local, regular symmetric Dirichlet form over a complete, locally compact geodesic metric space, full off-diagonal heat kernel estimates with walk dimension strictly larger than two (*sub-Gaussian* estimates) imply the singularity of the energy measures with respect to the symmetric measure, verifying a conjecture by M. T. Barlow in (*Contemp. Math.* **338** (2003) 11–40). We also prove that in the contrary case of walk dimension two, that is, where full off-diagonal *Gaussian* estimates of the heat kernel hold, the symmetric measure and the energy measures are mutually absolutely continuous in the sense that a Borel subset of the state space has measure zero for the symmetric measure if and only if it has measure zero for the energy measures of all functions in the domain of the Dirichlet form.

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A COMPARISON PRINCIPLE FOR RANDOM WALK ON DYNAMICAL PERCOLATION

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We consider the model of random walk on dynamical percolation introduced by Peres, Stauffer and Steif in (*Probab. Theory Related Fields* **162** (2015) 487–530). We obtain comparison results for this model for hitting and mixing times and for the spectral gap and log-Sobolev constant with the corresponding quantities for simple random walk on the underlying graph G, for general graphs. When G is the torus $\mathbb{Z}_n^d$, we recover the results of Peres et al., and we also extend them to the critical case. We also obtain bounds in the cases where G is a transitive graph of moderate growth and also when it is the hypercube.

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CAPACITY OF THE RANGE IN DIMENSION 5

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We prove a central limit theorem for the capacity of the range of a symmetric random walk on $\mathbb{Z}^5$, under only a moment condition on the step distribution. The result is analogous to the central limit theorem for the size of the range in dimension three, obtained by Jain and Pruitt in 1971. In particular, an atypical logarithmic correction appears in the scaling of the variance. The proof is based on new asymptotic estimates, which hold in any dimension $d \geq 5$, for the probability that the ranges of two independent random walks intersect. The latter are then used for computing covariances of some intersection events at the leading order.

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WELL-POSEDNESS, STABILITY AND SENSITIVITIES FOR STOCHASTIC DELAY EQUATIONS: A GENERALIZED COUPLING APPROACH

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We develop a new generalized coupling approach to the study of stochastic delay equations with Hölder continuous coefficients, for which analytical PDE-based methods are not available. We prove that such equations possess unique weak solutions, and establish weak ergodic rates for the corresponding segment processes. We also prove, under additional smoothness assumptions on the coefficients, stabilization rates for the sensitivities in the initial value of the corresponding semigroups.

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THE EXCLUSION PROCESS MIXES (ALMOST) FASTER THAN INDEPENDENT PARTICLES

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Oliveira conjectured that the order of the mixing time of the exclusion process with k-particles on an arbitrary n-vertex graph is at most that of the mixing-time of k independent particles. We verify this up to a constant factor for d-regular graphs when each edge rings at rate $1/d$ in various cases:

(1) when $d = \Omega(\log_{n/k} n)$,

(2) when gap := the spectral-gap of a single walk is $O(1/\log^4 n)$ and $k \geq n^{\Omega(1)}$,

(3) when $k \asymp n^a$ for some constant $0 < a < 1$.

In these cases, our analysis yields a probabilistic proof of a weaker version of Aldous' famous spectral-gap conjecture (resolved by Caputo et al.). We also prove a general bound of $O(\log n \log\log n/\text{gap})$, which is within a $\log\log n$ factor from Oliveira's conjecture when $k \geq n^{\Omega(1)}$. As applications, we get new mixing bounds:

(a) $O(\log n \log\log n)$ for expanders,

(b) order $d \log(dk)$ for the hypercube $\{0, 1\}^d$,

(c) order $(\text{Diameter})^2 \log k$ for vertex-transitive graphs of moderate growth and for supercritical percolation on a fixed dimensional torus.

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SYMMETRIC EXCLUSION AS A RANDOM ENVIRONMENT: INVARIANCE PRINCIPLE

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We establish an invariance principle for a one-dimensional random walk in a dynamic random environment given by a speed-change exclusion process. The jump probabilities of the walk depend on the configuration of the exclusion in a finite box around the walker. The environment starts from equilibrium. After a suitable space-time rescaling, the random walk converges to a sum of two independent processes: a Brownian motion and a Gaussian process with stationary increments.

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