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# PHASE TRANSITIONS IN THE ASEP AND STOCHASTIC SIX-VERTEX MODEL

BY AMOL AGGARWAL[1] AND ALEXEI BORODIN[2]

*Harvard University and Massachusetts Institute of Technology*

In this paper, we consider two models in the Kardar–Parisi–Zhang (KPZ) universality class, the asymmetric simple exclusion process (ASEP) and the stochastic six-vertex model. We introduce a new class of initial data (which we call *shape generalized step Bernoulli initial data*) for both of these models that generalizes the step Bernoulli initial data studied in a number of recent works on the ASEP. Under this class of initial data, we analyze the current fluctuations of both the ASEP and stochastic six-vertex model and establish the existence of a phase transition along a characteristic line, across which the fluctuation exponent changes from $1/2$ to $1/3$. On the characteristic line, the current fluctuations converge to the general (rank $k$) Baik–Ben–Arous–Péché distribution for the law of the largest eigenvalue of a critically spiked covariance matrix. For $k = 1$, this was established for the ASEP by Tracy and Widom; for $k > 1$ (and also $k = 1$, for the stochastic six-vertex model), the appearance of these distributions in both models is new.

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# LIOUVILLE FIRST-PASSAGE PERCOLATION: SUBSEQUENTIAL SCALING LIMITS AT HIGH TEMPERATURE

BY JIAN DING[1] AND ALEXANDER DUNLAP[2]

*University of Pennsylvania and Stanford University*

Let $\{Y_{\mathfrak{B}}(x) : x \in \mathfrak{B}\}$ be a discrete Gaussian free field in a two-dimensional box $\mathfrak{B}$ of side length $S$ with Dirichlet boundary conditions. We study Liouville first-passage percolation: the shortest-path metric in which each vertex $x$ is given a weight of $e^{\gamma Y_{\mathfrak{B}}(x)}$ for some $\gamma > 0$. We show that for sufficiently small but fixed $\gamma > 0$, for any sequence of scales $\{S_k\}$ there exists a subsequence along which the appropriately scaled and interpolated Liouville FPP metric converges in the Gromov–Hausdorff sense to a random metric on the unit square in $\mathbf{R}^2$. In addition, all possible (conjecturally unique) scaling limits are homeomorphic by bi-Hölder-continuous homeomorphisms to the unit square with the Euclidean metric.

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# DERIVATIVE AND DIVERGENCE FORMULAE FOR DIFFUSION SEMIGROUPS[1]

BY ANTON THALMAIER AND JAMES THOMPSON

*University of Luxembourg*

For a semigroup $P_t$ generated by an elliptic operator on a smooth manifold $M$, we use straightforward martingale arguments to derive probabilistic formulae for $P_t(V(f))$, not involving derivatives of $f$, where $V$ is a vector field on $M$. For nonsymmetric generators, such formulae correspond to the derivative of the heat kernel in the *forward* variable. As an application, these formulae can be used to derive various *shift-Harnack* inequalities.

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# LOW-DIMENSIONAL LONELY BRANCHING RANDOM WALKS DIE OUT

BY MATTHIAS BIRKNER[1] AND RONGFENG SUN[2]

*Johannes-Gutenberg-Universität Mainz and National University of Singapore*

The lonely branching random walks on $\mathbb{Z}^d$ is an interacting particle system where each particle moves as an independent random walk and undergoes critical binary branching when it is alone. We show that if the symmetrized walk is recurrent, lonely branching random walks die out locally. Furthermore, the same result holds if additional branching is allowed when the walk is not alone.

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# BOUNDARY REGULARITY OF STOCHASTIC PDES

BY MÁTÉ GERENCSÉR

*IST Austria*

The boundary behaviour of solutions of stochastic PDEs with Dirichlet boundary conditions can be surprisingly—and in a sense, arbitrarily—bad: as shown by Krylov [*SIAM J. Math. Anal.* **34** (2003) 1167–1182], for any $\alpha > 0$ one can find a simple 1-dimensional constant coefficient linear equation whose solution at the boundary is not $\alpha$-Hölder continuous.

We obtain a positive counterpart of this: under some mild regularity assumptions on the coefficients, solutions of semilinear SPDEs on $\mathcal{C}^1$ domains are proved to be $\alpha$-Hölder continuous up to the boundary with *some* $\alpha > 0$.

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# LIMIT THEORY FOR GEOMETRIC STATISTICS OF POINT PROCESSES HAVING FAST DECAY OF CORRELATIONS

BY B. BŁASZCZYSZYN, D. YOGESHWARAN[1] AND J. E. YUKICH[2]

*Inria/ENS, Indian Statistical Institute and Lehigh University*

Let $\mathcal{P}$ be a simple, stationary point process on $\mathbb{R}^d$ having fast decay of correlations, that is, its correlation functions factorize up to an additive error decaying faster than any power of the separation distance. Let $\mathcal{P}_n := \mathcal{P} \cap W_n$ be its restriction to windows $W_n := [-\frac{1}{2}n^{1/d}, \frac{1}{2}n^{1/d}]^d \subset \mathbb{R}^d$. We consider the statistic $H_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n)$ where $\xi(x, \mathcal{P}_n)$ denotes a score function representing the interaction of $x$ with respect to $\mathcal{P}_n$. When $\xi$ depends on local data in the sense that its radius of stabilization has an exponential tail, we establish expectation asymptotics, variance asymptotics and central limit theorems for $H_n^\xi$ and, more generally, for statistics of the re-scaled, possibly signed, $\xi$-weighted point measures $\mu_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n)\delta_{n^{-1/d}x}$, as $W_n \uparrow \mathbb{R}^d$. This gives the limit theory for nonlinear geometric statistics (such as clique counts, the number of Morse critical points, intrinsic volumes of the Boolean model and total edge length of the $k$-nearest neighbors graph) of $\alpha$-determinantal point processes (for $-1/\alpha \in \mathbb{N}$) having fast decreasing kernels, including the $\beta$-Ginibre ensembles, extending the Gaussian fluctuation results of Soshnikov [*Ann. Probab.* **30** (2002) 171–187] to nonlinear statistics. It also gives the limit theory for geometric $U$-statistics of $\alpha$-permanental point processes (for $1/\alpha \in \mathbb{N}$) as well as the zero set of Gaussian entire functions, extending the central limit theorems of Nazarov and Sodin [*Comm. Math. Phys.* **310** (2012) 75–98] and Shirai and Takahashi [*J. Funct. Anal.* **205** (2003) 414–463], which are also confined to linear statistics. The proof of the central limit theorem relies on a factorial moment expansion originating in [*Stochastic Process. Appl.* **56** (1995) 321–335; *Statist. Probab. Lett.* **36** (1997) 299–306] to show the fast decay of the correlations of $\xi$-weighted point measures. The latter property is shown to imply a condition equivalent to Brillinger mixing, and consequently yields the asymptotic normality of $\mu_n^\xi$ via an extension of the cumulant method.

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# DIFFERENTIAL SUBORDINATION UNDER CHANGE OF LAW

BY KOMLA DOMELEVO AND STEFANIE PETERMICHL[1,2]

*Université Paul Sabatier*

We prove optimal $L^2$ bounds for a pair of Hilbert space valued differentially subordinate martingales under a change of law. The change of law is given by a process called a weight and sharpness, and in this context refers to the optimal growth with respect to the characteristic of the weight. The pair of martingales is adapted, uniformly integrable and càdlàg. Differential subordination is in the sense of Burkholder, defined through the use of the square bracket. In the scalar dyadic setting with underlying Lebesgue measure, this was proved by Wittwer [*Math. Res. Lett.* **7** (2000) 1–12], where homogeneity was heavily used. Recent progress by Thiele–Treil–Volberg [*Adv. Math.* **285** (2015) 1155–1188] and Lacey [*Israel J. Math.* **217** (2017) 181–195] independently resolved the so-called nonhomogenous case using discrete in time filtrations, where one martingale is a predictable multiplier of the other. The general case for continuous-in-time filtrations and pairs of martingales that are not necessarily predictable multipliers, remained open and is addressed here. As a very useful second main result, we give the explicit expression of a Bellman function of four variables for the weighted estimate of subordinate martingales with jumps. This construction includes an analysis of the regularity of this function as well as a very precise convexity, needed to deal with the jump part.

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# CENTRAL LIMIT THEOREMS FOR EMPIRICAL TRANSPORTATION COST IN GENERAL DIMENSION[1]

BY EUSTASIO DEL BARRIO AND JEAN-MICHEL LOUBES

*Universidad de Valladolid and Université de Toulouse*

We consider the problem of optimal transportation with quadratic cost between a empirical measure and a general target probability on $\mathbb{R}^d$, with $d \geq 1$. We provide new results on the uniqueness and stability of the associated optimal transportation potentials, namely, the minimizers in the dual formulation of the optimal transportation problem. As a consequence, we show that a CLT holds for the empirical transportation cost under mild moment and smoothness requirements. The limiting distributions are Gaussian and admit a simple description in terms of the optimal transportation potentials.

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# DETERMINANTAL SPANNING FORESTS ON PLANAR GRAPHS[1]

BY RICHARD KENYON

*Brown University*

We generalize the uniform spanning tree to construct a family of determinantal measures on essential spanning forests on periodic planar graphs in which every component tree is bi-infinite. Like the uniform spanning tree, these measures arise naturally from the Laplacian on the graph.

More generally, these results hold for the "massive" Laplacian determinant which counts rooted spanning forests with weight $M$ per finite component. These measures typically have a form of conformal invariance, unlike the usual rooted spanning tree measure. We show that the spectral curve for these models is always a simple Harnack curve; this fact controls the decay of edge-edge correlations in these models.

We construct a limit shape theory in these settings, where the limit shapes are defined by measured foliations of fixed isotopy type.

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*Key words and phrases.* Graph Laplacian, spanning forest, determinantal process, limit shape.

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# COMPARISON PRINCIPLE FOR STOCHASTIC HEAT EQUATION ON $\mathbb{R}^d$

BY LE CHEN AND JINGYU HUANG

*University of Nevada, Las Vegas and University of Utah*

We establish the strong comparison principle and strict positivity of solutions to the following nonlinear stochastic heat equation on $\mathbb{R}^d$

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right)u(t,x) = \rho\big(u(t,x)\big)\dot{M}(t,x),$$

for measure-valued initial data, where $\dot{M}$ is a spatially homogeneous Gaussian noise that is white in time and $\rho$ is Lipschitz continuous. These results are obtained under the condition that $\int_{\mathbb{R}^d}(1+|\xi|^2)^{\alpha-1}\hat{f}(\mathrm{d}\xi) < \infty$ for some $\alpha \in (0,1]$, where $\hat{f}$ is the spectral measure of the noise. The weak comparison principle and nonnegativity of solutions to the same equation are obtained under Dalang's condition, that is, $\alpha = 0$. As some intermediate results, we obtain handy upper bounds for $L^p(\Omega)$-moments of $u(t,x)$ for all $p \geq 2$, and also prove that $u$ is a.s. Hölder continuous with order $\alpha - \varepsilon$ in space and $\alpha/2 - \varepsilon$ in time for any small $\varepsilon > 0$.

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# KIRILLOV–FRENKEL CHARACTER FORMULA FOR LOOP GROUPS, RADIAL PART AND BROWNIAN SHEET

BY MANON DEFOSSEUX

*Université Paris 5*

We consider the coadjoint action of a Loop group of a compact group on the dual of the corresponding centrally extended Loop algebra and prove that a Brownian motion in a Cartan subalgebra conditioned to remain in an affine Weyl chamber—which can be seen as a space time conditioned Brownian motion—is distributed as the radial part process of a Brownian sheet on the underlying Lie algebra.

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# HEAT KERNEL UPPER BOUNDS FOR INTERACTING PARTICLE SYSTEMS

BY ARIANNA GIUNTI, YU GU[1] AND JEAN-CHRISTOPHE MOURRAT

*University of Bonn, Carnegie Mellon University and CNRS*

We show a diffusive upper bound on the transition probability of a tagged particle in the symmetric simple exclusion process. The proof relies on optimal spectral gap estimates for the dynamics in finite volume, which are of independent interest. We also show off-diagonal estimates of Carne–Varopoulos type.

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# PARACONTROLLED QUASILINEAR SPDES

BY MARCO FURLAN AND MASSIMILIANO GUBINELLI

*PSL-Université Paris-Dauphine and IAM & HCM, Universität Bonn*

We introduce a nonlinear paracontrolled calculus and use it to renormalise a class of singular SPDEs including certain quasilinear variants of the periodic two-dimensional parabolic Anderson model.

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# ERDŐS–FELLER–KOLMOGOROV–PETROWSKY LAW OF THE ITERATED LOGARITHM FOR SELF-NORMALIZED MARTINGALES: A GAME-THEORETIC APPROACH

BY TAKEYUKI SASAI, KENSHI MIYABE AND AKIMICHI TAKEMURA[1,2]

*University of Tokyo, Meiji University and Shiga University*

We prove an Erdős–Feller–Kolmogorov–Petrowsky law of the iterated logarithm for self-normalized martingales. Our proof is given in the framework of the game-theoretic probability of Shafer and Vovk. Like many other game-theoretic proofs, our proof is self-contained and explicit.

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# CRITICAL RADIUS AND SUPREMUM OF RANDOM SPHERICAL HARMONICS

BY RENJIE FENG[1,2] AND ROBERT J. ADLER[1,3]

*Peking University and Technion*

We first consider *deterministic* immersions of the $d$-dimensional sphere into high dimensional Euclidean spaces, where the immersion is via spherical harmonics of level $n$. The main result of the article is the, a priori unexpected, fact that there is a uniform lower bound to the critical radius of the immersions as $n \to \infty$. This fact has immediate implications for *random* spherical harmonics with fixed $L^2$-norm. In particular, it leads to an exact and explicit formulae for the tail probability of their (large deviation) suprema by the tube formula, and also relates this to the expected Euler characteristic of their upper level sets.

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# COMPONENT SIZES FOR LARGE QUANTUM ERDŐS–RÉNYI GRAPH NEAR CRITICALITY

BY AMIR DEMBO[1], ANNA LEVIT AND SREEKAR VADLAMANI

*Stanford University, University of British Columbia and TIFR-CAM*

The $N$ vertices of a quantum random graph are each a circle independently punctured at Poisson points of arrivals, with parallel connections derived through for each pair of these punctured circles by yet another independent Poisson process. Considering these graphs at their critical parameters, we show that the joint law of the rescaled by $N^{2/3}$ and ordered sizes of their connected components, converges to that of the ordered lengths of excursions above zero for a reflected Brownian motion with drift. Thereby, this work forms the first example of an inhomogeneous random graph, beyond the case of effectively rank-1 models, which is rigorously shown to be in the Erdős–Rényi graphs universality class in terms of Aldous's results.

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