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## Articles

- Maximum of the Ginzburg–Landau fields . . . . . DAVID BELIUS AND WEI WU 2647
- The Riemann zeta function and Gaussian multiplicative chaos: Statistics on the critical  
line . . . . . EERO SAKSMAN AND CHRISTIAN WEBB 2680
- Localization in Gaussian disordered systems at low temperature  
ERIK BATES AND SOURAV CHATTERJEE 2755
- Spectral gaps for reversible Markov processes with chaotic invariant measures: The Kac  
process with hard sphere collisions in three dimensions  
ERIC CARLEN, MARIA CARVALHO AND MICHAEL LOSS 2807
- A covariance formula for topological events of smooth Gaussian fields  
DMITRY BELIAEV, STEPHEN MUIRHEAD AND ALEJANDRO RIVERA 2845
- Comparison theorem for some extremal eigenvalue statistics  
BENJAMIN LANDON, PATRICK LOPATTO AND JAKE MARCINEK 2894
- On singularity of energy measures for symmetric diffusions with full off-diagonal heat  
kernel estimates . . . . . NAOTAKA KAJINO AND MATHAV MURUGAN 2920
- A comparison principle for random walk on dynamical percolation  
JONATHAN HERMON AND PERLA SOUSI 2952
- Capacity of the range in dimension 5 . . . . . BRUNO SCHAPIRA 2988
- Well-posedness, stability and sensitivities for stochastic delay equations: A generalized  
coupling approach . . . . . ALEXEI KULIK AND MICHAEL SCHEUTZOW 3041
- The exclusion process mixes (almost) faster than independent particles  
JONATHAN HERMON AND RICHARD PYMAR 3077
- Symmetric exclusion as a random environment: Invariance principle  
MILTON JARA AND OTÁVIO MENEZES 3124

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## MAXIMUM OF THE GINZBURG–LANDAU FIELDS

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We study a two-dimensional massless field in a box with potential  $V(\nabla\phi(\cdot))$  and zero boundary condition, where  $V$  is any symmetric and uniformly convex function. Naddaf–Spencer (*Comm. Math. Phys.* **183** (1997) 55–84) and Miller (*Comm. Math. Phys.* **308** (2011) 591–639) proved that the rescaled macroscopic averages of this field converge to a continuum Gaussian free field. In this paper, we prove that the distribution of local marginal  $\phi(x)$ , for any  $x$  in the bulk, has a Gaussian tail. We further characterize the leading order of the maximum and the dimension of high points of this field, thus generalizing the results of Bolthausen–Deuschel–Giacomin (*Ann. Probab.* **29** (2001) 1670–1692) and Daviaud (*Ann. Probab.* **34** (2006) 962–986) for the discrete Gaussian free field.

### REFERENCES

- [1] AÏDÉKON, E. (2013). Convergence in law of the minimum of a branching random walk. *Ann. Probab.* **41** 1362–1426. MR3098680 <https://doi.org/10.1214/12-AOP750>
- [2] AÏDÉKON, E., BERESTYCKI, J., BRUNET, É. and SHI, Z. (2013). Branching Brownian motion seen from its tip. *Probab. Theory Related Fields* **157** 405–451. MR3101852 <https://doi.org/10.1007/s00440-012-0461-0>
- [3] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2018). Green kernel asymptotics for two-dimensional random walks under random conductances. Preprint. Available at [arXiv:1808.08126](https://arxiv.org/abs/1808.08126).
- [4] ARGUIN, L.-P., BELIUS, D. and BOURGADE, P. (2017). Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.* **349** 703–751. MR3594368 <https://doi.org/10.1007/s00220-016-2740-6>
- [5] ARGUIN, L.-P., BELIUS, D. and HARPER, A. J. (2017). Maxima of a randomized Riemann zeta function, and branching random walks. *Ann. Appl. Probab.* **27** 178–215. MR3619786 <https://doi.org/10.1214/16-AAP1201>
- [6] ARGUIN, L.-P., BOVIER, A. and KISTLER, N. (2013). The extremal process of branching Brownian motion. *Probab. Theory Related Fields* **157** 535–574. MR3129797 <https://doi.org/10.1007/s00440-012-0464-x>
- [7] ARMSTRONG, S. and WU, W. (2019).  $C^2$  regularity of the surface tension for the  $\nabla\phi$  interface model. Preprint. Available at [arXiv:1909.13325](https://arxiv.org/abs/1909.13325).
- [8] BELIUS, D. and KISTLER, N. (2017). The subleading order of two dimensional cover times. *Probab. Theory Related Fields* **167** 461–552. MR3602852 <https://doi.org/10.1007/s00440-015-0689-6>
- [9] BISKUP, M. and LOUIDOR, O. (2014). Conformal symmetries in the extremal process of two-dimensional discrete Gaussian free field. Preprint. Available at [arXiv:1410.4676](https://arxiv.org/abs/1410.4676).
- [10] BISKUP, M. and LOUIDOR, O. (2016). Extreme local extrema of two-dimensional discrete Gaussian free field. *Comm. Math. Phys.* **345** 271–304. MR3509015 <https://doi.org/10.1007/s00220-015-2565-8>
- [11] BISKUP, M. and LOUIDOR, O. (2018). Full extremal process, cluster law and freezing for the two-dimensional discrete Gaussian free field. *Adv. Math.* **330** 589–687. MR3787554 <https://doi.org/10.1016/j.aim.2018.02.018>
- [12] BOLTHAUSEN, E., DEUSCHEL, J.-D. and GIACOMIN, G. (2001). Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.* **29** 1670–1692. MR1880237 <https://doi.org/10.1214/aop/1015345767>
- [13] BRAMSON, M. (1983). Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.* **44** iv+190. MR0705746 <https://doi.org/10.1090/memo/0285>

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- [14] BRAMSON, M., DING, J. and ZEITOUNI, O. (2016). Convergence in law of the maximum of nonlattice branching random walk. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1897–1924. MR3573300 <https://doi.org/10.1214/15-AIHP703>
- [15] BRAMSON, M., DING, J. and ZEITOUNI, O. (2016). Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **69** 62–123. MR3433630 <https://doi.org/10.1002/cpa.21621>
- [16] BRAMSON, M. and ZEITOUNI, O. (2012). Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **65** 1–20. MR2846636 <https://doi.org/10.1002/cpa.20390>
- [17] BRAMSON, M. D. (1978). Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31** 531–581. MR0494541 <https://doi.org/10.1002/cpa.3160310502>
- [18] BRASCAMP, H. J. and LIEB, E. H. (1976). On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Funct. Anal.* **22** 366–389. MR0450480 [https://doi.org/10.1016/0022-1236\(76\)90004-5](https://doi.org/10.1016/0022-1236(76)90004-5)
- [19] BRASCAMP, H. J., LIEB, E. H. and LEBOWITZ, J. L. (1975). The statistical mechanics of anharmonic lattices. In *Statistical Mechanics* 379–390. Springer, Berlin.
- [20] BRYDGES, D. and YAU, H.-T. (1990). Grad  $\phi$  perturbations of massless Gaussian fields. *Comm. Math. Phys.* **129** 351–392. MR1048698
- [21] CHHAIBI, R., MADAULE, T. and NAJNUDEL, J. (2018). On the maximum of the C $\beta$ E field. *Duke Math. J.* **167** 2243–2345. MR3848391 <https://doi.org/10.1215/00127094-2018-0016>
- [22] CONLON, J. G. and SPENCER, T. (2014). A strong central limit theorem for a class of random surfaces. *Comm. Math. Phys.* **325** 1–15. MR3182484 <https://doi.org/10.1007/s00220-013-1843-6>
- [23] DAVIAUD, O. (2006). Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.* **34** 962–986. MR2243875 <https://doi.org/10.1214/009117906000000061>
- [24] DEUSCHEL, J.-D. and GIACOMIN, G. (2000). Entropic repulsion for massless fields. *Stochastic Process. Appl.* **89** 333–354. MR1780295 [https://doi.org/10.1016/S0304-4149\(00\)00030-2](https://doi.org/10.1016/S0304-4149(00)00030-2)
- [25] DEUSCHEL, J.-D., GIACOMIN, G. and IOFFE, D. (2000). Large deviations and concentration properties for  $\nabla\phi$  interface models. *Probab. Theory Related Fields* **117** 49–111. MR1759509 <https://doi.org/10.1007/s004400050266>
- [26] DING, J., ROY, R. and ZEITOUNI, O. (2017). Convergence of the centered maximum of log-correlated Gaussian fields. *Ann. Probab.* **45** 3886–3928. MR3729618 <https://doi.org/10.1214/16-AOP1152>
- [27] DING, J. and ZEITOUNI, O. (2014). Extreme values for two-dimensional discrete Gaussian free field. *Ann. Probab.* **42** 1480–1515. MR3262484 <https://doi.org/10.1214/13-AOP859>
- [28] DUPLANTIER, B. and SHEFFIELD, S. (2011). Liouville quantum gravity and KPZ. *Invent. Math.* **185** 333–393. MR2819163 <https://doi.org/10.1007/s00222-010-0308-1>
- [29] FUNAKI, T. and SPOHN, H. (1997). Motion by mean curvature from the Ginzburg–Landau  $\nabla\phi$  interface model. *Comm. Math. Phys.* **185** 1–36. MR1463032 <https://doi.org/10.1007/s002200050080>
- [30] GAWEDZKI, K. and KUPIAINEN, A. (1980). A rigorous block spin approach to massless lattice theories. *Comm. Math. Phys.* **77** 31–64. MR0588686
- [31] GIACOMIN, G., OLLA, S. and SPOHN, H. (2001). Equilibrium fluctuations for  $\nabla\phi$  interface model. *Ann. Probab.* **29** 1138–1172. MR1872740 <https://doi.org/10.1214/aop/1015345600>
- [32] HELFFER, B. (2002). *Semiclassical Analysis, Witten Laplacians, and Statistical Mechanics. Series in Partial Differential Equations and Applications* **1**. World Scientific, River Edge, NJ. MR1936110 <https://doi.org/10.1142/9789812776891>
- [33] HELFFER, B. and SJÖSTRAND, J. (1994). On the correlation for Kac-like models in the convex case. *J. Stat. Phys.* **74** 349–409. MR1257821 <https://doi.org/10.1007/BF02186817>
- [34] HU, X., MILLER, J. and PERES, Y. (2010). Thick points of the Gaussian free field. *Ann. Probab.* **38** 896–926. MR2642894 <https://doi.org/10.1214/09-AOP498>
- [35] KENYON, R. (2000). Conformal invariance of domino tiling. *Ann. Probab.* **28** 759–795. MR1782431 <https://doi.org/10.1214/aop/1019160260>
- [36] KENYON, R. (2001). Dominos and the Gaussian free field. *Ann. Probab.* **29** 1128–1137. MR1872739 <https://doi.org/10.1214/aop/1015345599>
- [37] KISTLER, N. (2015). Derrida’s random energy models. In *Correlated Random Systems: Five Different Methods: CIRM Jean-Morlet Chair, Spring 2013* (V. Gayrard and N. Kistler, eds.) 71–120. Springer, Cham. [https://doi.org/10.1007/978-3-319-17674-1\\_3](https://doi.org/10.1007/978-3-319-17674-1_3)
- [38] LAMBERT, G. and PAQUETTE, E. (2019). The law of large numbers for the maximum of almost Gaussian log-correlated fields coming from random matrices. *Probab. Theory Related Fields* **173** 157–209. MR3916106 <https://doi.org/10.1007/s00440-018-0832-2>
- [39] LAWLER, G. F. (2008). *Conformally Invariant Processes in the Plane. Mathematical Surveys and Monographs* **114**. Amer. Math. Soc., Providence, RI. MR2129588

- [40] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk: A Modern Introduction. Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. MR2677157 <https://doi.org/10.1017/CBO9780511750854>
- [41] MADAULE, T. (2017). Convergence in law for the branching random walk seen from its tip. *J. Theoret. Probab.* **30** 27–63. MR3615081 <https://doi.org/10.1007/s10959-015-0636-6>
- [42] MILLER, J. (2010). Universality for SLE (4). Preprint. Available at arXiv:1010.1356.
- [43] MILLER, J. (2011). Fluctuations for the Ginzburg–Landau  $\nabla\phi$  interface model on a bounded domain. *Comm. Math. Phys.* **308** 591–639. MR2855536 <https://doi.org/10.1007/s00220-011-1315-9>
- [44] NADDAF, A. and SPENCER, T. (1997). On homogenization and scaling limit of some gradient perturbations of a massless free field. *Comm. Math. Phys.* **183** 55–84. MR1461951 <https://doi.org/10.1007/BF02509796>
- [45] PAQUETTE, E. and ZEITOUNI, O. (2018). The maximum of the CUE field. *Int. Math. Res. Not. IMRN* **16** 5028–5119. MR3848227 <https://doi.org/10.1093/imrn/rnx033>
- [46] ROBERT, R. and VARGAS, V. (2008). Hydrodynamic turbulence and intermittent random fields. *Comm. Math. Phys.* **284** 649–673. MR2452591 <https://doi.org/10.1007/s00220-008-0642-y>
- [47] SHEFFIELD, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** 521–541. MR2322706 <https://doi.org/10.1007/s00440-006-0050-1>
- [48] SHEFFIELD, S. and WERNER, W. (2012). Conformal loop ensembles: The Markovian characterization and the loop-soup construction. *Ann. of Math. (2)* **176** 1827–1917. MR2979861 <https://doi.org/10.4007/annals.2012.176.3.8>

# THE RIEMANN ZETA FUNCTION AND GAUSSIAN MULTIPLICATIVE CHAOS: STATISTICS ON THE CRITICAL LINE

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We prove that if  $\omega$  is uniformly distributed on  $[0, 1]$ , then as  $T \rightarrow \infty$ ,  $t \mapsto \zeta(i\omega T + it + 1/2)$  converges to a nontrivial random generalized function, which in turn is identified as a product of a very well-behaved random smooth function and a random generalized function known as a complex Gaussian multiplicative chaos distribution. This demonstrates a novel rigorous connection between probabilistic number theory and the theory of multiplicative chaos—the latter is known to be connected to various branches of modern probability theory and mathematical physics.

We also investigate the statistical behavior of the zeta function on the mesoscopic scale. We prove that if we let  $\delta_T$  approach zero slowly enough as  $T \rightarrow \infty$ , then  $t \mapsto \zeta(1/2 + i\delta_T t + i\omega T)$  is asymptotically a product of a divergent scalar quantity suggested by Selberg’s central limit theorem and a strictly Gaussian multiplicative chaos. We also prove a similar result for the characteristic polynomial of a Haar distributed random unitary matrix, where the scalar quantity is slightly different but the multiplicative chaos part is identical. This says that up to scalar multiples, the zeta function and the characteristic polynomial of a Haar distributed random unitary matrix have an identical distribution on the mesoscopic scale.

## REFERENCES

- [1] ARGUIN, L.-P., BELIUS, D. and BOURGADE, P. (2017). Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.* **349** 703–751. MR3594368 <https://doi.org/10.1007/s00220-016-2740-6>
- [2] ARGUIN, L.-P., BELIUS, D., BOURGADE, P., RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2019). Maximum of the Riemann zeta function on a short interval of the critical line. *Comm. Pure Appl. Math.* **72** 500–535. MR3911893 <https://doi.org/10.1002/cpa.21791>
- [3] ARGUIN, L.-P., BELIUS, D. and HARPER, A. J. (2017). Maxima of a randomized Riemann zeta function, and branching random walks. *Ann. Appl. Probab.* **27** 178–215. MR3619786 <https://doi.org/10.1214/16-AAP1201>
- [4] ASTALA, K., JONES, P., KUPIAINEN, A. and SAKSMAN, E. (2011). Random conformal weldings. *Acta Math.* **207** 203–254. MR2892610 <https://doi.org/10.1007/s11511-012-0069-3>
- [5] BACRY, E. and MUZY, J. F. (2003). Log-infinitely divisible multifractal processes. *Comm. Math. Phys.* **236** 449–475. MR2021198 <https://doi.org/10.1007/s00220-003-0827-3>
- [6] BAGCHI, B. (1981). The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series. PhD Thesis, Calcutta, Indian Statistical Institute.
- [7] BARKS, C. Junior Woodchucks Guidebook. The Walt Disney Company.
- [8] BARRAL, J., JIN, X. and MANDELBROT, B. (2010). Uniform convergence for complex  $[0, 1]$ -martingales. *Ann. Appl. Probab.* **20** 1205–1218. MR2676937 <https://doi.org/10.1214/09-AAP664>
- [9] BARRAL, J., JIN, X. and MANDELBROT, B. (2010). Convergence of complex multiplicative cascades. *Ann. Appl. Probab.* **20** 1219–1252. MR2676938 <https://doi.org/10.1214/09-AAP665>
- [10] BARRAL, J., KUPIAINEN, A., NIKULA, M., SAKSMAN, E. and WEBB, C. (2015). Basic properties of critical lognormal multiplicative chaos. *Ann. Probab.* **43** 2205–2249. MR3395460 <https://doi.org/10.1214/14-AOP931>

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- [11] BERCU, B., DELYON, B. and RIO, E. (2015). *Concentration Inequalities for Sums and Martingales*. SpringerBriefs in Mathematics. Springer, Cham. MR3363542 <https://doi.org/10.1007/978-3-319-22099-4>
- [12] BERESTYCKI, N. (2017). An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.* **22** Paper No. 27, 12. MR3652040 <https://doi.org/10.1214/17-ECP58>
- [13] BERESTYCKI, N., SHEFFIELD, S. and SUN, X. (2014). Equivalence of Liouville measure and Gaussian free field. Preprint [arXiv:1410.5407](https://arxiv.org/abs/1410.5407).
- [14] BETTIN, S. (2010). The second moment of the Riemann zeta function with unbounded shifts. *Int. J. Number Theory* **6** 1933–1944. MR2755480 <https://doi.org/10.1142/S1793042110003861>
- [15] BOHR, H. and JESSEN, B. (1930). Über die Werteverteilung der Riemannschen Zetafunktion. *Acta Math.* **54** 1–35. MR1555301 <https://doi.org/10.1007/BF02547516>
- [16] BOHR, H. and JESSEN, B. (1932). Über die Werteverteilung der Riemannschen Zetafunktion. *Acta Math.* **58** 1–55. MR1555343 <https://doi.org/10.1007/BF02547773>
- [17] BOURGADE, P. (2010). Mesoscopic fluctuations of the zeta zeros. *Probab. Theory Related Fields* **148** 479–500. MR2678896 <https://doi.org/10.1007/s00440-009-0237-3>
- [18] BOURGADE, P. and KUAN, J. (2014). Strong Szegő asymptotics and zeros of the zeta-function. *Comm. Pure Appl. Math.* **67** 1028–1044. MR3193966 <https://doi.org/10.1002/cpa.21475>
- [19] BOURGAIN, J. (2017). Decoupling, exponential sums and the Riemann zeta function. *J. Amer. Math. Soc.* **30** 205–224. MR3556291 <https://doi.org/10.1090/jams/860>
- [20] CARPENTIER, D. and LE DOUSSAL, P. (2001). Glass transition of a particle in a random potential, front selection in non linear RG and entropic phenomena in Liouville and SinhGordon models. *Phys. Rev. E* **63** 026110.
- [21] CHANDEE, V. (2011). On the correlation of shifted values of the Riemann zeta function. *Q. J. Math.* **62** 545–572. MR2825471 <https://doi.org/10.1093/qmath/haq008>
- [22] CHHAIBI, R., MADAULE, T. and NAJNUDEL, J. (2018). On the maximum of the  $C\beta E$  field. *Duke Math. J.* **167** 2243–2345. MR3848391 <https://doi.org/10.1215/00127094-2018-0016>
- [23] CHHAIBI, R., NAJNUDEL, J. and NIKEGHBALI, A. (2017). The circular unitary ensemble and the Riemann zeta function: The microscopic landscape and a new approach to ratios. *Invent. Math.* **207** 23–113. MR3592756 <https://doi.org/10.1007/s00222-016-0669-1>
- [24] CONREY, B. (2016). Lectures on the Riemann zeta function [book review of MR3241276]. *Bull. Amer. Math. Soc. (N.S.)* **53** 507–512. MR3497796 <https://doi.org/10.1090/bull/1525>
- [25] CONREY, B. and KEATING, J. P. Moments of zeta and correlations of divisor-sums I–IV: [arXiv:1506.06842](https://arxiv.org/abs/1506.06842), [arXiv:1506.06844](https://arxiv.org/abs/1506.06844) and [arXiv:1603.06893](https://arxiv.org/abs/1603.06893).
- [26] CONREY, J. B., FARMER, D. W., KEATING, J. P., RUBINSTEIN, M. O. and SNAITH, N. C. (2003). Autocorrelation of random matrix polynomials. *Comm. Math. Phys.* **237** 365–395. MR1993332 <https://doi.org/10.1007/s00220-003-0852-2>
- [27] DAVID, F., KUPIAINEN, A., RHODES, R. and VARGAS, V. (2016). Liouville quantum gravity on the Riemann sphere. *Comm. Math. Phys.* **342** 869–907. MR3465434 <https://doi.org/10.1007/s00220-016-2572-4>
- [28] DEIFT, P., ITS, A. and KRASOVSKY, I. (2014). On the asymptotics of a Toeplitz determinant with singularities. In *Random Matrix Theory, Interacting Particle Systems, and Integrable Systems*. Math. Sci. Res. Inst. Publ. **65** 93–146. Cambridge Univ. Press, New York. MR3380684
- [29] DIACONIS, P. and SHAHSHAHANI, M. (1994). On the eigenvalues of random matrices *J. Appl. Probab.* **31A** 49–62. MR1274717 <https://doi.org/10.2307/3214948>
- [30] DIESTEL, J. and UHL, J. J. JR. (1977). *Vector Measures*. Mathematical Surveys **15**. Amer. Math. Soc., Providence, RI. With a foreword by B. J. Pettis. MR0453964
- [31] DUPLANTIER, B., RHODES, R., SHEFFIELD, S. and VARGAS, V. (2014). Renormalization of critical Gaussian multiplicative chaos and KPZ relation. *Comm. Math. Phys.* **330** 283–330. MR3215583 <https://doi.org/10.1007/s00220-014-2000-6>
- [32] DUPLANTIER, B., RHODES, R., SHEFFIELD, S. and VARGAS, V. (2014). Critical Gaussian multiplicative chaos: Convergence of the derivative martingale. *Ann. Probab.* **42** 1769–1808. MR3262492 <https://doi.org/10.1214/13-AOP890>
- [33] DUPLANTIER, B. and SHEFFIELD, S. (2011). Liouville quantum gravity and KPZ. *Invent. Math.* **185** 333–393. MR2819163 <https://doi.org/10.1007/s00222-010-0308-1>
- [34] EHRHARDT, T. (2001). A status report on the asymptotic behavior of Toeplitz determinants with Fisher–Hartwig singularities. In *Recent Advances in Operator Theory (Groningen, 1998)*. Oper. Theory Adv. Appl. **124** 217–241. Birkhäuser, Basel. MR1839838
- [35] FARMER, D. W., GONEK, S. M. and HUGHES, C. P. (2007). The maximum size of  $L$ -functions. *J. Reine Angew. Math.* **609** 215–236. MR2350784 <https://doi.org/10.1515/CRELLE.2007.064>



- [36] FYODOROV, Y. V., HIARY, G. and KEATING, J. P. (2012). Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta-function. *Phys. Rev. Lett.* **108** 170601.
- [37] FYODOROV, Y. V. and KEATING, J. P. (2014). Freezing transitions and extreme values: Random matrix theory, and disordered landscapes. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372** 20120503, 32. MR3151088 <https://doi.org/10.1098/rsta.2012.0503>
- [38] GRAFAKOS, L. (2009). *Modern Fourier Analysis*, 2nd ed. *Graduate Texts in Mathematics* **250**. Springer, New York. MR2463316 <https://doi.org/10.1007/978-0-387-09434-2>
- [39] HARPER, A. Sharp conditional bounds for the moments of the Riemann zeta function. arXiv:1305.4618.
- [40] HARPER, A. The Riemann zeta function in short intervals [after Najnudel, and Arguin, Belius, Bourgade, Radziwiłł, and Soundararajan]. Preprint, arXiv:1904.08204.
- [41] HEATH-BROWN, D. R. (1981). Fractional moments of the Riemann zeta function. *J. Lond. Math. Soc.* (2) **24** 65–78. MR0623671 <https://doi.org/10.1112/jlms/s2-24.1.65>
- [42] HEDENMALM, H., LINDQVIST, P. and SEIP, K. (1997). A Hilbert space of Dirichlet series and systems of dilated functions in  $L^2(0, 1)$ . *Duke Math. J.* **86** 1–37. MR1427844 <https://doi.org/10.1215/S0012-7094-97-08601-4>
- [43] HEINÄVAARA, O., HUANG, Y., JUNNILA, J., SAKSMAN, E. and WEBB, C. Properties of Hardy chaos. Manuscript in preparation.
- [44] HUGHES, C. P., KEATING, J. P. and O’CONNELL, N. (2001). On the characteristic polynomial of a random unitary matrix. *Comm. Math. Phys.* **220** 429–451. MR1844632 <https://doi.org/10.1007/s002200100453>
- [45] HUGHES, C. P., NIKEGBALI, A. and YOR, M. (2008). An arithmetic model for the total disorder process. *Probab. Theory Related Fields* **141** 47–59. MR2372965 <https://doi.org/10.1007/s00440-007-0079-9>
- [46] HYTÖNEN, T., VAN NEERVEN, J., VERAAR, M. and WEIS, L. Analysis in Banach spaces.
- [47] INGHAM, A. E. (1927). Mean-Value Theorems in the Theory of the Riemann Zeta-Function. *Proc. London Math. Soc.* (2) **27** 273–300. MR1575391 <https://doi.org/10.1112/plms/s2-27.1.273>
- [48] IVIĆ, A. (1985). *The Riemann Zeta-Function: The Theory of the Riemann Zeta-Function with Applications*. A Wiley-Interscience Publication. Wiley, New York. MR0792089
- [49] JUNNILA, J. On the multiplicative chaos of non-Gaussian log-correlated fields. Preprint, arXiv:1606.08986.
- [50] JUNNILA, J. and SAKSMAN, E. (2017). Uniqueness of critical Gaussian chaos. *Electron. J. Probab.* **22** Paper No. 11, 31. MR3613704 <https://doi.org/10.1214/17-EJP28>
- [51] KAHANE, J.-P. (1985). *Some Random Series of Functions*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **5**. Cambridge Univ. Press, Cambridge. MR0833073
- [52] KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR0829798
- [53] KEATING, J. P. and SNAITH, N. C. (2000). Random matrix theory and  $\zeta(1/2 + it)$ . *Comm. Math. Phys.* **214** 57–89. MR1794265 <https://doi.org/10.1007/s002200000261>
- [54] LACONIN, H., RHODES, R. and VARGAS, V. (2015). Complex Gaussian multiplicative chaos. *Comm. Math. Phys.* **337** 569–632. MR3339158 <https://doi.org/10.1007/s00220-015-2362-4>
- [55] LAMBERT, G., OSTROVSKY, D. and SIMM, N. (2018). Subcritical multiplicative chaos for regularized counting statistics from random matrix theory. *Comm. Math. Phys.* **360** 1–54. MR3795187 <https://doi.org/10.1007/s00220-018-3130-z>
- [56] LAURINČIKAS, A. (1985). The Riemann zeta function on the critical line. *Litovsk. Mat. Sb.* **25** 114–118. MR0814461
- [57] LAURINČIKAS, A. (1996). *Limit Theorems for the Riemann Zeta-Function*. *Mathematics and Its Applications* **352**. Kluwer Academic, Dordrecht. MR1376140 <https://doi.org/10.1007/978-94-017-2091-5>
- [58] MAPLES, K. and RODGERS, B. (2015). Bootstrapped zero density estimates and a central limit theorem for the zeros of the zeta function. *Int. J. Number Theory* **11** 2087–2107. MR3440448 <https://doi.org/10.1142/S1793042115500918>
- [59] MILLER, J. and SHEFFIELD, S. Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. Preprint arXiv:1605.03563.
- [60] MILLER, J. and SHEFFIELD, S. Liouville quantum gravity and the Brownian map III: the conformal structure is determined. Preprint arXiv:1608.05391.
- [61] MILLER, J. and SHEFFIELD, S. (2016). Quantum Loewner evolution. *Duke Math. J.* **165** 3241–3378. MR3572845 <https://doi.org/10.1215/00127094-3627096>
- [62] MILLER, J. and SHEFFIELD, S. (2020). Liouville quantum gravity and the Brownian map I: The QLE(8/3, 0) metric. *Invent. Math.* **219** 75–152. MR4050102 <https://doi.org/10.1007/s00222-019-00905-1>
- [63] NAJNUDEL, J. (2018). On the extreme values of the Riemann zeta function on random intervals of the critical line. *Probab. Theory Related Fields* **172** 387–452. MR3851835 <https://doi.org/10.1007/s00440-017-0812-y>

- [64] PAQUETTE, E. and ZEITOUNI, O. (2018). The maximum of the CUE field. *Int. Math. Res. Not. IMRN* **16** 5028–5119. MR3848227 <https://doi.org/10.1093/imrn/rnx033>
- [65] RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2013). Continuous lower bounds for moments of zeta and  $L$ -functions. *Mathematika* **59** 119–128. MR3028174 <https://doi.org/10.1112/S0025579312001088>
- [66] RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2017). Selberg’s central limit theorem for  $\log |\zeta(1/2 + it)|$ . *Enseign. Math.* **63** 1–19. MR3832861 <https://doi.org/10.4171/LEM/63-1/2-1>
- [67] RAMACHANDRA, K. (1978). Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. I. *Hardy-Ramanujan J.* **1** 15. MR0565298
- [68] RAMACHANDRA, K. (1995). *On the Mean-Value and Omega-Theorems for the Riemann Zeta-Function. Tata Institute of Fundamental Research Lectures on Mathematics and Physics* **85**. Springer, Berlin. Published for the Tata Institute of Fundamental Research, Bombay. MR1332493
- [69] RHODES, R. and VARGAS, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** 315–392. MR3274356 <https://doi.org/10.1214/13-PS218>
- [70] RODGERS, B. (2014). A central limit theorem for the zeroes of the zeta function. *Int. J. Number Theory* **10** 483–511. MR3189991 <https://doi.org/10.1142/S1793042113501054>
- [71] RUDIN, W. (1987). *Real and Complex Analysis*, 3rd ed. McGraw-Hill, New York. MR0924157
- [72] SAKSMAN, E. and WEBB, C. The Riemann zeta function and Gaussian multiplicative chaos: Statistics on the critical line. Preprint, [arXiv:1609.00027](https://arxiv.org/abs/1609.00027).
- [73] SELBERG, A. (1946). Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvidensk.* **48** 89–155. MR0020594
- [74] SHAMOV, A. (2016). On Gaussian multiplicative chaos. *J. Funct. Anal.* **270** 3224–3261. MR3475456 <https://doi.org/10.1016/j.jfa.2016.03.001>
- [75] SHEFFIELD, S. (2016). Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.* **44** 3474–3545. MR3551203 <https://doi.org/10.1214/15-AOP1055>
- [76] STEIN, E. M. (1993). *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Mathematical Series* **43**. Princeton Univ. Press, Princeton, NJ. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR1232192
- [77] TITCHMARSH, E. C. (1951). *The Theory of the Riemann Zeta-Function*. Clarendon Press, Oxford. MR0046485
- [78] VILLANI, C. (2009). *Optimal Transport: Old and New. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **338**. Springer, Berlin. MR2459454 <https://doi.org/10.1007/978-3-540-71050-9>
- [79] WEBB, C. (2015). The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos—the  $L^2$ -phase. *Electron. J. Probab.* **20** no. 104, 21. MR3407221 <https://doi.org/10.1214/EJP.v20-4296>

# LOCALIZATION IN GAUSSIAN DISORDERED SYSTEMS AT LOW TEMPERATURE

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For a broad class of Gaussian disordered systems at low temperature, we show that the Gibbs measure is asymptotically localized in small neighborhoods of a small number of states. From a single argument, we obtain: (i) a version of “complete” path localization for directed polymers that is not available even for exactly solvable models, and (ii) a result about the exhaustiveness of Gibbs states in spin glasses not requiring the Ghirlanda–Guerra identities.

## REFERENCES

- [1] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry. Springer Monographs in Mathematics*. Springer, New York. MR2319516
- [2] AIZENMAN, M. and CONTUCCI, P. (1998). On the stability of the quenched state in mean-field spin-glass models. *J. Stat. Phys.* **92** 765–783. MR1657840 <https://doi.org/10.1023/A:1023080223894>
- [3] AIZENMAN, M., LEBOWITZ, J. L. and RUELLE, D. (1987). Some rigorous results on the Sherrington–Kirkpatrick spin glass model. *Comm. Math. Phys.* **112** 3–20. MR0904135
- [4] ARGUIN, L.-P. and ZINDY, O. (2014). Poisson–Dirichlet statistics for the extremes of a log-correlated Gaussian field. *Ann. Appl. Probab.* **24** 1446–1481. MR3211001 <https://doi.org/10.1214/13-AAP952>
- [5] AUFFINGER, A. and CHEN, W.-K. (2015). On properties of Parisi measures. *Probab. Theory Related Fields* **161** 817–850. MR3334282 <https://doi.org/10.1007/s00440-014-0563-y>
- [6] AUFFINGER, A. and CHEN, W.-K. (2018). On concentration properties of disordered Hamiltonians. *Proc. Amer. Math. Soc.* **146** 1807–1815. MR3754362 <https://doi.org/10.1090/proc/13864>
- [7] AUFFINGER, A., CHEN, W.-K. and ZENG, Q. (2020). The SK model is full-step replica symmetry breaking at zero temperature. *Comm. Pure Appl. Math.* **73** 921–943. <https://doi.org/10.1002/cpa.21886>
- [8] AUFFINGER, A. and LOUIDOR, O. (2011). Directed polymers in a random environment with heavy tails. *Comm. Pure Appl. Math.* **64** 183–204. MR2766526 <https://doi.org/10.1002/cpa.20348>
- [9] AUFFINGER, A. and ZENG, Q. (2019). Existence of two-step replica symmetry breaking for the spherical mixed  $p$ -spin glass at zero temperature. *Comm. Math. Phys.* **370** 377–402. MR3982699 <https://doi.org/10.1007/s00220-018-3252-3>
- [10] BACRY, E. and MUZY, J. F. (2003). Log-infinitely divisible multifractal processes. *Comm. Math. Phys.* **236** 449–475. MR2021198 <https://doi.org/10.1007/s00220-003-0827-3>
- [11] BARRAL, J. and MANDELBROT, B. B. (2002). Multifractal products of cylindrical pulses. *Probab. Theory Related Fields* **124** 409–430. MR1939653 <https://doi.org/10.1007/s004400200220>
- [12] BARRAL, J., RHODES, R. and VARGAS, V. (2012). Limiting laws of supercritical branching random walks. *C. R. Math. Acad. Sci. Paris* **350** 535–538. MR2929063 <https://doi.org/10.1016/j.crma.2012.05.013>
- [13] BARRAQUAND, G. and CORWIN, I. (2017). Random-walk in beta-distributed random environment. *Probab. Theory Related Fields* **167** 1057–1116. MR3627433 <https://doi.org/10.1007/s00440-016-0699-z>
- [14] BATES, E. (2018). Localization of directed polymers with general reference walk. *Electron. J. Probab.* **23** Paper No. 30, 45. MR3785400 <https://doi.org/10.1214/18-EJP158>
- [15] BATES, E. and CHATTERJEE, S. (2020). The endpoint distribution of directed polymers. *Ann. Probab.* **48** 817–871. MR4089496 <https://doi.org/10.1214/19-AOP1376>
- [16] BERGER, Q. and TORRI, N. (2019). Directed polymers in heavy-tail random environment. *Ann. Probab.* **47** 4024–4076. MR4038048 <https://doi.org/10.1214/19-aop1353>
- [17] BOLTHAUSEN, E. (2007). Random media and spin glasses: An introduction into some mathematical results and problems. In *Spin Glasses. Lecture Notes in Math.* **1900** 1–44. Springer, Berlin. MR2309596 [https://doi.org/10.1007/978-3-540-40908-3\\_1](https://doi.org/10.1007/978-3-540-40908-3_1)

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- [18] BOVIER, A. (2006). *Statistical Mechanics of Disordered Systems: A Mathematical Perspective*. Cambridge Series in Statistical and Probabilistic Mathematics **18**. Cambridge Univ. Press, Cambridge. MR2252929 <https://doi.org/10.1017/CBO9780511616808>
- [19] CARMONA, P. and HU, Y. (2002). On the partition function of a directed polymer in a Gaussian random environment. *Probab. Theory Related Fields* **124** 431–457. MR1939654 <https://doi.org/10.1007/s004400200213>
- [20] CHATTERJEE, S. (2014). *Superconcentration and Related Topics*. Springer Monographs in Mathematics. Springer, Cham. MR3157205 <https://doi.org/10.1007/978-3-319-03886-5>
- [21] CHATTERJEE, S. (2019). Proof of the path localization conjecture for directed polymers. *Comm. Math. Phys.* **370** 703–717. MR3994583 <https://doi.org/10.1007/s00220-019-03533-1>
- [22] CHEN, W.-K. (2013). The Aizenman–Sims–Starr scheme and Parisi formula for mixed  $p$ -spin spherical models. *Electron. J. Probab.* **18** no. 94, 14. MR3126577 <https://doi.org/10.1214/EJP.v18-2580>
- [23] CHEN, W.-K. and SEN, A. (2017). Parisi formula, disorder chaos and fluctuation for the ground state energy in the spherical mixed  $p$ -spin models. *Comm. Math. Phys.* **350** 129–173. MR3606472 <https://doi.org/10.1007/s00220-016-2808-3>
- [24] COMETS, F. (2017). *Directed Polymers in Random Environments*. Lecture Notes in Math. **2175**. Springer, Cham. MR3444835 <https://doi.org/10.1007/978-3-319-50487-2>
- [25] COMETS, F. and COSCO, C. (2018). Brownian polymers in Poissonian environment: A survey. Preprint. Available at [arXiv:1805.10899](https://arxiv.org/abs/1805.10899).
- [26] COMETS, F. and CRANSTON, M. (2013). Overlaps and pathwise localization in the Anderson polymer model. *Stochastic Process. Appl.* **123** 2446–2471. MR3038513 <https://doi.org/10.1016/j.spa.2013.02.010>
- [27] COMETS, F. and NEVEU, J. (1995). The Sherrington–Kirkpatrick model of spin glasses and stochastic calculus: The high temperature case. *Comm. Math. Phys.* **166** 549–564. MR1312435
- [28] COMETS, F., SHIGA, T. and YOSHIDA, N. (2003). Directed polymers in a random environment: Path localization and strong disorder. *Bernoulli* **9** 705–723. MR1996276 <https://doi.org/10.3150/bj/1066223275>
- [29] COMETS, F. and YOSHIDA, N. (2005). Brownian directed polymers in random environment. *Comm. Math. Phys.* **254** 257–287. MR2117626 <https://doi.org/10.1007/s00220-004-1203-7>
- [30] COMETS, F. and YOSHIDA, N. (2013). Localization transition for polymers in Poissonian medium. *Comm. Math. Phys.* **323** 417–447. MR3085670 <https://doi.org/10.1007/s00220-013-1744-8>
- [31] CORWIN, I., SEPPÄLÄINEN, T. and SHEN, H. (2015). The strict-weak lattice polymer. *J. Stat. Phys.* **160** 1027–1053. MR3373650 <https://doi.org/10.1007/s10955-015-1267-0>
- [32] CRISANTI, A. and SOMMERS, H. J. (1992). The spherical  $p$ -spin interaction spin glass model: The statics. *Z. Phys. B, Condens. Matter* **87** 341–354. <https://doi.org/10.1007/BF01309287>
- [33] DERRIDA, B. (1980). Random-energy model: Limit of a family of disordered models. *Phys. Rev. Lett.* **45** 79–82. MR0575260 <https://doi.org/10.1103/PhysRevLett.45.79>
- [34] DERRIDA, B. (1981). Random-energy model: An exactly solvable model of disordered systems. *Phys. Rev. B* (3) **24** 2613–2626. MR0627810 <https://doi.org/10.1103/physrevb.24.2613>
- [35] DERRIDA, B. (1985). A generalization of the random energy model which includes correlations between energies. *J. Phys. Lett., Paris* **46** 401–407.
- [36] DERRIDA, B. and SPOHN, H. (1988). Polymers on disordered trees, spin glasses, and traveling waves. *J. Stat. Phys.* **51** 817–840. MR0971033 <https://doi.org/10.1007/BF01014886>
- [37] DEY, P. S. and ZYGOURAS, N. (2016). High temperature limits for  $(1 + 1)$ -dimensional directed polymer with heavy-tailed disorder. *Ann. Probab.* **44** 4006–4048. MR3572330 <https://doi.org/10.1214/15-AOP1067>
- [38] EDWARDS, S. F. and ANDERSON, P. W. (1975). Theory of spin glasses. *J. Phys. F, Met. Phys.* **5** 965–974.
- [39] GEORGIU, N., RASSOUL-AGHA, F. and SEPPÄLÄINEN, T. (2016). Variational formulas and cocycle solutions for directed polymer and percolation models. *Comm. Math. Phys.* **346** 741–779. MR3535900 <https://doi.org/10.1007/s00220-016-2613-z>
- [40] GUERRA, F. (2003). Broken replica symmetry bounds in the mean field spin glass model. *Comm. Math. Phys.* **233** 1–12. MR1957729 <https://doi.org/10.1007/s00220-002-0773-5>
- [41] JAGANNATH, A. (2016). On the overlap distribution of branching random walks. *Electron. J. Probab.* **21** Paper No. 50, 16. MR3539644 <https://doi.org/10.1214/16-EJP3>
- [42] JAGANNATH, A. (2017). Approximate ultrametricity for random measures and applications to spin glasses. *Comm. Pure Appl. Math.* **70** 611–664. MR3628881 <https://doi.org/10.1002/cpa.21685>
- [43] JAGANNATH, A. and TOBASCO, I. (2017). Low temperature asymptotics of spherical mean field spin glasses. *Comm. Math. Phys.* **352** 979–1017. MR3631397 <https://doi.org/10.1007/s00220-017-2864-3>
- [44] KOSTERLITZ, J. M., THOULESS, D. J. and JONES, R. C. (1976). Spherical model of a spin-glass. *Phys. Rev. Lett.* **36** 1217–1220. <https://doi.org/10.1103/PhysRevLett.36.1217>

- [45] MÉZARD, M., PARISI, G. and VIRASORO, M. A. (1987). *Spin Glass Theory and Beyond*. World Scientific Lecture Notes in Physics **9**. World Scientific Co., Inc., Teaneck, NJ. MR1026102
- [46] O'CONNELL, N. and ORTMANN, J. (2015). Tracy–Widom asymptotics for a random polymer model with gamma-distributed weights. *Electron. J. Probab.* **20** no. 25, 18. MR3325095 <https://doi.org/10.1214/EJP.v20-3787>
- [47] PANCHENKO, D. (2008). On differentiability of the Parisi formula. *Electron. Commun. Probab.* **13** 241–247. MR2399285 <https://doi.org/10.1214/ECP.v13-1365>
- [48] PANCHENKO, D. (2010). The Ghirlanda–Guerra identities for mixed  $p$ -spin model. *C. R. Math. Acad. Sci. Paris* **348** 189–192. MR2600075 <https://doi.org/10.1016/j.crma.2010.02.004>
- [49] PANCHENKO, D. (2013). The Parisi ultrametricity conjecture. *Ann. of Math. (2)* **177** 383–393. MR2999044 <https://doi.org/10.4007/annals.2013.177.1.8>
- [50] PANCHENKO, D. (2013). *The Sherrington–Kirkpatrick Model*. Springer Monographs in Mathematics. Springer, New York. MR3052333 <https://doi.org/10.1007/978-1-4614-6289-7>
- [51] PANCHENKO, D. (2014). The Parisi formula for mixed  $p$ -spin models. *Ann. Probab.* **42** 946–958. MR3189062 <https://doi.org/10.1214/12-AOP800>
- [52] PANCHENKO, D. and TALAGRAND, M. (2007). On the overlap in the multiple spherical SK models. *Ann. Probab.* **35** 2321–2355. MR2353390 <https://doi.org/10.1214/009117907000000015>
- [53] PARISI, G. (1980). A sequence of approximated solutions to the S–K model for spin glasses. *J. Phys. A: Math. Gen.* **13** L115–L121.
- [54] PARISI, G. (1979). Infinite number of order parameters for spin-glasses. *Phys. Rev. Lett.* **43** 1754–1756. <https://doi.org/10.1103/PhysRevLett.43.1754>
- [55] RASSOUL-AGHA, F. and SEPPÄLÄINEN, T. (2014). Quenched point-to-point free energy for random walks in random potentials. *Probab. Theory Related Fields* **158** 711–750. MR3176363 <https://doi.org/10.1007/s00440-013-0494-z>
- [56] SEPPÄLÄINEN, T. (2012). Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.* **40** 19–73. MR2917766 <https://doi.org/10.1214/10-AOP617>
- [57] SHERRINGTON, D. and KIRKPATRICK, S. (1975). Solvable model of a spin-glass. *Phys. Rev. Lett.* **35** 1792–1796.
- [58] SUBAG, E. (2017). The geometry of the Gibbs measure of pure spherical spin glasses. *Invent. Math.* **210** 135–209. MR3698341 <https://doi.org/10.1007/s00222-017-0726-4>
- [59] TALAGRAND, M. (2003). On the meaning of Parisi's functional order parameter. *C. R. Math. Acad. Sci. Paris* **337** 625–628. MR2017738 <https://doi.org/10.1016/j.crma.2003.09.013>
- [60] TALAGRAND, M. (2003). *Spin Glasses: A Challenge for Mathematicians: Cavity and Mean Field Models*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]* **46**. Springer, Berlin. MR1993891
- [61] TALAGRAND, M. (2006). Parisi measures. *J. Funct. Anal.* **231** 269–286. MR2195333 <https://doi.org/10.1016/j.jfa.2005.03.001>
- [62] TALAGRAND, M. (2006). The Parisi formula. *Ann. of Math. (2)* **163** 221–263. MR2195134 <https://doi.org/10.4007/annals.2006.163.221>
- [63] TALAGRAND, M. (2006). Free energy of the spherical mean field model. *Probab. Theory Related Fields* **134** 339–382. MR2226885 <https://doi.org/10.1007/s00440-005-0433-8>
- [64] TALAGRAND, M. (2010). Construction of pure states in mean field models for spin glasses. *Probab. Theory Related Fields* **148** 601–643. MR2678900 <https://doi.org/10.1007/s00440-009-0242-6>
- [65] TALAGRAND, M. (2011). *Mean Field Models for Spin Glasses. Volume I: Basic Examples*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]* **54**. Springer, Berlin. MR2731561 <https://doi.org/10.1007/978-3-642-15202-3>
- [66] TALAGRAND, M. (2011). *Mean Field Models for Spin Glasses. Volume II: Advanced Replica-Symmetry and Low Temperature*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]* **55**. Springer, Heidelberg. MR3024566
- [67] THIERY, T. and LE DOUSSAL, P. (2015). On integrable directed polymer models on the square lattice. *J. Phys. A* **48** 465001, 41. MR3418005 <https://doi.org/10.1088/1751-8113/48/46/465001>
- [68] TORRI, N. (2016). Pinning model with heavy tailed disorder. *Stochastic Process. Appl.* **126** 542–571. MR3434993 <https://doi.org/10.1016/j.spa.2015.09.010>
- [69] VARGAS, V. (2007). Strong localization and macroscopic atoms for directed polymers. *Probab. Theory Related Fields* **138** 391–410. MR2299713 <https://doi.org/10.1007/s00440-006-0030-5>



# SPECTRAL GAPS FOR REVERSIBLE MARKOV PROCESSES WITH CHAOTIC INVARIANT MEASURES: THE KAC PROCESS WITH HARD SPHERE COLLISIONS IN THREE DIMENSIONS

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We develop a method for producing estimates on the spectral gaps of reversible Markov jump processes with chaotic invariant measures, that is effective in the case of degenerate jump rates, and we apply it to prove the Kac conjecture for hard sphere collision in three dimensions.

## REFERENCES

- [1] CAPUTO, P. (2004). Spectral gap inequalities in product spaces with conservation laws. In *Stochastic Analysis on Large Scale Interacting Systems. Adv. Stud. Pure Math.* **39** 53–88. Math. Soc. Japan, Tokyo. [MR2073330](#) <https://doi.org/10.2969/aspm/03910053>
- [2] CAPUTO, P. (2008). On the spectral gap of the Kac walk and other binary collision processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **4** 205–222. [MR2429910](#)
- [3] CARLEN, E., CARVALHO, M. C. and LOSS, M. (2000). Many-body aspects of approach to equilibrium. In *Journées “Équations aux Dérivées Partielles” (La Chapelle sur Erdre, 2000)* Exp. No. XI, 12. Univ. Nantes, Nantes. [MR1775687](#)
- [4] CARLEN, E. A., CARRILLO, J. A. and CARVALHO, M. C. (2009). Strong convergence towards homogeneous cooling states for dissipative Maxwell models. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** 1675–1700. [MR2566705](#) <https://doi.org/10.1016/j.anihpc.2008.10.005>
- [5] CARLEN, E. A., CARVALHO, M. C. and LOSS, M. (2003). Determination of the spectral gap for Kac’s master equation and related stochastic evolution. *Acta Math.* **191** 1–54. [MR2020418](#) <https://doi.org/10.1007/BF02392695>
- [6] CARLEN, E. A., CARVALHO, M. C. and LOSS, M. (2014). Spectral gap for the Kac model with hard sphere collisions. *J. Funct. Anal.* **266** 1787–1832. [MR3146836](#) <https://doi.org/10.1016/j.jfa.2013.08.024>
- [7] CARLEN, E. A., GERONIMO, J. S. and LOSS, M. (2008). Determination of the spectral gap in the Kac model for physical momentum and energy-conserving collisions. *SIAM J. Math. Anal.* **40** 327–364. [MR2403324](#) <https://doi.org/10.1137/070695423>
- [8] FELLER, W. (1950). *An Introduction to Probability Theory and Its Applications. Vol. I.* Wiley, New York, NY. [MR0038583](#)
- [9] GIROUX, G. and FERLAND, R. (2008). Global spectral gap for Dirichlet–Kac random motions. *J. Stat. Phys.* **132** 561–567. [MR2415119](#) <https://doi.org/10.1007/s10955-008-9571-6>
- [10] GRIGO, A., KHANIN, K. and SZÁSZ, D. (2012). Mixing rates of particle systems with energy exchange. *Nonlinearity* **25** 2349–2376. [MR2956578](#) <https://doi.org/10.1088/0951-7715/25/8/2349>
- [11] JANVRESSE, E. (2001). Spectral gap for Kac’s model of Boltzmann equation. *Ann. Probab.* **29** 288–304. [MR1825150](#) <https://doi.org/10.1214/aop/1008956330>
- [12] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, Vol. III* 171–197. Univ. California Press, Berkeley and Los Angeles. [MR0084985](#)
- [13] KAC, M. (1959). *Probability and Related Topics in Physical Sciences.* Interscience Publishers, London. [MR0102849](#)
- [14] MEHLER, F. G. (1866). Ueber die Entwicklung einer Function von beliebig vielen Variablen nach Laplaceschen Functionen höherer Ordnung. *J. Reine Angew. Math.* **66** 161–176. [MR1579340](#) <https://doi.org/10.1515/crll.1866.66.161>

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- [15] MISCHLER, S. and MOUHOT, C. (2013). Kac's program in kinetic theory. *Invent. Math.* **193** 1–147. MR3069113 <https://doi.org/10.1007/s00222-012-0422-3>
- [16] SASADA, M. (2013). On the spectral gap of the Kac walk and other binary collision processes on  $d$ -dimensional lattice. In *Symmetries, Integrable Systems and Representations. Springer Proc. Math. Stat.* **40** 543–560. Springer, Heidelberg. MR3077700 [https://doi.org/10.1007/978-1-4471-4863-0\\_23](https://doi.org/10.1007/978-1-4471-4863-0_23)
- [17] SASADA, M. (2015). Spectral gap for stochastic energy exchange model with nonuniformly positive rate function. *Ann. Probab.* **43** 1663–1711. MR3353812 <https://doi.org/10.1214/14-AOP916>
- [18] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 <https://doi.org/10.1007/BFb0085169>
- [19] VILLANI, C. (2003). Cercignani's conjecture is sometimes true and always almost true. *Comm. Math. Phys.* **234** 455–490. MR1964379 <https://doi.org/10.1007/s00220-002-0777-1>

## A COVARIANCE FORMULA FOR TOPOLOGICAL EVENTS OF SMOOTH GAUSSIAN FIELDS

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We derive a covariance formula for the class of ‘topological events’ of smooth Gaussian fields on manifolds; these are events that depend only on the topology of the level sets of the field, for example, (i) crossing events for level or excursion sets, (ii) events measurable with respect to the number of connected components of level or excursion sets of a given diffeomorphism class and (iii) persistence events. As an application of the covariance formula, we derive strong mixing bounds for topological events, as well as lower concentration inequalities for additive topological functionals (e.g., the number of connected components) of the level sets that satisfy a law of large numbers. The covariance formula also gives an alternate justification of the Harris criterion, which conjecturally describes the boundary of the percolation universality class for level sets of stationary Gaussian fields. Our work is inspired by (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 1679–1711), in which a correlation inequality was derived for certain topological events on the plane, as well as by (*Asymptotic Methods in the Theory of Gaussian Processes and Fields* (1996) Amer. Math. Soc.), in which a similar covariance formula was established for finite-dimensional Gaussian vectors.

### REFERENCES

- [1] ALEXANDER, K. S. (1996). Boundedness of level lines for two-dimensional random fields. *Ann. Probab.* **24** 1653–1674. MR1415224 <https://doi.org/10.1214/aop/1041903201>
- [2] AURZADA, F. and SIMON, T. (2015). Persistence probabilities and exponents. In *Lévy Matters. V. Lecture Notes in Math.* **2149** 183–224. Springer, Cham. MR3468226 [https://doi.org/10.1007/978-3-319-23138-9\\_3](https://doi.org/10.1007/978-3-319-23138-9_3)
- [3] AZAÏS, J.-M. and WSCHBOR, M. (2009). *Level Sets and Extrema of Random Processes and Fields*. Wiley, Hoboken, NJ. MR2478201 <https://doi.org/10.1002/9780470434642>
- [4] BASU, R., DEMBO, A., FELDHEIM, N. and ZEITOUNI, O. (2020). Exponential concentration for zeroes of stationary Gaussian processes. *Int. Math. Res. Not.* To appear.
- [5] BEFFARA, V. and GAYET, D. (2017). Percolation without FKG. Preprint. Available at [arXiv:1710.10644](https://arxiv.org/abs/1710.10644).
- [6] BEFFARA, V. and GAYET, D. (2017). Percolation of random nodal lines. *Publ. Math. Inst. Hautes Études Sci.* **126** 131–176. MR3735866 <https://doi.org/10.1007/s10240-017-0093-0>
- [7] BELIAEV, D., MCAULEY, M. and MUIRHEAD, S. (2020). On the number of excursion sets of planar Gaussian fields. *Probab. Theory Related Fields*. To appear. <https://doi.org/10.1007/s00440-020-00984-9>
- [8] BELIAEV, D. and MUIRHEAD, S. (2018). Discretisation schemes for level sets of planar Gaussian fields. *Comm. Math. Phys.* **359** 869–913. MR3784534 <https://doi.org/10.1007/s00220-018-3084-1>
- [9] BELIAEV, D., MUIRHEAD, S. and WIGMAN, I. (2017). Russo–Seymour–Welsh estimates for the Kostlan ensemble of random polynomials. Preprint. Available at [arxiv:1709.08961](https://arxiv.org/abs/1709.08961).
- [10] BELIAEV, D. and WIGMAN, I. (2018). Volume distribution of nodal domains of random band-limited functions. *Probab. Theory Related Fields* **172** 453–492. MR3851836 <https://doi.org/10.1007/s00440-017-0813-x>
- [11] BOGOMOLNY, E. and SCHMIT, C. (2007). Random wavefunctions and percolation. *J. Phys. A* **40** 14033–14043. MR2438110 <https://doi.org/10.1088/1751-8113/40/47/001>
- [12] BOLLOBÁS, B. and RIORDAN, O. (2006). *Percolation*. Cambridge Univ. Press, New York. MR2283880 <https://doi.org/10.1017/CBO9781139167383>

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- [13] BRADLEY, R. C. (2005). Basic properties of strong mixing conditions. A survey and some open questions. *Probab. Surv.* **2** 107–144. MR2178042 <https://doi.org/10.1214/154957805100000104>
- [14] CANZANI, Y. and SARNAK, P. (2014). On the topology of the zero sets of monochromatic random waves. Preprint. Available at arXiv:1412.4437.
- [15] CHATTERJEE, S. (2008). Chaos, concentration, and multiple valleys. Preprint. Available at arXiv:0810.4221.
- [16] DEMBO, A. and MUKHERJEE, S. (2015). No zero-crossings for random polynomials and the heat equation. *Ann. Probab.* **43** 85–118. MR3298469 <https://doi.org/10.1214/13-AOP852>
- [17] DOUKHAN, P. (1994). *Mixing: Properties and Examples. Lecture Notes in Statistics* **85**. Springer, New York. MR1312160 <https://doi.org/10.1007/978-1-4612-2642-0>
- [18] EDWARDS, R. D. and KIRBY, R. C. (1971). Deformations of spaces of imbeddings. *Ann. of Math.* (2) **93** 63–88. MR0283802 <https://doi.org/10.2307/1970753>
- [19] EKSTRÖM, M. (2014). A general central limit theorem for strong mixing sequences. *Statist. Probab. Lett.* **94** 236–238. MR3257385 <https://doi.org/10.1016/j.spl.2014.07.024>
- [20] FELDHEIM, N., FELDHEIM, O. and NITZAN, S. (2017). Persistence of Gaussian stationary processes: A spectral perspective. Preprint. Available at arXiv:1709.00204.
- [21] GAYET, D. and WELSCHINGER, J.-Y. (2014). Lower estimates for the expected Betti numbers of random real hypersurfaces. *J. Lond. Math. Soc.* (2) **90** 105–120. MR3245138 <https://doi.org/10.1112/jlms/jdu018>
- [22] GOLUBITSKY, M. and GUILLEMIN, V. (1980). *Stable Mappings and Their Singularities. Graduate Texts in Mathematics* **14**. Springer, New York. MR0341518
- [23] GORESKY, M. and MACPHERSON, R. (1988). *Stratified Morse Theory. Ergebnisse der Mathematik und Ihrer Grenzgebiete* (3) [Results in Mathematics and Related Areas (3)] **14**. Springer, Berlin. MR0932724 <https://doi.org/10.1007/978-3-642-71714-7>
- [24] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **321**. Springer, Berlin. MR1707339 <https://doi.org/10.1007/978-3-662-03981-6>
- [25] IBRAGIMOV, I. A. and LINNIK, YU. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff Publishing, Groningen. MR0322926
- [26] KAC, M. (1943). On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.* **49** 314–320. MR0007812 <https://doi.org/10.1090/S0002-9904-1943-07912-8>
- [27] KESTEN, H. (1987). Scaling relations for 2D-percolation. *Comm. Math. Phys.* **109** 109–156. MR0879034
- [28] KOLMOGOROV, A. N. and ROZANOV, JU. A. (1960). On a strong mixing condition for stationary Gaussian processes. *Teor. Veroyatn. Primen.* **5** 222–227. MR0133175
- [29] KOSTLAN, E. (1993). On the distribution of roots of random polynomials. In *From Topology to Computation: Proceedings of the Smalefest (Berkeley, CA, 1990)* 419–431. Springer, New York. MR1246137
- [30] KOSTLAN, E. (2002). On the expected number of real roots of a system of random polynomial equations. In *Foundations of Computational Mathematics (Hong Kong, 2000)* 149–188. World Sci., River Edge, NJ. MR2021981
- [31] KURLBERG, P. and WIGMAN, I. (2018). Variation of the Nazarov–Sodin constant for random plane waves and arithmetic random waves. *Adv. Math.* **330** 516–552. MR3787552 <https://doi.org/10.1016/j.aim.2018.03.026>
- [32] LIN, Z. and LU, C. (1996). *Limit Theory for Mixing Dependent Random Variables. Mathematics and Its Applications* **378**. Kluwer Academic, Dordrecht; Science Press Beijing, New York. MR1486580
- [33] LOYNES, R. M. (1965). Extreme values in uniformly mixing stationary stochastic processes. *Ann. Math. Stat.* **36** 993–999. MR0176530 <https://doi.org/10.1214/aoms/1177700071>
- [34] MATHER, J. N. (1973). Stratifications and mappings. In *Dynamical Systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)* 195–232. MR0368064
- [35] MOLCHANOV, S. A. and STEPANOV, A. K. (1983). Percolation in random fields. II. *Theoret. Math. Phys.* **55** 592–599.
- [36] MUIRHEAD, S. and VANNEUVILLE, H. (2020). The sharp phase transition for level set percolation of smooth planar Gaussian fields. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 1358–1390. MR4076787 <https://doi.org/10.1214/19-AIHP1006>
- [37] NAZAROV, F. and SODIN, M. (2011). Fluctuations in random complex zeroes: Asymptotic normality revisited. *Int. Math. Res. Not.* **2011** 5720–5759. MR2863379 <https://doi.org/10.1093/imrn/rnr007>
- [38] NAZAROV, F. and SODIN, M. (2016). Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. *Zh. Mat. Fiz. Anal. Geom.* **12** 205–278. MR3522141 <https://doi.org/10.15407/mag12.03.205>
- [39] NICOLAESCU, L. (2011). *An Invitation to Morse Theory*, 2nd ed. *Universitext*. Springer, New York. MR2883440 <https://doi.org/10.1007/978-1-4614-1105-5>

- [40] PITERBARG, V. I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields. Translations of Mathematical Monographs* **148**. Amer. Math. Soc., Providence, RI. [MR1361884](#)
- [41] PITT, L. D. (1982). Positively correlated normal variables are associated. *Ann. Probab.* **10** 496–499. [MR0665603](#)
- [42] RICE, S. O. (1945). Mathematical analysis of random noise. *Bell Syst. Tech. J.* **24** 46–156. [MR0011918](#) <https://doi.org/10.1002/j.1538-7305.1945.tb00453.x>
- [43] RIVERA, A. (2017). Hole probability for nodal sets of the cut-off Gaussian free field. *Adv. Math.* **319** 1–39. [MR3695866](#) <https://doi.org/10.1016/j.aim.2017.08.002>
- [44] RIVERA, A. and VANNEUVILLE, H. (2019). Quasi-independence for nodal lines. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 1679–1711. [MR4010948](#) <https://doi.org/10.1214/18-aihp931>
- [45] RIVERA, A. and VANNEUVILLE, H. (2020). The critical threshold for Bargmann–Fock percolation. *Ann. Henri Lebesgue* **3** 169–215. <https://doi.org/10.5802/agk.29>
- [46] ROSENBLATT, M. (1956). A central limit theorem and a strong mixing condition. *Proc. Natl. Acad. Sci. USA* **42** 43–47. [MR0074711](#) <https://doi.org/10.1073/pnas.42.1.43>
- [47] SARNAK, P. and WIGMAN, I. (2016). Topologies of nodal sets of random band limited functions. In *Advances in the Theory of Automorphic Forms and Their L-Functions. Contemp. Math.* **664** 351–365. Amer. Math. Soc., Providence, RI. [MR3502990](#) <https://doi.org/10.1090/conm/664/13040>
- [48] VANNEUVILLE, H. (2019). Reading group on random nodal lines at ETHZ. Available at [https://metaphor.ethz.ch/x/2019/hs/401-4600-69L/Reading\\_group\\_nodal\\_1\\_2.pdf](https://metaphor.ethz.ch/x/2019/hs/401-4600-69L/Reading_group_nodal_1_2.pdf).
- [49] WEINRIB, A. (1984). Long-range correlated percolation. *Phys. Rev. B* **29** 387–395. [MR0729982](#) <https://doi.org/10.1103/physrevb.29.387>

## COMPARISON THEOREM FOR SOME EXTREMAL EIGENVALUE STATISTICS

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We introduce a method for the comparison of some extremal eigenvalue statistics of random matrices. For example, it allows one to compare the maximal eigenvalue gap in the bulk of two generalized Wigner ensembles, provided that the first four moments of their matrix entries match. As an application, we extend results of Ben Arous–Bourgade and Feng–Wei that identify the limit of the maximal eigenvalue gap in the bulk of the GUE to all complex Hermitian generalized Wigner matrices.

### REFERENCES

- [1] BEN AROUS, G. and BOURGADE, P. (2013). Extreme gaps between eigenvalues of random matrices. *Ann. Probab.* **41** 2648–2681. MR3112927 <https://doi.org/10.1214/11-AOP710>
- [2] BENAYCH-GEORGES, F. and KNOWLES, A. (2018). Lectures on the local semicircle law for Wigner matrices. Preprint. Available at [arXiv:1601.04055](https://arxiv.org/abs/1601.04055).
- [3] BOURGADE, P. (2019). Extreme gaps between eigenvalues of Wigner matrices. Preprint. Available at [arXiv:1812.10376](https://arxiv.org/abs/1812.10376).
- [4] BOURGADE, P., ERDŐS, L., YAU, H.-T. and YIN, J. (2016). Fixed energy universality for generalized Wigner matrices. *Comm. Pure Appl. Math.* **69** 1815–1881. MR3541852 <https://doi.org/10.1002/cpa.21624>
- [5] BOURGADE, P. and YAU, H.-T. (2017). The eigenvector moment flow and local quantum unique ergodicity. *Comm. Math. Phys.* **350** 231–278. MR3606475 <https://doi.org/10.1007/s00220-016-2627-6>
- [6] CHATTERJEE, S. (2006). A generalization of the Lindeberg principle. *Ann. Probab.* **34** 2061–2076. MR2294976 <https://doi.org/10.1214/009117906000000575>
- [7] CHE, Z. and LOPATTO, P. (2019). Universality of the least singular value for sparse random matrices. *Electron. J. Probab.* **24** Paper No. 9, 53. MR3916329 <https://doi.org/10.1214/19-EJP269>
- [8] DIACONIS, P. (2003). Patterns in eigenvalues: The 70th Josiah Willard Gibbs lecture. *Bull. Amer. Math. Soc. (N.S.)* **40** 155–178. MR1962294 <https://doi.org/10.1090/S0273-0979-03-00975-3>
- [9] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2012). Spectral statistics of Erdős–Rényi Graphs II: Eigenvalue spacing and the extreme eigenvalues. *Comm. Math. Phys.* **314** 587–640. MR2964770 <https://doi.org/10.1007/s00220-012-1527-7>
- [10] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2013). The local semicircle law for a general class of random matrices. *Electron. J. Probab.* **18** no. 59, 58. MR3068390 <https://doi.org/10.1214/EJP.v18-2473>
- [11] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2013). Spectral statistics of Erdős–Rényi graphs I: Local semicircle law. *Ann. Probab.* **41** 2279–2375. MR3098073 <https://doi.org/10.1214/11-AOP734>
- [12] ERDŐS, L., PÉCHÉ, S., RAMÍREZ, J. A., SCHLEIN, B. and YAU, H.-T. (2010). Bulk universality for Wigner matrices. *Comm. Pure Appl. Math.* **63** 895–925. MR2662426 <https://doi.org/10.1002/cpa.20317>
- [13] ERDŐS, L., SCHLEIN, B. and YAU, H.-T. (2010). Wegner estimate and level repulsion for Wigner random matrices. *Int. Math. Res. Not. IMRN* **2010** 436–479. MR2587574 <https://doi.org/10.1093/imrn/rnp136>
- [14] ERDŐS, L., SCHLEIN, B. and YAU, H.-T. (2011). Universality of random matrices and local relaxation flow. *Invent. Math.* **185** 75–119. MR2810797 <https://doi.org/10.1007/s00222-010-0302-7>
- [15] ERDŐS, L. and YAU, H.-T. (2012). Universality of local spectral statistics of random matrices. *Bull. Amer. Math. Soc. (N.S.)* **49** 377–414. MR2917064 <https://doi.org/10.1090/S0273-0979-2012-01372-1>

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- [16] ERDŐS, L. and YAU, H.-T. (2015). Gap universality of generalized Wigner and  $\beta$ -ensembles. *J. Eur. Math. Soc. (JEMS)* **17** 1927–2036. MR3372074 <https://doi.org/10.4171/JEMS/548>
- [17] ERDŐS, L. and YAU, H.-T. (2017). *A Dynamical Approach to Random Matrix Theory*. Courant Lecture Notes in Mathematics **28**. Courant Institute of Mathematical Sciences, New York. MR3699468
- [18] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Bulk universality for generalized Wigner matrices. *Probab. Theory Related Fields* **154** 341–407. MR2981427 <https://doi.org/10.1007/s00440-011-0390-3>
- [19] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.* **229** 1435–1515. MR2871147 <https://doi.org/10.1016/j.aim.2011.12.010>
- [20] FENG, R., TIAN, G. and WEI, D. (2019). Small gaps of GOE. *Geom. Funct. Anal.* **29** 1794–1827. MR4034920 <https://doi.org/10.1007/s00039-019-00520-5>
- [21] FENG, R. and WEI, D. (2018). Large gaps of CUE and GUE. Preprint. Available at [arXiv:1807.02149](https://arxiv.org/abs/1807.02149).
- [22] FENG, R. and WEI, D. (2018). Small gaps of circular  $\beta$ -ensemble. Preprint. Available at [arXiv:1806.01555](https://arxiv.org/abs/1806.01555).
- [23] FIGALLI, A. and GUIONNET, A. (2016). Universality in several-matrix models via approximate transport maps. *Acta Math.* **217** 81–176. MR3646880 <https://doi.org/10.1007/s11511-016-0142-4>
- [24] HUANG, J., LANDON, B. and YAU, H.-T. (2015). Bulk universality of sparse random matrices. *J. Math. Phys.* **56** 123301, 19. MR3429490 <https://doi.org/10.1063/1.4936139>
- [25] KNOWLES, A. and YIN, J. (2013). Eigenvector distribution of Wigner matrices. *Probab. Theory Related Fields* **155** 543–582. MR3034787 <https://doi.org/10.1007/s00440-011-0407-y>
- [26] KORADA, S. B. and MONTANARI, A. (2011). Applications of the Lindeberg principle in communications and statistical learning. *IEEE Trans. Inf. Theory* **57** 2440–2450. MR2809100 <https://doi.org/10.1109/TIT.2011.2112231>
- [27] LANDON, B., SOSOE, P. and YAU, H.-T. (2019). Fixed energy universality of Dyson Brownian motion. *Adv. Math.* **346** 1137–1332. MR3914908 <https://doi.org/10.1016/j.aim.2019.02.010>
- [28] LANDON, B. and YAU, H.-T. (2017). Convergence of local statistics of Dyson Brownian motion. *Comm. Math. Phys.* **355** 949–1000. MR3687212 <https://doi.org/10.1007/s00220-017-2955-1>
- [29] LOPATTO, P. and LUH, K. (2019). Tail bounds for gaps between eigenvalues of sparse random matrices. Preprint. Available at [arXiv:1901.05948](https://arxiv.org/abs/1901.05948).
- [30] LUH, K. and VU, V. (2018). Sparse random matrices have simple spectrum. Preprint. Available at [arXiv:1802.03662](https://arxiv.org/abs/1802.03662).
- [31] NGUYEN, H., TAO, T. and VU, V. (2017). Random matrices: Tail bounds for gaps between eigenvalues. *Probab. Theory Related Fields* **167** 777–816. MR3627428 <https://doi.org/10.1007/s00440-016-0693-5>
- [32] ODLYZKO, A. M. (1987). On the distribution of spacings between zeros of the zeta function. *Math. Comp.* **48** 273–308. MR0866115 <https://doi.org/10.2307/2007890>
- [33] TAO, T. and VU, V. (2010). Random matrices: Universality of local eigenvalue statistics up to the edge. *Comm. Math. Phys.* **298** 549–572. MR2669449 <https://doi.org/10.1007/s00220-010-1044-5>
- [34] TAO, T. and VU, V. (2011). Random matrices: Universality of local eigenvalue statistics. *Acta Math.* **206** 127–204. MR2784665 <https://doi.org/10.1007/s11511-011-0061-3>
- [35] VINSON, J. P. (2001). Closest Spacing of Eigenvalues. Ph.D. thesis, Princeton University, Princeton, NJ. MR2702214

# ON SINGULARITY OF ENERGY MEASURES FOR SYMMETRIC DIFFUSIONS WITH FULL OFF-DIAGONAL HEAT KERNEL ESTIMATES

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We show that, for a strongly local, regular symmetric Dirichlet form over a complete, locally compact geodesic metric space, full off-diagonal heat kernel estimates with walk dimension strictly larger than two (*sub-Gaussian* estimates) imply the singularity of the energy measures with respect to the symmetric measure, verifying a conjecture by M. T. Barlow in (*Contemp. Math.* **338** (2003) 11–40). We also prove that in the contrary case of walk dimension two, that is, where full off-diagonal *Gaussian* estimates of the heat kernel hold, the symmetric measure and the energy measures are mutually absolutely continuous in the sense that a Borel subset of the state space has measure zero for the symmetric measure if and only if it has measure zero for the energy measures of all functions in the domain of the Dirichlet form.

## REFERENCES

- [1] ALONSO-RUIZ, P., BAUDOIN, F., CHEN, L., ROGERS, L., SHANMUGALINGAM, N. and TEPLYAEV, A. (2020). Besov class via heat semigroup on Dirichlet spaces II: BV functions and Gaussian heat kernel estimates. *Calc. Var. Partial Differential Equations* **59** Art. ID 103. MR4102351 <https://doi.org/10.1007/s00526-020-01750-4>
- [2] ANDRES, S. and BARLOW, M. T. (2015). Energy inequalities for cutoff functions and some applications. *J. Reine Angew. Math.* **699** 183–215. MR3305925 <https://doi.org/10.1515/crelle-2013-0009>
- [3] BARLOW, M. T. (1998). Diffusions on fractals. In *Lectures on Probability Theory and Statistics (Saint-Flour, 1995)*. *Lecture Notes in Math.* **1690** 1–121. Springer, Berlin. MR1668115 <https://doi.org/10.1007/BFb0092537>
- [4] BARLOW, M. T. (2003). Heat kernels and sets with fractal structure. In *Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces (Paris, 2002)*. *Contemp. Math.* **338** 11–40. Amer. Math. Soc., Providence, RI. MR2039950 <https://doi.org/10.1090/conm/338/06069>
- [5] BARLOW, M. T. (2019). Personal communication, July 17.
- [6] BARLOW, M. T. and BASS, R. F. (1989). The construction of Brownian motion on the Sierpiński carpet. *Ann. Inst. Henri Poincaré Probab. Stat.* **25** 225–257. MR1023950
- [7] BARLOW, M. T. and BASS, R. F. (1992). Transition densities for Brownian motion on the Sierpiński carpet. *Probab. Theory Related Fields* **91** 307–330. MR1151799 <https://doi.org/10.1007/BF01192060>
- [8] BARLOW, M. T. and BASS, R. F. (1999). Brownian motion and harmonic analysis on Sierpinski carpets. *Canad. J. Math.* **51** 673–744. MR1701339 <https://doi.org/10.4153/CJM-1999-031-4>
- [9] BARLOW, M. T. and BASS, R. F. (2004). Stability of parabolic Harnack inequalities. *Trans. Amer. Math. Soc.* **356** 1501–1533. MR2034316 <https://doi.org/10.1090/S0002-9947-03-03414-7>
- [10] BARLOW, M. T., BASS, R. F. and KUMAGAI, T. (2006). Stability of parabolic Harnack inequalities on metric measure spaces. *J. Math. Soc. Japan* **58** 485–519. MR2228569
- [11] BARLOW, M. T., BASS, R. F., KUMAGAI, T. and TEPLYAEV, A. (2010). Uniqueness of Brownian motion on Sierpiński carpets. *J. Eur. Math. Soc. (JEMS)* **12** 655–701. MR2639315 <https://doi.org/10.4171/jems/211>
- [12] BARLOW, M. T., GRIGOR'YAN, A. and KUMAGAI, T. (2012). On the equivalence of parabolic Harnack inequalities and heat kernel estimates. *J. Math. Soc. Japan* **64** 1091–1146. MR2998918 <https://doi.org/10.2969/jmsj/06441091>

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- [13] BARLOW, M. T. and HAMBLY, B. M. (1997). Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets. *Ann. Inst. Henri Poincaré Probab. Stat.* **33** 531–557. MR1473565 [https://doi.org/10.1016/S0246-0203\(97\)80104-5](https://doi.org/10.1016/S0246-0203(97)80104-5)
- [14] BARLOW, M. T. and PERKINS, E. A. (1988). Brownian motion on the Sierpiński gasket. *Probab. Theory Related Fields* **79** 543–623. MR0966175 <https://doi.org/10.1007/BF00318785>
- [15] BEN-BASSAT, O., STRICHARTZ, R. S. and TEPLYAEV, A. (1999). What is not in the domain of the Laplacian on Sierpinski gasket type fractals. *J. Funct. Anal.* **166** 197–217. MR1707752 <https://doi.org/10.1006/jfan.1999.3431>
- [16] BURAGO, D., BURAGO, Y. and IVANOV, S. (2001). *A Course in Metric Geometry. Graduate Studies in Mathematics* **33**. Amer. Math. Soc., Providence, RI. MR1835418 <https://doi.org/10.1090/gsm/033>
- [17] CHEN, Z.-Q. and FUKUSHIMA, M. (2012). *Symmetric Markov Processes, Time Change, and Boundary Theory. London Mathematical Society Monographs Series* **35**. Princeton Univ. Press, Princeton, NJ. MR2849840
- [18] FITZSIMMONS, P. J., HAMBLY, B. M. and KUMAGAI, T. (1994). Transition density estimates for Brownian motion on affine nested fractals. *Comm. Math. Phys.* **165** 595–620. MR1301625
- [19] FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (2011). *Dirichlet Forms and Symmetric Markov Processes*, 2nd revised and extended ed. *de Gruyter Studies in Mathematics* **19**. de Gruyter, Berlin. MR2778606
- [20] GRIGOR'YAN, A. (2009). *Heat Kernel and Analysis on Manifolds. AMS/IP Studies in Advanced Mathematics* **47**. Amer. Math. Soc., Providence, RI; International Press, Boston, MA. MR2569498
- [21] GRIGOR'YAN, A. and HU, J. (2014). Heat kernels and Green functions on metric measure spaces. *Canad. J. Math.* **66** 641–699. MR3194164 <https://doi.org/10.4153/CJM-2012-061-5>
- [22] GRIGOR'YAN, A., HU, J. and LAU, K.-S. (2015). Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric measure spaces. *J. Math. Soc. Japan* **67** 1485–1549. MR3417504 <https://doi.org/10.2969/jmsj/06741485>
- [23] GRIGOR'YAN, A. and KAJINO, N. (2017). Localized upper bounds of heat kernels for diffusions via a multiple Dynkin–Hunt formula. *Trans. Amer. Math. Soc.* **369** 1025–1060. MR3572263 <https://doi.org/10.1090/tran/6784>
- [24] GRIGOR'YAN, A. and TELCS, A. (2012). Two-sided estimates of heat kernels on metric measure spaces. *Ann. Probab.* **40** 1212–1284. MR2962091 <https://doi.org/10.1214/11-AOP645>
- [25] HAMBLY, B. M. (1992). Brownian motion on a homogeneous random fractal. *Probab. Theory Related Fields* **94** 1–38. MR1189083 <https://doi.org/10.1007/BF01222507>
- [26] HAMBLY, B. M. (2000). Heat kernels and spectral asymptotics for some random Sierpinski gaskets. In *Fractal Geometry and Stochastics, II (Greifswald/Koserow, 1998). Progress in Probability* **46** 239–267. Birkhäuser, Basel. MR1786351
- [27] HEINONEN, J. (2001). *Lectures on Analysis on Metric Spaces. Universitext*. Springer, New York. MR1800917 <https://doi.org/10.1007/978-1-4613-0131-8>
- [28] HEINONEN, J. and KOSKELA, P. (1998). Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.* **181** 1–61. MR1654771 <https://doi.org/10.1007/BF02392747>
- [29] HINO, M. (2005). On singularity of energy measures on self-similar sets. *Probab. Theory Related Fields* **132** 265–290. MR2199293 <https://doi.org/10.1007/s00440-004-0396-1>
- [30] HINO, M. (2010). Energy measures and indices of Dirichlet forms, with applications to derivatives on some fractals. *Proc. Lond. Math. Soc.* (3) **100** 269–302. MR2578475 <https://doi.org/10.1112/plms/pdp032>
- [31] HINO, M. and NAKAHARA, K. (2006). On singularity of energy measures on self-similar sets. II. *Bull. Lond. Math. Soc.* **38** 1019–1032. MR2285256 <https://doi.org/10.1112/S0024609306019072>
- [32] KAJINO, N. (2020). An elementary proof of walk dimension being greater than two for Brownian motion on Sierpiński carpets. Preprint. Available at 2005.02524.
- [33] KAJINO, N. On singularity of energy measures for symmetric diffusions with full off-diagonal heat kernel estimates II: Some borderline examples. In preparation.
- [34] KANAI, M. (1985). Rough isometries, and combinatorial approximations of geometries of noncompact Riemannian manifolds. *J. Math. Soc. Japan* **37** 391–413. MR0792983 <https://doi.org/10.2969/jmsj/03730391>
- [35] KIGAMI, J. (2001). *Analysis on Fractals. Cambridge Tracts in Mathematics* **143**. Cambridge Univ. Press, Cambridge. MR1840042 <https://doi.org/10.1017/CBO9780511470943>
- [36] KIGAMI, J. (2012). Resistance forms, quasisymmetric maps and heat kernel estimates. *Mem. Amer. Math. Soc.* **216** vi + 132. MR2919892 <https://doi.org/10.1090/S0065-9266-2011-00632-5>
- [37] KOSKELA, P., SHANMUGALINGAM, N. and TYSON, J. T. (2004). Dirichlet forms, Poincaré inequalities, and the Sobolev spaces of Korevaar and Schoen. *Potential Anal.* **21** 241–262. MR2075670 <https://doi.org/10.1023/B:POTA.0000033331.88514.6e>

- [38] KUMAGAI, T. (1993). Estimates of transition densities for Brownian motion on nested fractals. *Probab. Theory Related Fields* **96** 205–224. [MR1227032](#) <https://doi.org/10.1007/BF01192133>
- [39] KUSUOKA, S. (1989). Dirichlet forms on fractals and products of random matrices. *Publ. Res. Inst. Math. Sci.* **25** 659–680. [MR1025071](#) <https://doi.org/10.2977/prims/1195173187>
- [40] KUSUOKA, S. (1993). Lecture on diffusion processes on nested fractals. In *Statistical Mechanics and Fractals. Lecture Notes in Math.* **1567** 39–98. Springer, Berlin. [MR1295841](#) <https://doi.org/10.1007/BFb0074240>
- [41] KUSUOKA, S. and ZHOU, X. Y. (1992). Dirichlet forms on fractals: Poincaré constant and resistance. *Probab. Theory Related Fields* **93** 169–196. [MR1176724](#) <https://doi.org/10.1007/BF01195228>
- [42] LIERL, J. (2015). Scale-invariant boundary Harnack principle on inner uniform domains in fractal-type spaces. *Potential Anal.* **43** 717–747. [MR3432457](#) <https://doi.org/10.1007/s11118-015-9494-1>
- [43] MA, Z.-M. and RÖCKNER, M. (1992). *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Universitext.* Springer, Berlin. [MR1214375](#) <https://doi.org/10.1007/978-3-642-77739-4>
- [44] MURUGAN, M. (2020). On the length of chains in a metric space. *J. Funct. Anal.* **279** Art. ID 108627. [MR4099475](#) <https://doi.org/10.1016/j.jfa.2020.108627>
- [45] RUDIN, W. (1987). *Real and Complex Analysis*, 3rd ed. McGraw-Hill, New York. [MR0924157](#)
- [46] SALOFF-COSTE, L. (2002). *Aspects of Sobolev-Type Inequalities. London Mathematical Society Lecture Note Series* **289**. Cambridge Univ. Press, Cambridge. [MR1872526](#)
- [47] STURM, K.-T. (1995). Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations. *Osaka J. Math.* **32** 275–312. [MR1355744](#)
- [48] STURM, K.-T. (1995). On the geometry defined by Dirichlet forms. In *Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993). Progress in Probability* **36** 231–242. Birkhäuser, Basel. [MR1360279](#)
- [49] STURM, K.-T. (1996). Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality. *J. Math. Pures Appl.* (9) **75** 273–297. [MR1387522](#)

# A COMPARISON PRINCIPLE FOR RANDOM WALK ON DYNAMICAL PERCOLATION

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We consider the model of random walk on dynamical percolation introduced by Peres, Stauffer and Steif in (*Probab. Theory Related Fields* **162** (2015) 487–530). We obtain comparison results for this model for hitting and mixing times and for the spectral gap and log-Sobolev constant with the corresponding quantities for simple random walk on the underlying graph  $G$ , for general graphs. When  $G$  is the torus  $\mathbb{Z}_n^d$ , we recover the results of Peres et al., and we also extend them to the critical case. We also obtain bounds in the cases where  $G$  is a transitive graph of moderate growth and also when it is the hypercube.

## REFERENCES

- [1] ALDOUS, D. and FILL, J. (2002). Reversible Markov chains and random walks on graphs. Unfinished manuscript. Available at <http://www.stat.berkeley.edu/~aldous/RWG/book.html>.
- [2] ALDOUS, D. J. (1982). Some inequalities for reversible Markov chains. *J. Lond. Math. Soc.* (2) **25** 564–576. MR0657512 <https://doi.org/10.1112/jlms/s2-25.3.564>
- [3] AVENA, L., GÜLDAŞ, H., VAN DER HOFSTAD, R. and DEN HOLLANDER, F. (2018). Mixing times of random walks on dynamic configuration models. *Ann. Appl. Probab.* **28** 1977–2002. MR3843821 <https://doi.org/10.1214/17-AAP1289>
- [4] BASU, R., HERMON, J. and PERES, Y. (2017). Characterization of cutoff for reversible Markov chains. *Ann. Probab.* **45** 1448–1487. MR3650406 <https://doi.org/10.1214/16-AOP1090>
- [5] BISKUP, M. and RODRIGUEZ, P.-F. (2018). Limit theory for random walks in degenerate time-dependent random environments. *J. Funct. Anal.* **274** 985–1046. MR3743189 <https://doi.org/10.1016/j.jfa.2017.12.002>
- [6] BREUILLARD, E. and TOINTON, M. C. H. (2016). Nilprogressions and groups with moderate growth. *Adv. Math.* **289** 1008–1055. MR3439705 <https://doi.org/10.1016/j.aim.2015.11.025>
- [7] DIACONIS, P. and SALOFF-COSTE, L. (1994). Moderate growth and random walk on finite groups. *Geom. Funct. Anal.* **4** 1–36. MR1254308 <https://doi.org/10.1007/BF01898359>
- [8] DIACONIS, P. and SALOFF-COSTE, L. (1996). Logarithmic Sobolev inequalities for finite Markov chains. *Ann. Appl. Probab.* **6** 695–750. MR1410112 <https://doi.org/10.1214/aoap/1034968224>
- [9] DING, J., LEE, J. R. and PERES, Y. (2012). Cover times, blanket times, and majorizing measures. *Ann. of Math.* (2) **175** 1409–1471. MR2912708 <https://doi.org/10.4007/annals.2012.175.3.8>
- [10] DING, J. and PERES, Y. (2013). Sensitivity of mixing times. *Electron. Commun. Probab.* **18** no. 88. MR3141797 <https://doi.org/10.1214/ECP.v18-2765>
- [11] DOYLE, P. G. and STEINER, J. (2011). Commuting time geometry of ergodic markov chains. Preprint. Available at [arXiv:1107.2612](https://arxiv.org/abs/1107.2612).
- [12] GAUILLIÈRE, A. and LANDIM, C. (2014). A Dirichlet principle for non reversible Markov chains and some recurrence theorems. *Probab. Theory Related Fields* **158** 55–89. MR3152780 <https://doi.org/10.1007/s00440-012-0477-5>
- [13] GOEL, S., MONTENEGRO, R. and TETALI, P. (2006). Mixing time bounds via the spectral profile. *Electron. J. Probab.* **11** 1–26. MR2199053 <https://doi.org/10.1214/EJP.v11-300>
- [14] HÄGGSTRÖM, O., PERES, Y. and STEIF, J. E. (1997). Dynamical percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **33** 497–528. MR1465800 [https://doi.org/10.1016/S0246-0203\(97\)80103-3](https://doi.org/10.1016/S0246-0203(97)80103-3)
- [15] HERMON, J. (2018). On sensitivity of uniform mixing times. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 234–248. MR3765888 <https://doi.org/10.1214/16-AIHP802>



- [16] HERMON, J. (2018). A spectral characterization for concentration of the cover time. *J. Theoret. Probab.* To appear. Preprint. Available at [arXiv:1809.00145](https://arxiv.org/abs/1809.00145).
- [17] HERMON, J. and PERES, Y. (2018). A characterization of  $L_2$  mixing and hypercontractivity via hitting times and maximal inequalities. *Probab. Theory Related Fields* **170** 769–800. [MR3773799 https://doi.org/10.1007/s00440-017-0769-x](https://doi.org/10.1007/s00440-017-0769-x)
- [18] HERMON, J. and PERES, Y. (2018). On sensitivity of mixing times and cutoff. *Electron. J. Probab.* **23** Paper No. 25. [MR3779818 https://doi.org/10.1214/18-EJP154](https://doi.org/10.1214/18-EJP154)
- [19] HERMON, J. and PYMAR, R. (2018). The exclusion process mixes (almost) faster than independent particles. Preprint. Available at [arXiv:1808.10846](https://arxiv.org/abs/1808.10846).
- [20] HERMON, J. and SOUSI, P. (2019). A comparison principle for random walk on dynamical percolation. Preprint. Available at [arXiv:1902.02770](https://arxiv.org/abs/1902.02770).
- [21] KOZMA, G. (2007). On the precision of the spectral profile. *ALEA Lat. Am. J. Probab. Math. Stat.* **3** 321–329. [MR2372888](https://doi.org/10.1214/07-ALEA328)
- [22] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. Second edition of [MR2466937]. [MR3726904](https://doi.org/10.1090/S0025-5718-2017-08111-0)
- [23] LOVÁSZ, L. and WINKLER, P. (1998). Mixing times. In *Microsurveys in Discrete Probability (Princeton, NJ, 1997)*. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **41** 85–133. Amer. Math. Soc., Providence, RI. [MR1630411](https://doi.org/10.1090/S0025-5718-1998-08111-0)
- [24] LYONS, R. and OVEIS GHARAN, S. (2018). Sharp bounds on random walk eigenvalues via spectral embedding. *Int. Math. Res. Not. IMRN* **2018** 7555–7605. [MR3892273 https://doi.org/10.1093/imrn/rnx082](https://doi.org/10.1093/imrn/rnx082)
- [25] MORRIS, B. and PERES, Y. (2005). Evolving sets, mixing and heat kernel bounds. *Probab. Theory Related Fields* **133** 245–266. [MR2198701 https://doi.org/10.1007/s00440-005-0434-7](https://doi.org/10.1007/s00440-005-0434-7)
- [26] PERES, Y., SOUSI, P. and STEIF, J. E. (2019). Quenched exit times for random walk on dynamical percolation. *Markov Process. Related Fields*. To appear.
- [27] PERES, Y., SOUSI, P. and STEIF, J. E. (2020). Mixing time for random walk on supercritical dynamical percolation. *Probab. Theory Related Fields* **176** 809–849. [MR4087484 https://doi.org/10.1007/s00440-019-00927-z](https://doi.org/10.1007/s00440-019-00927-z)
- [28] PERES, Y., STAUFFER, A. and STEIF, J. E. (2015). Random walks on dynamical percolation: Mixing times, mean squared displacement and hitting times. *Probab. Theory Related Fields* **162** 487–530. [MR3383336 https://doi.org/10.1007/s00440-014-0578-4](https://doi.org/10.1007/s00440-014-0578-4)
- [29] PERES, Y. and STEIF, J. E. Private communication.
- [30] SOUSI, P. and THOMAS, S. (2018). Cutoff for random walk on dynamical Erdős Rényi graph. Preprint. Available at [arXiv:1807.04719](https://arxiv.org/abs/1807.04719).
- [31] TESSERA, R. and TOINTON, M. (2019). A finitary structure theorem for vertex-transitive graphs of polynomial growth. Preprint. Available at [arXiv:1908.06044](https://arxiv.org/abs/1908.06044).
- [32] VAN DER HOFSTAD, R. and NACHMIAS, A. (2017). Hypercube percolation. *J. Eur. Math. Soc. (JEMS)* **19** 725–814. [MR3612867 https://doi.org/10.4171/JEMS/679](https://doi.org/10.4171/JEMS/679)

## CAPACITY OF THE RANGE IN DIMENSION 5

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We prove a central limit theorem for the capacity of the range of a symmetric random walk on  $\mathbb{Z}^5$ , under only a moment condition on the step distribution. The result is analogous to the central limit theorem for the size of the range in dimension three, obtained by Jain and Pruitt in 1971. In particular, an atypical logarithmic correction appears in the scaling of the variance. The proof is based on new asymptotic estimates, which hold in any dimension  $d \geq 5$ , for the probability that the ranges of two independent random walks intersect. The latter are then used for computing covariances of some intersection events at the leading order.

### REFERENCES

- [1] ASSELAH, A. and SCHAPIRA, B. (2017). Boundary of the range of transient random walk. *Probab. Theory Related Fields* **168** 691–719. MR3663629 <https://doi.org/10.1007/s00440-016-0722-4>
- [2] ASSELAH, A. and SCHAPIRA, B. (2017). Moderate deviations for the range of a transient random walk: Path concentration. *Ann. Sci. Éc. Norm. Supér. (4)* **50** 755–786. MR3665554 <https://doi.org/10.24033/asens.2331>
- [3] ASSELAH, A., SCHAPIRA, B. and SOUSI, P. (2018). Capacity of the range of random walk on  $\mathbb{Z}^d$ . *Trans. Amer. Math. Soc.* **370** 7627–7645. MR3852443 <https://doi.org/10.1090/tran/7247>
- [4] ASSELAH, A., SCHAPIRA, B. and SOUSI, P. (2019). Capacity of the range of random walk on  $\mathbb{Z}^4$ . *Ann. Probab.* **47** 1447–1497. MR3945751 <https://doi.org/10.1214/18-AOP1288>
- [5] BENJAMINI, I., KOZMA, G., YADIN, A. and YEHUDAYOFF, A. (2010). Entropy of random walk range. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 1080–1092. MR2744887 <https://doi.org/10.1214/09-AIHP345>
- [6] BERESTYCKI, N. and YADIN, A. (2019). Condensation of a self-attracting random walk. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 835–861. MR3949955 <https://doi.org/10.1214/18-aihp900>
- [7] CHANG, Y. (2017). Two observations on the capacity of the range of simple random walks on  $\mathbb{Z}^3$  and  $\mathbb{Z}^4$ . *Electron. Commun. Probab.* **22** Paper No. 25, 9. MR3652038 <https://doi.org/10.1214/17-ECP55>
- [8] DELIGIANNIDIS, G., GOUEZEL, S. and KOSLOFF, Z. Boundary of the range of a random walk and the Følner property. Available at [arXiv:1810.10454](https://arxiv.org/abs/1810.10454).
- [9] DURRETT, R. (2010). *Probability: Theory and Examples*, 4th ed. *Cambridge Series in Statistical and Probabilistic Mathematics* **31**. Cambridge Univ. Press, Cambridge. MR2722836 <https://doi.org/10.1017/CBO9780511779398>
- [10] DVORETZKY, A. and ERDÖS, P. (1951). Some problems on random walk in space. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950 353–367. Univ. California Press, Berkeley and Los Angeles. MR0047272
- [11] ERDÖS, P. and TAYLOR, S. J. (1960). Some intersection properties of random walk paths. *Acta Math. Acad. Sci. Hung.* **11** 231–248. MR0126299 <https://doi.org/10.1007/BF02020942>
- [12] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II. Second Edition*. Wiley, New York. MR0270403
- [13] JAIN, N. and OREY, S. (1968). On the range of random walk. *Israel J. Math.* **6** 373–380. MR0243623 <https://doi.org/10.1007/BF02771217>
- [14] JAIN, N. C. and PRUITT, W. E. (1971). The range of transient random walk. *J. Anal. Math.* **24** 369–393. MR0283890 <https://doi.org/10.1007/BF02790380>
- [15] LAWLER, G. F. (1991). *Intersections of Random Walks. Probability and Its Applications*. Birkhäuser, Boston, MA. MR1117680

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- [16] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk: A Modern Introduction*. *Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. MR2677157 <https://doi.org/10.1017/CBO9780511750854>
- [17] LE GALL, J.-F. (1986). Propriétés d'intersection des marches aléatoires. I. Convergence vers le temps local d'intersection. *Comm. Math. Phys.* **104** 471–507. MR0840748
- [18] OKADA, I. (2016). The inner boundary of random walk range. *J. Math. Soc. Japan* **68** 939–959. MR3523532 <https://doi.org/10.2969/jmsj/06830939>
- [19] SCHAPIRA, B. (2020). Supplement to “Capacity of the range in dimension 5.” <https://doi.org/10.1214/20-AOP1442SUPP>
- [20] SPITZER, F. (1976). *Principles of Random Walk*, 2nd ed. Springer, New York. MR0388547
- [21] UCHIYAMA, K. (1998). Green's functions for random walks on  $\mathbf{Z}^N$ . *Proc. Lond. Math. Soc.* (3) **77** 215–240. MR1625467 <https://doi.org/10.1112/S0024611598000458>

# WELL-POSEDNESS, STABILITY AND SENSITIVITIES FOR STOCHASTIC DELAY EQUATIONS: A GENERALIZED COUPLING APPROACH

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We develop a new generalized coupling approach to the study of stochastic delay equations with Hölder continuous coefficients, for which analytical PDE-based methods are not available. We prove that such equations possess unique weak solutions, and establish weak ergodic rates for the corresponding segment processes. We also prove, under additional smoothness assumptions on the coefficients, stabilization rates for the sensitivities in the initial value of the corresponding semigroups.

## REFERENCES

- [1] ABOURASHCHI, N. and VERETENNIKOV, A. Y. (2010). On stochastic averaging and mixing. *Theory Stoch. Process.* **16** 111–129. [MR2779833](#)
- [2] BESSAIH, H., KAPICA, R. and SZAREK, T. (2014). Criterion on stability for Markov processes applied to a model with jumps. *Semigroup Forum* **88** 76–92. [MR3164152](#) <https://doi.org/10.1007/s00233-013-9503-x>
- [3] BISMUT, J.-M. (1981). Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions. *Z. Wahrsch. Verw. Gebiete* **56** 469–505. [MR0621660](#) <https://doi.org/10.1007/BF00531428>
- [4] BODNARCHUK, S. V. and KULIK, O. M. (2009). Conditions for the existence and smoothness of a density of distribution for Ornstein–Uhlenbeck processes with Lévy noise. *Theory Probab. Math. Statist.* **79** 23–38.
- [5] BUTKOVSKY, O. (2014). Subgeometric rates of convergence of Markov processes in the Wasserstein metric. *Ann. Appl. Probab.* **24** 526–552. [MR3178490](#) <https://doi.org/10.1214/13-AAP922>
- [6] BUTKOVSKY, O., KULIK, A. and SCHEUTZOW, M. (2020). Generalized couplings and ergodic rates for SPDEs and other Markov models. *Ann. Appl. Probab.* **30** 1–39. [MR4068305](#) <https://doi.org/10.1214/19-AAP1485>
- [7] BUTKOVSKY, O. and SCHEUTZOW, M. (2017). Invariant measures for stochastic functional differential equations. *Electron. J. Probab.* **22** Paper No. 98, 23. [MR3724566](#) <https://doi.org/10.1214/17-EJP122>
- [8] CERRAI, S. (1999). Differentiability of Markov semigroups for stochastic reaction-diffusion equations and applications to control. *Stochastic Process. Appl.* **83** 15–37. [MR1705598](#) [https://doi.org/10.1016/S0304-4149\(99\)00014-9](https://doi.org/10.1016/S0304-4149(99)00014-9)
- [9] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. [MR1207136](#) <https://doi.org/10.1017/CBO9780511666223>
- [10] DUDLEY, R. M. (2002). *Real Analysis and Probability*. *Cambridge Studies in Advanced Mathematics* **74**. Cambridge Univ. Press, Cambridge. [MR1932358](#) <https://doi.org/10.1017/CBO9780511755347>
- [11] DURMUS, A., FORT, G. and MOULINES, É. (2016). Subgeometric rates of convergence in Wasserstein distance for Markov chains. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1799–1822. [MR3573296](#) <https://doi.org/10.1214/15-AIHP699>
- [12] ELWORTHY, K. D. and LI, X.-M. (1994). Formulae for the derivatives of heat semigroups. *J. Funct. Anal.* **125** 252–286. [MR1297021](#) <https://doi.org/10.1006/jfan.1994.1124>
- [13] ES-SARHIR, A., VON RENESSE, M.-K. and SCHEUTZOW, M. (2009). Harnack inequality for functional SDEs with bounded memory. *Electron. Commun. Probab.* **14** 560–565. [MR2570679](#) <https://doi.org/10.1214/ECP.v14-1513>
- [14] GIKHMAN, I. and SKOROHOD, A. V. (1972). *Stochastic Differential Equations*. Springer, New York. [MR0346904](#)

---

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- [15] HAIRER, M. (2002). Exponential mixing properties of stochastic PDEs through asymptotic coupling. *Probab. Theory Related Fields* **124** 345–380. MR1939651 <https://doi.org/10.1007/s004400200216>
- [16] HAIRER, M. and MATTINGLY, J. C. (2006). Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing. *Ann. of Math. (2)* **164** 993–1032. MR2259251 <https://doi.org/10.4007/annals.2006.164.993>
- [17] HAIRER, M., MATTINGLY, J. C. and SCHEUTZOW, M. (2011). Asymptotic coupling and a general form of Harris’ theorem with applications to stochastic delay equations. *Probab. Theory Related Fields* **149** 223–259. MR2773030 <https://doi.org/10.1007/s00440-009-0250-6>
- [18] KOMOROWSKI, T., PESZAT, S. and SZAREK, T. (2010). On ergodicity of some Markov processes. *Ann. Probab.* **38** 1401–1443. MR2663632 <https://doi.org/10.1214/09-AOP513>
- [19] KULIK, A. (2018). *Ergodic Behavior of Markov Processes: With Applications to Limit Theorems. De Gruyter Studies in Mathematics* **67**. de Gruyter, Berlin. MR3791835
- [20] LIPTSER, R. S. and SHIRYAYEV, A. N. (2013). *Statistics of Random Processes. I*. Springer, Berlin.
- [21] MOHAMMED, S. E. A. (1984). *Stochastic Functional Differential Equations. Research Notes in Mathematics* **99**. Pitman (Advanced Publishing Program), Boston, MA. MR0754561
- [22] PAGÈS, G. and PANLOUP, F. (2014). A mixed-step algorithm for the approximation of the stationary regime of a diffusion. *Stochastic Process. Appl.* **124** 522–565. MR3131304 <https://doi.org/10.1016/j.spa.2013.07.011>
- [23] PARDOUX, È. and VERETENNIKOV, A. Y. (2003). On Poisson equation and diffusion approximation. II. *Ann. Probab.* **31** 1166–1192. MR1988467 <https://doi.org/10.1214/aop/1055425774>
- [24] PESZAT, S. and ZABCZYK, J. (2007). *Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach. Encyclopedia of Mathematics and Its Applications* **113**. Cambridge Univ. Press, Cambridge. MR2356959 <https://doi.org/10.1017/CBO9780511721373>
- [25] REVUZ, D. and YOR, M. (2005). *Continuous Martingales and Brownian Motion*, 3rd ed. Springer, Berlin.
- [26] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **233**. Springer, Berlin. MR0532498
- [27] THORISSON, H. (2000). *Coupling, Stationarity, and Regeneration. Probability and Its Applications (New York)*. Springer, New York. MR1741181 <https://doi.org/10.1007/978-1-4612-1236-2>
- [28] TSYBAKOV, A. B. (2009). *Introduction to Nonparametric Estimation. Springer Series in Statistics*. Springer, New York. MR2724359 <https://doi.org/10.1007/b13794>
- [29] VON RENESSE, M.-K. and SCHEUTZOW, M. (2010). Existence and uniqueness of solutions of stochastic functional differential equations. *Random Oper. Stoch. Equ.* **18** 267–284. MR2718125 <https://doi.org/10.1515/ROSE.2010.015>
- [30] WANG, F.-Y. (2011). Harnack inequality for SDE with multiplicative noise and extension to Neumann semi-group on nonconvex manifolds. *Ann. Probab.* **39** 1449–1467. MR2857246 <https://doi.org/10.1214/10-AOP600>

# THE EXCLUSION PROCESS MIXES (ALMOST) FASTER THAN INDEPENDENT PARTICLES

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Oliveira conjectured that the order of the mixing time of the exclusion process with  $k$ -particles on an arbitrary  $n$ -vertex graph is at most that of the mixing-time of  $k$  independent particles. We verify this up to a constant factor for  $d$ -regular graphs when each edge rings at rate  $1/d$  in various cases:

- (1) when  $d = \Omega(\log_{n/k} n)$ ,
- (2) when  $\text{gap} :=$  the spectral-gap of a single walk is  $O(1/\log^4 n)$  and  $k \geq n^{\Omega(1)}$ ,
- (3) when  $k \asymp n^a$  for some constant  $0 < a < 1$ .

In these cases, our analysis yields a probabilistic proof of a weaker version of Aldous' famous spectral-gap conjecture (resolved by Caputo et al.). We also prove a general bound of  $O(\log n \log \log n / \text{gap})$ , which is within a  $\log \log n$  factor from Oliveira's conjecture when  $k \geq n^{\Omega(1)}$ . As applications, we get new mixing bounds:

- (a)  $O(\log n \log \log n)$  for expanders,
- (b) order  $d \log(dk)$  for the hypercube  $\{0, 1\}^d$ ,
- (c) order  $(\text{Diameter})^2 \log k$  for vertex-transitive graphs of moderate growth and for supercritical percolation on a fixed dimensional torus.

## REFERENCES

- [1] ALDOUS, D. and FILL, J. (2002). Reversible Markov chains and random walks on graphs.
- [2] ALON, G. and KOZMA, G. (2018). Comparing with octopi. Preprint. Available at [arXiv:1811.10537](https://arxiv.org/abs/1811.10537).
- [3] BENJAMINI, I. and HERMON, J. (2019). Rapid social connectivity. *Electron. J. Probab.* **24** Paper No. 32, 33. [MR3940762 https://doi.org/10.1214/19-EJP294](https://doi.org/10.1214/19-EJP294)
- [4] BOCZKOWSKI, L., PERES, Y. and SOUSI, P. (2018). Sensitivity of mixing times in Eulerian digraphs. *SIAM J. Discrete Math.* **32** 624–655. [MR3775128 https://doi.org/10.1137/16M1073376](https://doi.org/10.1137/16M1073376)
- [5] BORCEA, J., BRÄNDÉN, P. and LIGGETT, T. M. (2009). Negative dependence and the geometry of polynomials. *J. Amer. Math. Soc.* **22** 521–567. [MR2476782 https://doi.org/10.1090/S0894-0347-08-00618-8](https://doi.org/10.1090/S0894-0347-08-00618-8)
- [6] BREUILLARD, E. and TOINTON, M. C. H. (2016). Nilprogressions and groups with moderate growth. *Adv. Math.* **289** 1008–1055. [MR3439705 https://doi.org/10.1016/j.aim.2015.11.025](https://doi.org/10.1016/j.aim.2015.11.025)
- [7] CANCRINI, N. and MARTINELLI, F. (2000). On the spectral gap of Kawasaki dynamics under a mixing condition revisited. *J. Math. Phys.* **41** 1391–1423. [MR1757965 https://doi.org/10.1063/1.533192](https://doi.org/10.1063/1.533192)
- [8] CAPUTO, P., LIGGETT, T. M. and RICHTHAMMER, T. (2010). Proof of Aldous' spectral gap conjecture. *J. Amer. Math. Soc.* **23** 831–851. [MR2629990 https://doi.org/10.1090/S0894-0347-10-00659-4](https://doi.org/10.1090/S0894-0347-10-00659-4)
- [9] CONNOR, S. B. and PYMAR, R. J. (2019). Mixing times for exclusion processes on hypergraphs. *Electron. J. Probab.* **24** Paper No. 73, 48. [MR3978223 https://doi.org/10.1214/19-EJP332](https://doi.org/10.1214/19-EJP332)
- [10] DIACONIS, P. and SALOFF-COSTE, L. (1993). Comparison theorems for reversible Markov chains. *Ann. Appl. Probab.* **3** 696–730. [MR1233621](https://doi.org/10.1214/19-EJP332)
- [11] DIACONIS, P. and SALOFF-COSTE, L. (1994). Moderate growth and random walk on finite groups. *Geom. Funct. Anal.* **4** 1–36. [MR1254308 https://doi.org/10.1007/BF01898359](https://doi.org/10.1007/BF01898359)
- [12] DIACONIS, P. and SALOFF-COSTE, L. (1996). Logarithmic Sobolev inequalities for finite Markov chains. *Ann. Appl. Probab.* **6** 695–750. [MR1410112 https://doi.org/10.1214/aoap/1034968224](https://doi.org/10.1214/aoap/1034968224)
- [13] DIACONIS, P. and SALOFF-COSTE, L. (1996). Nash inequalities for finite Markov chains. *J. Theoret. Probab.* **9** 459–510. [MR1385408 https://doi.org/10.1007/BF02214660](https://doi.org/10.1007/BF02214660)

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- [14] GOEL, S., MONTENEGRO, R. and TETALI, P. (2006). Mixing time bounds via the spectral profile. *Electron. J. Probab.* **11** 1–26. MR2199053 <https://doi.org/10.1214/EJP.v11-300>
- [15] GREENBERG, S., PASCOE, A. and RANDALL, D. (2009). Sampling biased lattice configurations using exponential metrics. In *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms* 76–85. SIAM, Philadelphia, PA. MR2809307
- [16] HERMON, J. and PERES, Y. (2018). A characterization of  $L_2$  mixing and hypercontractivity via hitting times and maximal inequalities. *Probab. Theory Related Fields* **170** 769–800. MR3773799 <https://doi.org/10.1007/s00440-017-0769-x>
- [17] HERMON, J. and PYMAR, R. (2020). Supplement to “The exclusion process mixes (almost) faster than independent particles.” <https://doi.org/10.1214/20-AOP1455SUPP>
- [18] HERMON, J. and SALEZ, J. (2019). The interchange process on high-dimensional products. Preprint. Available at [arXiv:1905.02146](https://arxiv.org/abs/1905.02146).
- [19] HERMON, J. and SALEZ, J. (2019). A version of Aldous’ spectral-gap conjecture for the zero range process. *Ann. Appl. Probab.* **29** 2217–2229. MR3984254 <https://doi.org/10.1214/18-AAP1449>
- [20] JONASSON, J. (2012). Mixing times for the interchange process. *ALEA Lat. Am. J. Probab. Math. Stat.* **9** 667–683. MR3069380
- [21] KIPNIS, C., OLLA, S. and VARADHAN, S. R. S. (1989). Hydrodynamics and large deviation for simple exclusion processes. *Comm. Pure Appl. Math.* **42** 115–137. MR0978701 <https://doi.org/10.1002/cpa.3160420202>
- [22] KOZMA, G. (2007). On the precision of the spectral profile. *ALEA Lat. Am. J. Probab. Math. Stat.* **3** 321–329. MR2372888
- [23] LABBÉ, C. and LACONIN, H. (2018). Mixing time and cutoff for the weakly asymmetric simple exclusion process. Preprint. Available at [arXiv:1805.12213](https://arxiv.org/abs/1805.12213).
- [24] LABBÉ, C. and LACONIN, H. (2019). Cutoff phenomenon for the asymmetric simple exclusion process and the biased card shuffling. *Ann. Probab.* **47** 1541–1586. MR3945753 <https://doi.org/10.1214/18-AOP1290>
- [25] LACONIN, H. (2015). A product chain without cutoff. *Electron. Commun. Probab.* **20** no. 19, 9. MR3320407 <https://doi.org/10.1214/ECP.v20-3765>
- [26] LACONIN, H. (2016). The cutoff profile for the simple exclusion process on the circle. *Ann. Probab.* **44** 3399–3430. MR3551201 <https://doi.org/10.1214/15-AOP1053>
- [27] LACONIN, H. (2016). Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion. *Ann. Probab.* **44** 1426–1487. MR3474475 <https://doi.org/10.1214/15-AOP1004>
- [28] LACONIN, H. and LEBLOND, R. (2011). Cutoff phenomenon for the simple exclusion process on the complete graph. *ALEA Lat. Am. J. Probab. Math. Stat.* **8** 285–301. MR2869447
- [29] LEE, T.-Y. and YAU, H.-T. (1998). Logarithmic Sobolev inequality for some models of random walks. *Ann. Probab.* **26** 1855–1873. MR1675008 <https://doi.org/10.1214/aop/1022855885>
- [30] LEVIN, D. A. and PERES, Y. (2016). Mixing of the exclusion process with small bias. *J. Stat. Phys.* **165** 1036–1050. MR3575636 <https://doi.org/10.1007/s10955-016-1664-z>
- [31] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*. 2nd ed. Amer. Math. Soc., Providence, RI. MR3726904
- [32] LIGGETT, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **324**. Springer, Berlin. MR1717346 <https://doi.org/10.1007/978-3-662-03990-8>
- [33] LIGGETT, T. M. (2005). *Interacting Particle Systems. Classics in Mathematics*. Springer, Berlin. Reprint of the 1985 original. MR2108619 <https://doi.org/10.1007/b138374>
- [34] LYONS, R., MORRIS, B. J. and SCHRAMM, O. (2008). Ends in uniform spanning forests. *Electron. J. Probab.* **13** 1702–1725. MR2448128 <https://doi.org/10.1214/EJP.v13-566>
- [35] LYONS, R. and OVEIS GHARAN, S. (2018). Sharp bounds on random walk eigenvalues via spectral embedding. *Int. Math. Res. Not. IMRN* **24** 7555–7605. MR3892273 <https://doi.org/10.1093/imrn/rnx082>
- [36] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks. Cambridge Series in Statistical and Probabilistic Mathematics* **42**. Cambridge Univ. Press, New York. MR3616205 <https://doi.org/10.1017/9781316672815>
- [37] MORRIS, B. (2006). The mixing time for simple exclusion. *Ann. Appl. Probab.* **16** 615–635. MR2244427 <https://doi.org/10.1214/105051605000000728>
- [38] MORRIS, B. and PERES, Y. (2005). Evolving sets, mixing and heat kernel bounds. *Probab. Theory Related Fields* **133** 245–266. MR2198701 <https://doi.org/10.1007/s00440-005-0434-7>
- [39] OLIVEIRA, R. I. (2013). Mixing of the symmetric exclusion processes in terms of the corresponding single-particle random walk. *Ann. Probab.* **41** 871–913. MR3077529 <https://doi.org/10.1214/11-AOP714>
- [40] PETE, G. (2008). A note on percolation on  $\mathbb{Z}^d$ : Isoperimetric profile via exponential cluster repulsion. *Electron. Commun. Probab.* **13** 377–392. MR2415145 <https://doi.org/10.1214/ECP.v13-1390>

- [41] QUASTEL, J. (1992). Diffusion of color in the simple exclusion process. *Comm. Pure Appl. Math.* **45** 623–679. [MR1162368 https://doi.org/10.1002/cpa.3160450602](https://doi.org/10.1002/cpa.3160450602)
- [42] WILSON, D. B. (2004). Mixing times of Lozenge tiling and card shuffling Markov chains. *Ann. Appl. Probab.* **14** 274–325. [MR2023023 https://doi.org/10.1214/aoap/1075828054](https://doi.org/10.1214/aoap/1075828054)
- [43] YAU, H.-T. (1997). Logarithmic Sobolev inequality for generalized simple exclusion processes. *Probab. Theory Related Fields* **109** 507–538. [MR1483598 https://doi.org/10.1007/s004400050140](https://doi.org/10.1007/s004400050140)



# SYMMETRIC EXCLUSION AS A RANDOM ENVIRONMENT: INVARIANCE PRINCIPLE

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We establish an invariance principle for a one-dimensional random walk in a dynamic random environment given by a speed-change exclusion process. The jump probabilities of the walk depend on the configuration of the exclusion in a finite box around the walker. The environment starts from equilibrium. After a suitable space-time rescaling, the random walk converges to a sum of two independent processes: a Brownian motion and a Gaussian process with stationary increments.

## REFERENCES

- [1] AVENA, L. (2010). Random walks in dynamic random environments. Ph.D. thesis, Mathematical Institute, Faculty of Science, Leiden Univ.
- [2] AVENA, L. (2012). Symmetric exclusion as a model of non-elliptic dynamical random conductances. *Electron. Commun. Probab.* **17** no. 44, 8. MR2988390 <https://doi.org/10.1214/ECP.v17-2081>
- [3] AVENA, L., CHINO, Y., DA COSTA, C. and DEN HOLLANDER, F. (2019). Random walk in cooling random environment: Ergodic limits and concentration inequalities. *Electron. J. Probab.* **24** Paper No. 38, 35. MR3940768 <https://doi.org/10.1214/19-EJP296>
- [4] AVENA, L. and DEN HOLLANDER, F. (2019). Random walks in cooling random environments. In *Sojourns in Probability Theory and Statistical Physics—III* 23–42. Springer, Singapore.
- [5] AVENA, L., DEN HOLLANDER, F. and REDIG, F. (2010). Large deviation principle for one-dimensional random walk in dynamic random environment: Attractive spin-flips and simple symmetric exclusion. *Markov Process. Related Fields* **16** 139–168. MR2664339
- [6] AVENA, L., FRANCO, T., JARA, M. and VÖLLERING, F. (2015). Symmetric exclusion as a random environment: Hydrodynamic limits. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 901–916. MR3365966 <https://doi.org/10.1214/14-AIHP607>
- [7] AVENA, L., JARA, M. and VÖLLERING, F. (2018). Explicit LDP for a slowed RW driven by a symmetric exclusion process. *Probab. Theory Related Fields* **171** 865–915. MR3827224 <https://doi.org/10.1007/s00440-017-0797-6>
- [8] AVENA, L. and THOMANN, P. (2012). Continuity and anomalous fluctuations in random walks in dynamic random environments: Numerics, phase diagrams and conjectures. *J. Stat. Phys.* **147** 1041–1067. MR2949519 <https://doi.org/10.1007/s10955-012-0502-1>
- [9] BERTINI, L., DE SOLE, A., GABRIELLI, D., JONA-LASINIO, G. and LANDIM, C. (2003). Large deviations for the boundary driven symmetric simple exclusion process. *Math. Phys. Anal. Geom.* **6** 231–267. MR1997915 <https://doi.org/10.1023/A:1024967818899>
- [10] BETHUELSEN, S. A. (2018). The contact process as seen from a random walk. *ALEA Lat. Am. J. Probab. Math. Stat.* **15** 571–585. MR3800486 <https://doi.org/10.30757/alea.v15-23>
- [11] BILLINGSLEY, P. (2013). *Convergence of Probability Measures*. Wiley, New York. MR0233396
- [12] BLONDEL, O., HILÁRIO, M. R. and TEIXEIRA, A. (2020). Random walks on dynamical random environments with nonuniform mixing. *Ann. Probab.* **48** 2014–2051. MR4124532 <https://doi.org/10.1214/19-AOP1414>
- [13] BOUCHERON, S., LUGOSI, G. and MASSART, P. (2013). *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford Univ. Press, Oxford. MR3185193 <https://doi.org/10.1093/acprof:oso/9780199535255.001.0001>
- [14] CHANG, C. C. (1994). Equilibrium fluctuations of gradient reversible particle systems. *Probab. Theory Related Fields* **100** 269–283. MR1305583 <https://doi.org/10.1007/BF01193701>

- [15] DE MASI, A., PRESUTTI, E., SPOHN, H. and WICK, W. D. (1986). Asymptotic equivalence of fluctuation fields for reversible exclusion processes with speed change. *Ann. Probab.* **14** 409–423. [MR0832017](#)
- [16] ETHIER, S. N. and KURTZ, T. G. (2009). *Markov Processes: Characterization and Convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. Wiley, New York. [MR0838085](#) <https://doi.org/10.1002/9780470316658>
- [17] FARFAN, J., LANDIM, C. and MOURRAGUI, M. (2011). Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion processes. *Stochastic Process. Appl.* **121** 725–758. [MR2770905](#) <https://doi.org/10.1016/j.spa.2010.11.014>
- [18] FUNAKI, T., HANDA, K. and UCHIYAMA, K. (1991). Hydrodynamic limit of one-dimensional exclusion processes with speed change. *Ann. Probab.* **19** 245–265. [MR1085335](#)
- [19] GONÇALVES, P. and JARA, M. (2013). Scaling limits of additive functionals of interacting particle systems. *Comm. Pure Appl. Math.* **66** 649–677. [MR3028483](#) <https://doi.org/10.1002/cpa.21441>
- [20] HILÁRIO, M., KIOUS, D. and TEIXEIRA, A. (2019). Random walk on the simple symmetric exclusion process. Preprint. Available at [arXiv:1906.03167](https://arxiv.org/abs/1906.03167).
- [21] HILÁRIO, M. R., DEN HOLLANDER, F., DOS SANTOS, R. S., SIDORAVICIUS, V. and TEIXEIRA, A. (2015). Random walk on random walks. *Electron. J. Probab.* **20** no. 95, 35. [MR3399831](#) <https://doi.org/10.1214/EJP.v20-4437>
- [22] HOLLEY, R. A. and STROOCK, D. W. (1978). Generalized Ornstein–Uhlenbeck processes and infinite particle branching Brownian motions. *Publ. Res. Inst. Math. Sci.* **14** 741–788. [MR0527199](#) <https://doi.org/10.2977/prims/1195188837>
- [23] HUVENEERS, F. and SIMENHAUS, F. (2015). Random walk driven by the simple exclusion process. *Electron. J. Probab.* **20** no. 105, 42. [MR3407222](#) <https://doi.org/10.1214/EJP.v20-3906>
- [24] JACOD, J. and SHIRYAEV, A. N. (2013). *Limit Theorems for Stochastic Processes*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **288**. Springer, Berlin. [MR0959133](#) <https://doi.org/10.1007/978-3-662-02514-7>
- [25] JARA, M. D., LANDIM, C. and SETHURAMAN, S. (2009). Nonequilibrium fluctuations for a tagged particle in mean-zero one-dimensional zero-range processes. *Probab. Theory Related Fields* **145** 565–590. [MR2529439](#) <https://doi.org/10.1007/s00440-008-0178-2>
- [26] KARATZAS, I. and SHREVE, S. E. (2012). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. [MR1121940](#) <https://doi.org/10.1007/978-1-4612-0949-2>
- [27] KIPNIS, C. and LANDIM, C. (2013). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **320**. Springer, Berlin. [MR1707314](#) <https://doi.org/10.1007/978-3-662-03752-2>
- [28] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** 1–19. [MR0834478](#)
- [29] KOMOROWSKI, T., LANDIM, C. and OLLA, S. (2012). *Fluctuations in Markov Processes: Time Symmetry and Martingale Approximation*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **345**. Springer, Heidelberg. [MR2952852](#) <https://doi.org/10.1007/978-3-642-29880-6>
- [30] LANDIM, C. and VARES, M. E. (1994). Equilibrium fluctuations for exclusion processes with speed change. *Stochastic Process. Appl.* **52** 107–118. [MR1289171](#) [https://doi.org/10.1016/0304-4149\(94\)90103-1](https://doi.org/10.1016/0304-4149(94)90103-1)
- [31] PORT, S. C. (1966). A system of denumerably many transient Markov chains. *Ann. Math. Stat.* **37** 406–411. [MR0195152](#) <https://doi.org/10.1214/aoms/1177699522>
- [32] SELAMI, S. (1999). Equilibrium density fluctuations of a one-dimensional non-gradient reversible model: The generalized exclusion process. *Markov Process. Related Fields* **5** 21–51. [MR1673219](#)
- [33] SETHURAMAN, S. (2000). Central limit theorems for additive functionals of the simple exclusion process. *Ann. Probab.* **28** 277–302. [MR1756006](#) <https://doi.org/10.1214/aop/1019160120>
- [34] WHITT, W. (2007). Proofs of the martingale FCLT. *Probab. Surv.* **4** 268–302. [MR2368952](#) <https://doi.org/10.1214/07-PS122>
- [35] YAU, H.-T. (1991). Relative entropy and hydrodynamics of Ginzburg–Landau models. *Lett. Math. Phys.* **22** 63–80. [MR1121850](#) <https://doi.org/10.1007/BF00400379>
- [36] ZEITOUNI, O. (2009). Random walks in random environment. In *Encyclopedia of Complexity and Systems Science* 7520–7533.

THE ANNALS  
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VOLUME 48

2020

## Articles

AÏDÉKON, ELIE, HU, YUEYUN AND SHI, ZHAN. Points of infinite multiplicity of planar Brownian motion: Measures and local times . . . . .	1785–1825
AIMINO, ROMAIN AND LIVERANI, CARLANGÉLO. Deterministic walks in random environment . . . . .	2212–2257
ALBEVERIO, SERGIO, DE VECCHI, FRANCESCO C. AND GUBINELLI, MASSIMILIANO. Elliptic stochastic quantization . . . . .	1693–1741
ALT, JOHANNES, ERDŐS, LÁSZLÓ, KRÜGER, TORBEN AND SCHRÖDER, DOMINIK. Correlated random matrices: Band rigidity and edge universality . . . . .	963–1001
ANGST, JÜRGEN AND POLY, GUILLAUME. On the absolute continuity of random nodal volumes . . . . .	2145–2175
ASSELAH, AMINE AND SCHAPIRA, BRUNO. On the nature of the Swiss cheese in dimension 3 . . . . .	1002–1013
ATHREYA, SIVA, BUTKOVSKY, OLEG AND MYTNIK, LEONID. Strong existence and uniqueness for stable stochastic differential equations with distributional drift . . . . .	178–210
AUGERI, FANNY. Nonlinear large deviation bounds with applications to Wigner matrices and sparse Erdős–Rényi graphs . . . . .	2404–2448
BACKHOFF-VERAGUAS, JULIO, BEIGLBÖCK, MATHIAS, HUESMANN, MARTIN AND KÄLLBLAD, SIGRID. Martingale Benamou–Brenier: A probabilistic perspective . . . . .	2258–2289
BAHADORAN, C., MOUNTFORD, T., RAVISHANKAR, K. AND SAADA, E. Hydrodynamics in a condensation regime: The disordered asymmetric zero-range process . . . . .	404–444
BANDERIER, CYRIL, MARCHAL, PHILIPPE AND WALLNER, MICHAEL. Periodic Pólya urns, the density method and asymptotics of Young tableaux . . . . .	1921–1965
BARBU, VIOREL AND RÖCKNER, MICHAEL. From nonlinear Fokker–Planck equations to solutions of distribution dependent SDE . . . . .	1902–1920
BATES, ERIK AND CHATTERJEE, SOURAV. Localization in Gaussian disordered systems at low temperature . . . . .	2755–2806
BATES, ERIK AND CHATTERJEE, SOURAV. The endpoint distribution of directed polymers . . . . .	817–871
BEIGLBÖCK, MATHIAS, HUESMANN, MARTIN, KÄLLBLAD, SIGRID AND BACKHOFF-VERAGUAS, JULIO. Martingale Benamou–Brenier: A probabilistic perspective . . . . .	2258–2289
BELIAEV, DMITRY, MUIRHEAD, STEPHEN AND RIVERA, ALEJANDRO. A covariance formula for topological events of smooth Gaussian fields . . . . .	2845–2893
BELIUS, DAVID AND WU, WEI. Maximum of the Ginzburg–Landau fields . . . . .	2647–2679
BELLA, PETER AND SCHÄFFNER, MATHIAS. Quenched invariance principle for random walks among random degenerate conductances . . . . .	296–316
BEN AROUS, GÉRARD, GHEISSARI, REZA AND JAGANNATH, AUKOSH. Algorithmic thresholds for tensor PCA . . . . .	2052–2087
BENDER, CHRISTIAN. Itô’s formula for Gaussian processes with stochastic discontinuities . . . . .	458–492
BENOIST, STÉPHANE, DUMAZ, LAURE AND WERNER, WENDELIN. Near-critical spanning forests and renormalization . . . . .	1980–2013

BERESTYCKI, NATHANAËL, LASLIER, BENOÎT AND RAY, GOURAB. Dimers and imaginary geometry .....	1–52
BLANQUICETT, DANIEL. Anisotropic bootstrap percolation in three dimensions	2591–2614
BLONDEL, ORIANE, HILÁRIO, MARCELO R. AND TEIXEIRA, AUGUSTO. Random walks on dynamical random environments with nonuniform mixing .....	2014–2051
BOBKOV, S. G., CHISTYAKOV, G. P. AND GÖTZE, F. Normal approximation for weighted sums under a second-order correlation condition .....	1202–1219
BUTKOVSKY, OLEG, MYTNIK, LEONID AND ATHREYA, SIVA. Strong existence and uniqueness for stable stochastic differential equations with distributional drift .....	178–210
CAMPESE, SIMON, NOURDIN, IVAN AND NUALART, DAVID. Continuous Breuer–Major theorem: Tightness and nonstationarity .....	147–177
CAPUTO, PIETRO, LABBÉ, CYRIL AND LACONIN, HUBERT. Mixing time of the adjacent walk on the simplex .....	2449–2493
CARAVENNA, FRANCESCO, SUN, RONGFENG AND ZYGOURAS, NIKOS. The two-dimensional KPZ equation in the entire subcritical regime .....	1086–1127
CARLEN, ERIC, CARVALHO, MARIA AND LOSS, MICHAEL. Spectral gaps for reversible Markov processes with chaotic invariant measures: The Kac process with hard sphere collisions in three dimensions .....	2807–2844
CARVALHO, MARIA, LOSS, MICHAEL AND CARLEN, ERIC. Spectral gaps for reversible Markov processes with chaotic invariant measures: The Kac process with hard sphere collisions in three dimensions .....	2807–2844
ČERNÝ, JIŘÍ AND DREWITZ, ALEXANDER. Quenched invariance principles for the maximal particle in branching random walk in random environment and the parabolic Anderson model .....	94–146
CHATTERJEE, SOURAV AND BATES, ERIK. Localization in Gaussian disordered systems at low temperature .....	2755–2806
CHATTERJEE, SOURAV AND BATES, ERIK. The endpoint distribution of directed polymers .....	817–871
CHATTERJEE, SOURAV AND DUNLAP, ALEXANDER. Constructing a solution of the $(2 + 1)$ -dimensional KPZ equation .....	1014–1055
CHATTERJEE, SOURAV AND HAREL, MATAN. Localization in random geometric graphs with too many edges .....	574–621
CHIARINI, ALBERTO AND NITZSCHNER, MAXIMILIAN. Entropic repulsion for the occupation-time field of random interlacements conditioned on disconnection .....	1317–1351
CHISTYAKOV, G. P., GÖTZE, F. AND BOBKOV, S. G. Normal approximation for weighted sums under a second-order correlation condition .....	1202–1219
CHONG, CARSTEN AND KEVELI, PÉTER. The almost-sure asymptotic behavior of the solution to the stochastic heat equation with Lévy noise .....	1466–1494
DE LA RUE, THIERRY, JANVRESSE, ÉLISE AND ROY, EMMANUEL. Ergodic Poisson splittings .....	1266–1285
DE VECCHI, FRANCESCO C., GUBINELLI, MASSIMILIANO AND ALBEVERIO, SERGIO. Elliptic stochastic quantization .....	1693–1741
DELARUE, FRANÇOIS, LACKER, DANIEL AND RAMANAN, KAVITA. From the master equation to mean field game limit theory: Large deviations and concentration of measure .....	211–263

DEMBIN, BARBARA. The maximal flow from a compact convex subset to infinity in first passage percolation on $\mathbb{Z}^d$ .....	622–645
DETTE, HOLGER, TOMECKI, DOMINIK AND VENKER, MARTIN. Random moment problems under constraints .....	672–713
DEVRAJ, ADITHYA, KONTOYIANNIS, IOANNIS AND MEYN, SEAN. Geometric ergodicity in a weighted Sobolev space .....	380–403
DING, JIAN AND WIRTH, MATEO. Percolation for level-sets of Gaussian free fields on metric graphs .....	1411–1435
DÖRING, LEIF AND KYPRIANOU, ANDREAS E. Entrance and exit at infinity for stable jump diffusions .....	1220–1265
DREWITZ, ALEXANDER AND ČERNÝ, JIŘÍ. Quenched invariance principles for the maximal particle in branching random walk in random environment and the parabolic Anderson model .....	94–146
DUMAZ, LAURE, WERNER, WENDELIN AND BENOIST, STÉPHANE. Near-critical spanning forests and renormalization .....	1980–2013
DUMINIL-COPIN, HUGO, GANGULY, SHIRSHENDU, HAMMOND, ALAN AND MANOLESCU, IOAN. Bounding the number of self-avoiding walks: Hammersley–Welsh with polygon insertion .....	1644–1692
DUMINIL-COPIN, HUGO, KESTEN, HARRY, NAZAROV, FEDOR, PERES, YUVAL AND SIDORAVICIUS, VLADAS. On the number of maximal paths in directed last-passage percolation .....	2176–2188
DUNLAP, ALEXANDER AND CHATTERJEE, SOURAV. Constructing a solution of the $(2 + 1)$ -dimensional KPZ equation .....	1014–1055
ELDAN, RONEN, MIKULINCER, DAN AND ZHAI, ALEX. The CLT in high dimensions: Quantitative bounds via martingale embedding .....	2494–2524
ERDŐS, LÁSZLÓ, KRÜGER, TORBEN, SCHRÖDER, DOMINIK AND ALT, JOHANNES. Correlated random matrices: Band rigidity and edge universality	963–1001
FLANDOLI, FRANCO AND LUO, DEJUN. Convergence of transport noise to Ornstein–Uhlenbeck for 2D Euler equations under the enstrophy measure	264–295
FOURNIER, NICOLAS AND TARDIF, CAMILLE. Anomalous diffusion for multi-dimensional critical kinetic Fokker–Planck equations .....	2359–2403
GANGULY, SHIRSHENDU, HAMMOND, ALAN, MANOLESCU, IOAN AND DUMINIL-COPIN, HUGO. Bounding the number of self-avoiding walks: Hammersley–Welsh with polygon insertion .....	1644–1692
GERSTENBERG, JULIAN. Exchangeable interval hypergraphs and limits of ordered discrete structures .....	1128–1167
GHEISSARI, REZA, JAGANNATH, AUKOSH AND BEN AROUS, GÉRARD. Algorithmic thresholds for tensor PCA .....	2052–2087
GÖTZE, F., BOBKOV, S. G. AND CHISTYAKOV, G. P. Normal approximation for weighted sums under a second-order correlation condition .....	1202–1219
GUBINELLI, MASSIMILIANO, ALBEVERIO, SERGIO AND DE VECCHI, FRANCESCO C. Elliptic stochastic quantization .....	1693–1741
GUIONNET, ALICE AND HUSSON, JONATHAN. Large deviations for the largest eigenvalue of Rademacher matrices .....	1436–1465
GWYNNE, EWAIN, HOLDEN, NINA AND MILLER, JASON. An almost sure KPZ relation for SLE and Brownian motion .....	527–573
GWYNNE, EWAIN AND MILLER, JASON. Confluence of geodesics in Liouville quantum gravity for $\gamma \in (0, 2)$ .....	1861–1901

GWYNNE, EWAIN AND PFEFFER, JOSHUA. Connectivity properties of the adjacency graph of $SLE_\kappa$ bubbles for $\kappa \in (4, 8)$ . . . . .	1495–1519
HAIRER, MARTIN AND LI, XUE-MEI. Averaging dynamics driven by fractional Brownian motion . . . . .	1826–1860
HAMMOND, ALAN, MANOLESCU, IOAN, DUMINIL-COPIN, HUGO AND GANGULY, SHIRSHENDU. Bounding the number of self-avoiding walks: Hammersley–Welsh with polygon insertion . . . . .	1644–1692
HAREL, MATAN AND CHATTERJEE, SOURAV. Localization in random geometric graphs with too many edges . . . . .	574–621
HERMON, JONATHAN AND PYMAR, RICHARD. The exclusion process mixes (almost) faster than independent particles . . . . .	3077–3123
HERMON, JONATHAN AND SALEZ, JUSTIN. Cutoff for the mean-field zero-range process with bounded monotone rates . . . . .	742–759
HERMON, JONATHAN AND SOUSI, PERLA. A comparison principle for random walk on dynamical percolation . . . . .	2952–2987
HILÁRIO, MARCELO R., TEIXEIRA, AUGUSTO AND BLONDEL, ORIANE. Random walks on dynamical random environments with nonuniform mixing . . . . .	2014–2051
HOLDEN, NINA, MILLER, JASON AND GWYNNE, EWAIN. An almost sure KPZ relation for SLE and Brownian motion . . . . .	527–573
HOLROYD, ALEXANDER E., HUTCHCROFT, TOM AND LEVY, AVI. Mallows permutations and finite dependence . . . . .	343–379
HONG, JIELIANG, MYTNIK, LEONID AND PERKINS, EDWIN. On the topological boundary of the range of super-Brownian motion . . . . .	1168–1201
HU, YUEYUN, SHI, ZHAN AND AÏDÉKON, ELIE. Points of infinite multiplicity of planar Brownian motion: Measures and local times . . . . .	1785–1825
HUANG, JIAOYANG, LANDON, BENJAMIN AND YAU, HORNG-TZER. Transition from Tracy–Widom to Gaussian fluctuations of extremal eigenvalues of sparse Erdős–Rényi graphs . . . . .	916–962
HUESMANN, MARTIN, KÄLLBLAD, SIGRID, BACKHOFF-VERAGUAS, JULIO AND BEIGLBÖCK, MATHIAS. Martingale Benamou–Brenier: A probabilistic perspective . . . . .	2258–2289
HUSSON, JONATHAN AND GUIONNET, ALICE. Large deviations for the largest eigenvalue of Rademacher matrices . . . . .	1436–1465
HUTCHCROFT, TOM. Locality of the critical probability for transitive graphs of exponential growth . . . . .	1352–1371
HUTCHCROFT, TOM, LEVY, AVI AND HOLROYD, ALEXANDER E. Mallows permutations and finite dependence . . . . .	343–379
HUTZENTHALER, MARTIN AND JENTZEN, ARNULF. On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with nonglobally monotone coefficients . . . . .	53–93
JAGANNATH, AUKOSH, BEN AROUS, GÉRARD AND GHEISSARI, REZA. Algorithmic thresholds for tensor PCA . . . . .	2052–2087
JANJIGIAN, CHRISTOPHER AND RASSOUL-AGHA, FIRAS. Busemann functions and Gibbs measures in directed polymer models on $\mathbb{Z}^2$ . . . . .	778–816
JANVRESSE, ÉLISE, ROY, EMMANUEL AND DE LA RUE, THIERRY. Ergodic Poisson splittings . . . . .	1266–1285

JARA, MILTON AND MENEZES, OTÁVIO. Symmetric exclusion as a random environment: Invariance principle .....	3124–3149
JEGO, ANTOINE. Planar Brownian motion and Gaussian multiplicative chaos ..	1597–1643
JENTZEN, ARNULF AND HUTZENTHALER, MARTIN. On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with nonglobally monotone coefficients .....	53–93
KAJINO, NAOTAKA AND MURUGAN, MATHAV. On singularity of energy measures for symmetric diffusions with full off-diagonal heat kernel estimates	2920–2951
KÄLLBLAD, SIGRID, BACKHOFF-VERAGUAS, JULIO, BEIGLBÖCK, MATHIAS AND HUESMANN, MARTIN. Martingale Benamou–Brenier: A probabilistic perspective .....	2258–2289
KESTEN, HARRY, NAZAROV, FEDOR, PERES, YUVAL, SIDORAVICIUS, VLADAS AND DUMINIL-COPIN, HUGO. On the number of maximal paths in directed last-passage percolation .....	2176–2188
KEVEI, PÉTER AND CHONG, CARSTEN. The almost-sure asymptotic behavior of the solution to the stochastic heat equation with Lévy noise .....	1466–1494
KOMOROWSKI, TOMASZ, OLLA, STEFANO AND RYZHIK, LENYA. Fractional diffusion limit for a kinetic equation with an interface .....	2290–2322
KONTOYIANNIS, IOANNIS, MEYN, SEAN AND DEVRAJ, ADITHYA. Geometric ergodicity in a weighted Sobolev space .....	380–403
KOPEL, PHIL, O’ROURKE, SEAN AND VU, VAN. Random matrix products: Universality and least singular values .....	1372–1410
KOSLOFF, ZEMER AND SOO, TERRY. Finitary isomorphisms of Brownian motions .....	1966–1979
KRÜGER, TORBEN, SCHRÖDER, DOMINIK, ALT, JOHANNES AND ERDŐS, LÁSZLÓ. Correlated random matrices: Band rigidity and edge universality .....	963–1001
KULIK, ALEXEI AND SCHEUTZOW, MICHAEL. Well-posedness, stability and sensitivities for stochastic delay equations: A generalized coupling approach .....	3041–3076
KYPRIANOU, ANDREAS E. AND DÖRING, LEIF. Entrance and exit at infinity for stable jump diffusions .....	1220–1265
LABBÉ, CYRIL, LACOIN, HUBERT AND CAPUTO, PIETRO. Mixing time of the adjacent walk on the simplex .....	2449–2493
LACKER, DANIEL, RAMANAN, KAVITA AND DELARUE, FRANÇOIS. From the master equation to mean field game limit theory: Large deviations and concentration of measure .....	211–263
LACKER, DANIEL, SHKOLNIKOV, MYKHAYLO AND ZHANG, JIACHENG. Inverting the Markovian projection, with an application to local stochastic volatility models .....	2189–2211
LACOIN, HUBERT, CAPUTO, PIETRO AND LABBÉ, CYRIL. Mixing time of the adjacent walk on the simplex .....	2449–2493
LANDON, BENJAMIN, LOPATTO, PATRICK AND MARCINEK, JAKE. Comparison theorem for some extremal eigenvalue statistics .....	2894–2919
LANDON, BENJAMIN, YAU, HORNG-TZER AND HUANG, JIAOYANG. Transition from Tracy–Widom to Gaussian fluctuations of extremal eigenvalues of sparse Erdős–Rényi graphs .....	916–962



LASLIER, BENOÎT, RAY, GOURAB AND BERESTYCKI, NATHANAËL. Dimers and imaginary geometry .....	1–52
LE GALL, JEAN-FRANÇOIS AND RIERA, ARMAND. Growth-fragmentation processes in Brownian motion indexed by the Brownian tree .....	1742–1784
LE JAN, YVES AND RAIMOND, OLIVIER. Flows, coalescence and noise. A correction .....	1592–1595
LEE, JONG JUN, MUELLER, CARL AND NEUMAN, EYAL. Hitting probabilities of a Brownian flow with radial drift .....	646–671
LEVY, AVI, HOLROYD, ALEXANDER E. AND HUTCHCROFT, TOM. Mallows permutations and finite dependence .....	343–379
LI, XUE-MEI AND HAIRER, MARTIN. Averaging dynamics driven by fractional Brownian motion .....	1826–1860
LIVERANI, CARLANGELLO AND AIMINO, ROMAIN. Deterministic walks in random environment .....	2212–2257
LÖHR, WOLFGANG, MYTNIK, LEONID AND WINTER, ANITA. The Aldous chain on cladograms in the diffusion limit .....	2565–2590
LOPATTO, PATRICK, MARCINEK, JAKE AND LANDON, BENJAMIN. Comparison theorem for some extremal eigenvalue statistics .....	2894–2919
LOSS, MICHAEL, CARLEN, ERIC AND CARVALHO, MARIA. Spectral gaps for reversible Markov processes with chaotic invariant measures: The Kac process with hard sphere collisions in three dimensions .....	2807–2844
LUO, DEJUN AND FLANDOLI, FRANCO. Convergence of transport noise to Ornstein–Uhlenbeck for 2D Euler equations under the enstrophy measure	264–295
MANOLESCU, IOAN, DUMINIL-COPIN, HUGO, GANGULY, SHIRSHENDU AND HAMMOND, ALAN. Bounding the number of self-avoiding walks: Hammersley–Welsh with polygon insertion .....	1644–1692
MARCHAL, PHILIPPE, WALLNER, MICHAEL AND BANDERIER, CYRIL. Periodic Pólya urns, the density method and asymptotics of Young tableaux ..	1921–1965
MARCINEK, JAKE, LANDON, BENJAMIN AND LOPATTO, PATRICK. Comparison theorem for some extremal eigenvalue statistics .....	2894–2919
MARÊCHÉ, LAURE, MARTINELLI, FABIO AND TONINELLI, CRISTINA. Exact asymptotics for Duarte and supercritical rooted kinetically constrained models .....	317–342
MARTINELLI, FABIO, TONINELLI, CRISTINA AND MARÊCHÉ, LAURE. Exact asymptotics for Duarte and supercritical rooted kinetically constrained models .....	317–342
MENEZES, OTÁVIO AND JARA, MILTON. Symmetric exclusion as a random environment: Invariance principle .....	3124–3149
MÉSZAROS, ANDRÁS. Limiting entropy of determinantal processes .....	2615–2643
MEYN, SEAN, DEVRAJ, ADITHYA AND KONTOYIANNIS, IOANNIS. Geometric ergodicity in a weighted Sobolev space .....	380–403
MIKULINCER, DAN, ZHAI, ALEX AND ELKAN, RONEN. The CLT in high dimensions: Quantitative bounds via martingale embedding .....	2494–2524
MILLER, JASON AND GWYNNE, EWAIN. Confluence of geodesics in Liouville quantum gravity for $\gamma \in (0, 2)$ .....	1861–1901
MILLER, JASON, GWYNNE, EWAIN AND HOLDEN, NINA. An almost sure KPZ relation for SLE and Brownian motion .....	527–573

MOUNTFORD, T., RAVISHANKAR, K., SAADA, E. AND BAHADORAN, C. Hydrodynamics in a condensation regime: The disordered asymmetric zero-range process . . . . .	404–444
MOUSSET, FRANK, NOEVER, ANDREAS, PANAGIOTOU, KONSTANTINOS AND SAMOTIJ, WOJCIECH. On the probability of nonexistence in binomial subsets . . . . .	493–525
MUELLER, CARL, NEUMAN, EYAL AND LEE, JONG JUN. Hitting probabilities of a Brownian flow with radial drift . . . . .	646–671
MUIRHEAD, STEPHEN, RIVERA, ALEJANDRO AND BELIAEV, DMITRY. A covariance formula for topological events of smooth Gaussian fields . . . . .	2845–2893
MUKHERJEE, CHIRANJIB AND VARADHAN, S. R. S. Identification of the Polaron measure in strong coupling and the Pekar variational formula . . . . .	2119–2144
MURUGAN, MATHAV AND KAJINO, NAOTAKA. On singularity of energy measures for symmetric diffusions with full off-diagonal heat kernel estimates . . . . .	2920–2951
MYTNIK, LEONID, ATHREYA, SIVA AND BUTKOVSKY, OLEG. Strong existence and uniqueness for stable stochastic differential equations with distributional drift . . . . .	178–210
MYTNIK, LEONID, PERKINS, EDWIN AND HONG, JIELIANG. On the topological boundary of the range of super-Brownian motion . . . . .	1168–1201
MYTNIK, LEONID, WINTER, ANITA AND LÖHR, WOLFGANG. The Aldous chain on cladograms in the diffusion limit . . . . .	2565–2590
NADTOCHIY, SERGEY AND SHKOLNIKOV, MYKHAYLO. Mean field systems on networks, with singular interaction through hitting times . . . . .	1520–1556
NAM, KYEONGSIK. Large deviations and localization of the microcanonical ensembles given by multiple constraints . . . . .	2525–2564
NAZAROV, FEDOR, PERES, YUVAL, SIDORAVICIUS, VLADAS, DUMINIL-COPIN, HUGO AND KESTEN, HARRY. On the number of maximal paths in directed last-passage percolation . . . . .	2176–2188
NEUMAN, EYAL, LEE, JONG JUN AND MUELLER, CARL. Hitting probabilities of a Brownian flow with radial drift . . . . .	646–671
NICA, MIHAI, QUASTEL, JEREMY AND REMENIK, DANIEL. Solution of the Kolmogorov equation for TASEP . . . . .	2344–2358
NITZSCHNER, MAXIMILIAN AND CHIARINI, ALBERTO. Entropic repulsion for the occupation-time field of random interlacements conditioned on disconnection . . . . .	1317–1351
NOEVER, ANDREAS, PANAGIOTOU, KONSTANTINOS, SAMOTIJ, WOJCIECH AND MOUSSET, FRANK. On the probability of nonexistence in binomial subsets . . . . .	493–525
NOURDIN, IVAN, NUALART, DAVID AND CAMPESE, SIMON. Continuous Breuer–Major theorem: Tightness and nonstationarity . . . . .	147–177
NUALART, DAVID, CAMPESE, SIMON AND NOURDIN, IVAN. Continuous Breuer–Major theorem: Tightness and nonstationarity . . . . .	147–177
OLLA, STEFANO, RYZHIK, LENYA AND KOMOROWSKI, TOMASZ. Fractional diffusion limit for a kinetic equation with an interface . . . . .	2290–2322
O’ROURKE, SEAN, VU, VAN AND KOPEL, PHIL. Random matrix products: Universality and least singular values . . . . .	1372–1410
PANAGIOTOU, KONSTANTINOS, SAMOTIJ, WOJCIECH, MOUSSET, FRANK AND NOEVER, ANDREAS. On the probability of nonexistence in binomial subsets . . . . .	493–525

PERES, YUVAL, SIDORAVICIUS, VLADAS, DUMINIL-COPIN, HUGO, KESTEN, HARRY AND NAZAROV, FEDOR. On the number of maximal paths in directed last-passage percolation .....	2176–2188
PERKINS, EDWIN, HONG, JIELIANG AND MYTNIK, LEONID. On the topological boundary of the range of super-Brownian motion .....	1168–1201
PFEFFER, JOSHUA AND GWYNNE, EWAIN. Connectivity properties of the adjacency graph of $SLE_\kappa$ bubbles for $\kappa \in (4, 8)$ .....	1495–1519
POLY, GUILLAUME AND ANGST, JÜRGEN. On the absolute continuity of random nodal volumes .....	2145–2175
PYMAR, RICHARD AND HERMON, JONATHAN. The exclusion process mixes (almost) faster than independent particles .....	3077–3123
QUASTEL, JEREMY, REMENIK, DANIEL AND NICA, MIHAI. Solution of the Kolmogorov equation for TASEP .....	2344–2358
RAIMOND, OLIVIER AND LE JAN, YVES. Flows, coalescence and noise. A correction .....	1592–1595
RAMANAN, KAVITA, DELARUE, FRANÇOIS AND LACKER, DANIEL. From the master equation to mean field game limit theory: Large deviations and concentration of measure .....	211–263
RAOUFI, ARAN. Translation-invariant Gibbs states of the Ising model: General setting .....	760–777
RASSOUL-AGHA, FIRAS AND JANJIGIAN, CHRISTOPHER. Busemann functions and Gibbs measures in directed polymer models on $\mathbb{Z}^2$ .....	778–816
RAVISHANKAR, K., SAADA, E., BAHADORAN, C. AND MOUNTFORD, T. Hydrodynamics in a condensation regime: The disordered asymmetric zero-range process .....	404–444
RAY, GOURAB, BERESTYCKI, NATHANAËL AND LASLIER, BENOÎT. Dimers and imaginary geometry .....	1–52
REMENIK, DANIEL, NICA, MIHAI AND QUASTEL, JEREMY. Solution of the Kolmogorov equation for TASEP .....	2344–2358
REMY, GUILLAUME AND ZHU, TUNAN. The distribution of Gaussian multiplicative chaos on the unit interval .....	872–915
RIERA, ARMAND AND LE GALL, JEAN-FRANÇOIS. Growth-fragmentation processes in Brownian motion indexed by the Brownian tree .....	1742–1784
RIVERA, ALEJANDRO, BELIAEV, DMITRY AND MUIRHEAD, STEPHEN. A covariance formula for topological events of smooth Gaussian fields .....	2845–2893
RÖCKNER, MICHAEL AND BARBU, VIOREL. From nonlinear Fokker–Planck equations to solutions of distribution dependent SDE .....	1902–1920
ROY, EMMANUEL, DE LA RUE, THIERRY AND JANVRESSE, ÉLISE. Ergodic Poisson splittings .....	1266–1285
RYZHNIK, LENYA, KOMOROWSKI, TOMASZ AND OLLA, STEFANO. Fractional diffusion limit for a kinetic equation with an interface .....	2290–2322
SAADA, E., BAHADORAN, C., MOUNTFORD, T. AND RAVISHANKAR, K. Hydrodynamics in a condensation regime: The disordered asymmetric zero-range process .....	404–444
SABOT, CHRISTOPHE AND ZENG, XIAOLIN. Hitting times of interacting drifted Brownian motions and the vertex reinforced jump process .....	1057–1085
SAKSMAN, EERO AND WEBB, CHRISTIAN. The Riemann zeta function and Gaussian multiplicative chaos: Statistics on the critical line .....	2680–2754

SALEZ, JUSTIN AND HERMON, JONATHAN. Cutoff for the mean-field zero-range process with bounded monotone rates .....	742–759
SAMOTIJ, WOJCIECH, MOUSSET, FRANK, NOEVER, ANDREAS AND PANAGIOTOU, KONSTANTINOS. On the probability of nonexistence in binomial subsets .....	493–525
SCHÄFFNER, MATHIAS AND BELLA, PETER. Quenched invariance principle for random walks among random degenerate conductances .....	296–316
SCHAPIRA, BRUNO. Capacity of the range in dimension 5 .....	2988–3040
SCHAPIRA, BRUNO AND ASSELAH, AMINE. On the nature of the Swiss cheese in dimension 3 .....	1002–1013
SCHEUTZOW, MICHAEL AND KULIK, ALEXEI. Well-posedness, stability and sensitivities for stochastic delay equations: A generalized coupling approach .....	3041–3076
SCHMIDT, MARIUS A. A simple proof of the DPRZ theorem for 2d cover times .....	445–457
SCHRÖDER, DOMINIK, ALT, JOHANNES, ERDŐS, LÁSZLÓ AND KRÜGER, TORBEN. Correlated random matrices: Band rigidity and edge universality .....	963–1001
SCHWEIGER, FLORIAN. The maximum of the four-dimensional membrane model .....	714–741
SHI, ZHAN, AÏDÉKON, ELIE AND HU, YUEYUN. Points of infinite multiplicity of planar Brownian motion: Measures and local times .....	1785–1825
SHKOLNIKOV, MYKHAYLO AND NADTOCHIY, SERGEY. Mean field systems on networks, with singular interaction through hitting times .....	1520–1556
SHKOLNIKOV, MYKHAYLO, ZHANG, JIACHENG AND LACKER, DANIEL. Inverting the Markovian projection, with an application to local stochastic volatility models .....	2189–2211
SIDORAVICIUS, VLADAS, DUMINIL-COPIN, HUGO, KESTEN, HARRY, NAZAROV, FEDOR AND PERES, YUVAL. On the number of maximal paths in directed last-passage percolation .....	2176–2188
SOO, TERRY AND KOSLOFF, ZEMER. Finitary isomorphisms of Brownian motions .....	1966–1979
SOUSI, PERLA AND HERMON, JONATHAN. A comparison principle for random walk on dynamical percolation .....	2952–2987
SPINKA, YINON. Finitary codings for spatial mixing Markov random fields ...	1557–1591
SPINKA, YINON. Finitely dependent processes are finitary .....	2088–2117
SUN, RONGFENG, ZYGOURAS, NIKOS AND CARAVENNA, FRANCESCO. The two-dimensional KPZ equation in the entire subcritical regime .....	1086–1127
TARDIF, CAMILLE AND FOURNIER, NICOLAS. Anomalous diffusion for multi-dimensional critical kinetic Fokker–Planck equations .....	2359–2403
TEIXEIRA, AUGUSTO, BLONDEL, ORIANE AND HILÁRIO, MARCELO R. Random walks on dynamical random environments with nonuniform mixing .....	2014–2051
TEYSSIER, LUCAS. Limit profile for random transpositions .....	2323–2343
TOMECKI, DOMINIK, VENKER, MARTIN AND DETTE, HOLGER. Random moment problems under constraints .....	672–713
TONINELLI, CRISTINA, MARÉCHÉ, LAURE AND MARTINELLI, FABIO. Exact asymptotics for Duarte and supercritical rooted kinetically constrained models .....	317–342

VALKÓ, BENEDEK AND VIRÁG, BÁLINT. Operator limit of the circular beta ensemble.....	1286–1316
VARADHAN, S. R. S. AND MUKHERJEE, CHIRANJIB. Identification of the Polaron measure in strong coupling and the Pekar variational formula .....	2119–2144
VENKER, MARTIN, DETTE, HOLGER AND TOMECKI, DOMINIK. Random moment problems under constraints .....	672–713
VIRÁG, BÁLINT AND VALKÓ, BENEDEK. Operator limit of the circular beta ensemble.....	1286–1316
VU, VAN, KOPEL, PHIL AND O’ROURKE, SEAN. Random matrix products: Universality and least singular values .....	1372–1410
WALLNER, MICHAEL, BANDERIER, CYRIL AND MARCHAL, PHILIPPE. Periodic Pólya urns, the density method and asymptotics of Young tableaux ..	1921–1965
WEBB, CHRISTIAN AND SAKSMAN, EERO. The Riemann zeta function and Gaussian multiplicative chaos: Statistics on the critical line.....	2680–2754
WERNER, WENDELIN, BENOIST, STÉPHANE AND DUMAZ, LAURE. Near-critical spanning forests and renormalization.....	1980–2013
WINTER, ANITA, LÖHR, WOLFGANG AND MYTNIK, LEONID. The Aldous chain on cladograms in the diffusion limit .....	2565–2590
WIRTH, MATEO AND DING, JIAN. Percolation for level-sets of Gaussian free fields on metric graphs .....	1411–1435
WU, WEI AND BELIUS, DAVID. Maximum of the Ginzburg–Landau fields....	2647–2679
YAU, HORNG-TZER, HUANG, JIAOYANG AND LANDON, BENJAMIN. Transition from Tracy–Widom to Gaussian fluctuations of extremal eigenvalues of sparse Erdős–Rényi graphs .....	916–962
ZENG, XIAOLIN AND SABOT, CHRISTOPHE. Hitting times of interacting drifted Brownian motions and the vertex reinforced jump process .....	1057–1085
ZHAI, ALEX, ELKAN, RONEN AND MIKULINCER, DAN. The CLT in high dimensions: Quantitative bounds via martingale embedding .....	2494–2524
ZHANG, JIACHENG, LACKER, DANIEL AND SHKOLNIKOV, MYKHAYLO. Inverting the Markovian projection, with an application to local stochastic volatility models .....	2189–2211
ZHU, TUNAN AND REMY, GUILLAUME. The distribution of Gaussian multiplicative chaos on the unit interval .....	872–915
ZYGOURAS, NIKOS, CARAVENNA, FRANCESCO AND SUN, RONGFENG. The two-dimensional KPZ equation in the entire subcritical regime .....	1086–1127

### Errata

CARMONA, RENÉ, DELARUE, FRANÇOIS AND LACKER, DANIEL. Errata: Mean field games with common noise.....	2644–2646
DELARUE, FRANÇOIS, LACKER, DANIEL AND CARMONA, RENÉ. Errata: Mean field games with common noise.....	2644–2646
LACKER, DANIEL, CARMONA, RENÉ AND DELARUE, FRANÇOIS. Errata: Mean field games with common noise.....	2644–2646

# The Annals of Probability

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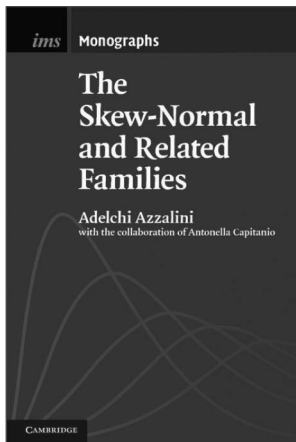
## Articles

- TAP free energy, spin glasses and variational inference  
ZHOU FAN, SONG MEI AND ANDREA MONTANARI
- Local laws and rigidity for Coulomb gases at any temperature  
SCOTT ARMSTRONG AND SYLVIA SERFATY
- 2D anisotropic KPZ at stationarity: Scaling, tightness and nontriviality  
GIUSEPPE CANNIZZARO, DIRK ERHARD AND PHILIPP SCHÖNBAUER
- Freeness over the diagonal for large random matrices . . . . BENSON AU, GUILLAUME CÉBRON,  
ANTOINE DAHLQVIST, FRANCK GABRIEL AND CAMILLE MALE
- The overlap gap property and approximate message passing algorithms for  $p$ -spin  
models . . . . . DAVID GAMARNIK AND AUKOSH JAGANNATH
- Distribution of the random walk conditioned on survival among quenched Bernoulli  
obstacles . . . . . JIAN DING, RYOKI FUKUSHIMA, RONGFENG SUN AND CHANGJI XU
- Survival and extinction of epidemics on random graphs with general degree  
SHANKAR BHAMIDI, DANNY NAM, OANH NGUYEN AND ALLAN SLY
- Gravitational allocation for uniform points on the sphere  
NINA HOLDEN, YUVAL PERES AND ALEX ZHAI
- Scaling limit of dynamical percolation on critical Erdős–Rényi random graphs  
RAPHAËL ROSSIGNOL
- On the uniqueness of global multiple SLEs  
VINCENT BEFFARA, EVELIINA PELTOLA AND HAO WU
- The 2D-directed spanning forest converges to the Brownian web  
DAVID COUPIER, KUMARJIT SAHA, ANISH SARKAR AND VIET CHI TRAN



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## ***The Skew-Normal and Related Families***

Adelchi Azzalini

in collaboration with Antonella Capitanio

Interest in the skew-normal and related families of distributions has grown enormously over recent years, as theory has advanced, challenges of data have grown, and computational tools have made substantial progress. This comprehensive treatment, blending theory and practice, will be the standard resource for statisticians and applied researchers. Assuming only basic knowledge of (non-measure-theoretic) probability and statistical inference, the book is accessible to the wide range of researchers who use statistical modelling techniques. Guiding readers through the main concepts and results, it covers both the probability and the statistics sides of the subject, in the univariate and multivariate settings. The theoretical development is complemented by numerous illustrations and applications to a range of fields including quantitative finance, medical statistics, environmental risk studies, and industrial and business efficiency.

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