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PFAFFIAN SCHUR PROCESSES AND LAST PASSAGE PERCOLATION IN A HALF-QUADRANT[1]

BY JINHO BAIK*,[2], GUILLAUME BARRAQUAND†,[3], IVAN CORWIN†,[4] AND TOUFIC SUIDAN

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We study last passage percolation in a half-quadrant, which we analyze within the framework of Pfaffian Schur processes. For the model with exponential weights, we prove that the fluctuations of the last passage time to a point on the diagonal are either GSE Tracy–Widom distributed, GOE Tracy–Widom distributed or Gaussian, depending on the size of weights along the diagonal. Away from the diagonal, the fluctuations of passage times follow the GUE Tracy–Widom distribution. We also obtain a two-dimensional crossover between the GUE, GOE and GSE distribution by studying the multipoint distribution of last passage times close to the diagonal when the size of the diagonal weights is simultaneously scaled close to the critical point. We expect that this crossover arises universally in KPZ growth models in half-space. Along the way, we introduce a method to deal with diverging correlation kernels of point processes where points collide in the scaling limit.

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PATHWISE UNIQUENESS OF THE STOCHASTIC HEAT EQUATION WITH SPATIALLY INHOMOGENEOUS WHITE NOISE

BY EYAL NEUMAN

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We study the solutions of the stochastic heat equation driven by spatially inhomogeneous multiplicative white noise based on a fractal measure. We prove pathwise uniqueness for solutions of this equation when the noise coefficient is Hölder continuous of index $\gamma > 1 - \frac{\eta}{2(\eta+1)}$. Here $\eta \in (0, 1)$ is a constant that defines the spatial regularity of the noise.

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MSC2010 subject classifications. Primary 60H10; secondary 60H40, 60J80.

Key words and phrases. Uniqueness, white noise, stochastic partial differential equations, heat equation, catalytic superprocesses.

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A QUANTITATIVE CENTRAL LIMIT THEOREM FOR THE EULER–POINCARÉ CHARACTERISTIC OF RANDOM SPHERICAL EIGENFUNCTIONS[1]

BY VALENTINA CAMMAROTA AND DOMENICO MARINUCCI

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We establish here a quantitative central limit theorem (in Wasserstein distance) for the Euler–Poincaré characteristic of excursion sets of random spherical eigenfunctions in dimension 2. Our proof is based upon a decomposition of the Euler–Poincaré characteristic into different Wiener-chaos components: we prove that its asymptotic behaviour is dominated by a single term, corresponding to the chaotic component of order two. As a consequence, we show how the asymptotic dependence on the threshold level u is fully degenerate, that is, the Euler–Poincaré characteristic converges to a single random variable times a deterministic function of the threshold. This deterministic function has a zero at the origin, where the variance is thus asymptotically of smaller order. We discuss also a possible unifying framework for the Lipschitz–Killing curvatures of the excursion sets for Gaussian spherical harmonics.

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REPRESENTATIONS AND ISOMORPHISM IDENTITIES FOR INFINITELY DIVISIBLE PROCESSES

BY JAN ROSIŃSKI[1]

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We propose isomorphism-type identities for nonlinear functionals of general infinitely divisible processes. Such identities can be viewed as an analogy of the Cameron–Martin formula for Poissonian infinitely divisible processes but with random translations. The applicability of such tools relies on precise understanding of Lévy measures of infinitely divisible processes and their representations, which are studied here in full generality. We illustrate this approach on examples of squared Bessel processes, Feller diffusions, permanental processes, as well as Lévy processes.

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COUPLING IN THE HEISENBERG GROUP AND ITS APPLICATIONS TO GRADIENT ESTIMATES

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We construct a non-Markovian coupling for hypoelliptic diffusions which are Brownian motions in the three-dimensional Heisenberg group. We then derive properties of this coupling such as estimates on the coupling rate, and upper and lower bounds on the total variation distance between the laws of the Brownian motions. Finally, we use these properties to prove gradient estimates for harmonic functions for the hypoelliptic Laplacian which is the generator of Brownian motion in the Heisenberg group.

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FIRST-PASSAGE TIMES FOR RANDOM WALKS WITH NONIDENTICALLY DISTRIBUTED INCREMENTS

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We consider random walks with independent but not necessarily identical distributed increments. Assuming that the increments satisfy the well-known Lindeberg condition, we investigate the asymptotic behaviour of first-passage times over moving boundaries. Furthermore, we prove that a properly rescaled random walk conditioned to stay above the boundary up to time n converges, as $n \to \infty$, towards the Brownian meander.

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CANONICAL SUPERMARTINGALE COUPLINGS

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Two probability distributions μ and ν in second stochastic order can be coupled by a supermartingale, and in fact by many. Is there a canonical choice? We construct and investigate two couplings which arise as optimizers for constrained Monge–Kantorovich optimal transport problems where only supermartingales are allowed as transports. Much like the Hoeffding–Fréchet coupling of classical transport and its symmetric counterpart, the antitone coupling, these can be characterized by order-theoretic minimality properties, as simultaneous optimal transports for certain classes of reward (or cost) functions, and through no-crossing conditions on their supports; however, our two couplings have asymmetric geometries. Remarkably, supermartingale optimal transport decomposes into classical and martingale transport in several ways.

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A WEAK VERSION OF PATH-DEPENDENT FUNCTIONAL ITÔ CALCULUS

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We introduce a variational theory for processes adapted to the multidimensional Brownian motion filtration that provides a differential structure allowing to describe infinitesimal evolution of Wiener functionals at very small scales. The main novel idea is to compute the "sensitivities" of processes, namely derivatives of martingale components and a weak notion of infinitesimal generator, via a finite-dimensional approximation procedure based on controlled inter-arrival times and approximating martingales. The theory comes with convergence results that allow to interpret a large class of Wiener functionals beyond semimartingales as limiting objects of differential forms which can be computed path wisely over finite-dimensional spaces. The theory reveals that solutions of BSDEs are minimizers of energy functionals w.r.t. Brownian motion driving noise.

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LOWER BOUNDS FOR THE SMALLEST SINGULAR VALUE OF STRUCTURED RANDOM MATRICES[1]

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We obtain lower tail estimates for the smallest singular value of random matrices with independent but nonidentically distributed entries. Specifically, we consider $n \times n$ matrices with complex entries of the form

$$M = A \circ X + B = (a_{ij}\xi_{ij} + b_{ij}),$$

where $X = (\xi_{ij})$ has i.i.d. centered entries of unit variance and A and B are fixed matrices. In our main result, we obtain polynomial bounds on the smallest singular value of M for the case that A has bounded (possibly zero) entries, and $B = Z\sqrt{n}$ where Z is a diagonal matrix with entries bounded away from zero. As a byproduct of our methods we can also handle general perturbations B under additional hypotheses on A, which translate to connectivity hypotheses on an associated graph. In particular, we extend a result of Rudelson and Zeitouni for Gaussian matrices to allow for general entry distributions satisfying some moment hypotheses. Our proofs make use of tools which (to our knowledge) were previously unexploited in random matrix theory, in particular Szemerédi's regularity lemma, and a version of the restricted invertibility theorem due to Spielman and Srivastava.

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THE SCALING LIMITS OF THE MINIMAL SPANNING TREE AND INVASION PERCOLATION IN THE PLANE

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We prove that the Minimal Spanning Tree and the Invasion Percolation Tree on a version of the triangular lattice in the complex plane have unique scaling limits, which are invariant under rotations, scalings, and, in the case of the MST, also under translations. However, they are not expected to be conformally invariant. We also prove some geometric properties of the limiting MST. The topology of convergence is the space of spanning trees introduced by Aizenman et al. [*Random Structures Algorithms* **15** (1999) 319–365], and the proof relies on the existence and conformal covariance of the scaling limit of the near-critical percolation ensemble, established in our earlier works.

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QUENCHED CENTRAL LIMIT THEOREM FOR RANDOM WALKS IN DOUBLY STOCHASTIC RANDOM ENVIRONMENT[1]

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We prove the quenched version of the central limit theorem for the displacement of a random walk in doubly stochastic random environment, under the H_{-1}-condition, with slightly stronger, $\mathcal{L}^{2+\varepsilon}$ (rather than $\mathcal{L}^2$) integrability condition on the stream tensor. On the way we extend Nash's moment bound to the nonreversible, divergence-free drift case, with unbounded ($\mathcal{L}^{2+\varepsilon}$) stream tensor. This paper is a sequel of [*Ann. Probab.* **45** (2017) 4307–4347] and relies on technical results quoted from there.

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THE KLS ISOPERIMETRIC CONJECTURE FOR GENERALIZED ORLICZ BALLS

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What is the optimal way to cut a convex bounded domain K in Euclidean space $(\mathbb{R}^n, |\cdot|)$ into two halves of equal volume, so that the interface between the two halves has least surface area? A conjecture of Kannan, Lovász and Simonovits asserts that, if one does not mind gaining a universal numerical factor (independent of n) in the surface area, one might as well dissect K using a hyperplane. This conjectured essential equivalence between the former nonlinear isoperimetric inequality and its latter linear relaxation, has been shown over the last two decades to be of fundamental importance to the understanding of volume-concentration and spectral properties of convex domains. In this work, we address the conjecture for the subclass of generalized Orlicz balls

$$K = \left\{x \in \mathbb{R}^n; \sum_{i=1}^n V_i(x_i) \leq E\right\},$$

confirming its validity for certain levels $E \in \mathbb{R}$ under a mild technical assumption on the growth of the convex functions V_i at infinity [without which we confirm the conjecture up to a $\log(1+n)$ factor]. In sharp contrast to previous approaches for tackling the KLS conjecture, we emphasize that no symmetry is required from K. This significantly enlarges the subclass of convex bodies for which the conjecture is confirmed.

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