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GAUSSIAN AND NON-GAUSSIAN FLUCTUATIONS FOR MESOSCOPIC LINEAR STATISTICS IN DETERMINANTAL PROCESSES[1]

BY KURT JOHANSSON AND GAULTIER LAMBERT

KTH Royal Institute of Technology and University of Zürich

We study mesoscopic linear statistics for a class of determinantal point processes which interpolate between Poisson and random matrix statistics. These processes are obtained by modifying the spectrum of the correlation kernel of the Gaussian Unitary Ensemble (GUE) eigenvalue process. An example of such a system comes from considering the distribution of noncolliding Brownian motions in a cylindrical geometry, or a grand canonical ensemble of free fermions in a quadratic well at positive temperature. When the scale of the modification of the spectrum of the correlation kernel, related to the size of the cylinder or the temperature, is different from the scale in the mesoscopic linear statistic, we obtain a central limit theorem (CLT) of either Poisson or GUE type. On the other hand, in the critical regime where the scales are the same, we observe a non-Gaussian process in the limit. Its distribution is characterized by explicit but complicated formulae for the cumulants of smooth linear statistics. These results rely on an asymptotic sine-kernel approximation of the GUE kernel which is valid at all mesoscopic scales, and a generalization of cumulant computations of Soshnikov for the sine process. Analogous determinantal processes on the circle are also considered with similar results.

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ON GLOBAL FLUCTUATIONS FOR NON-COLLIDING PROCESSES[1]

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We study the global fluctuations for a class of determinantal point processes coming from large systems of non-colliding processes and non-intersecting paths. Our main assumption is that the point processes are constructed by biorthogonal families that satisfy finite term recurrence relations. The central observation of the paper is that the fluctuations of multi-time or multi-layer linear statistics can be efficiently expressed in terms of the associated recurrence matrices. As a consequence, we prove that different models that share the same asymptotic behavior of the recurrence matrices, also share the same asymptotic behavior for the global fluctuations. An important special case is when the recurrence matrices have limits along the diagonals, in which case we prove Central Limit Theorems for the linear statistics. We then show that these results prove Gaussian Free Field fluctuations for the random surfaces associated to these systems. To illustrate the results, several examples will be discussed, including non-colliding processes for which the invariant measures are the classical orthogonal polynomial ensembles and random lozenge tilings of a hexagon.

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MULTIVARIATE APPROXIMATION IN TOTAL VARIATION, I: EQUILIBRIUM DISTRIBUTIONS OF MARKOV JUMP PROCESSES

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For integer valued random variables, the translated Poisson distributions form a flexible family for approximation in total variation, in much the same way that the normal family is used for approximation in Kolmogorov distance. Using the Stein–Chen method, approximation can often be achieved with error bounds of the same order as those for the CLT. In this paper, an analogous theory, again based on Stein's method, is developed in the multivariate context. The approximating family consists of the equilibrium distributions of a collection of Markov jump processes, whose analogues in one dimension are the immigration-death processes with Poisson distributions as equilibria. The method is illustrated by providing total variation error bounds for the approximation of the equilibrium distribution of one Markov jump process by that of another. In a companion paper, it is shown how to use the method for discrete normal approximation in $\mathbb{Z}^d$.

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MULTIVARIATE APPROXIMATION IN TOTAL VARIATION, II: DISCRETE NORMAL APPROXIMATION

BY A. D. BARBOUR[1], M. J. LUCZAK[2] AND A. XIA[3]

Universität Zürich, Queen Mary University of London and University of Melbourne

The paper applies the theory developed in Part I to the discrete normal approximation in total variation of random vectors in $\mathbb{Z}^d$. We illustrate the use of the method for sums of independent integer valued random vectors, and for random vectors exhibiting an exchangeable pair. We conclude with an application to random colourings of regular graphs.

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A GAUSSIAN SMALL DEVIATION INEQUALITY FOR CONVEX FUNCTIONS

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Let Z be an n-dimensional Gaussian vector and let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. We prove that

$$\mathbb{P}\big(f(Z) \le \mathbb{E}f(Z) - t\sqrt{\operatorname{Var} f(Z)}\big) \le \exp(-ct^2),$$

for all $t > 1$ where $c > 0$ is an absolute constant. As an application we derive variance-sensitive small ball probabilities for Gaussian processes.

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A VARIATIONAL APPROACH TO DISSIPATIVE SPDES WITH SINGULAR DRIFT[1]

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We prove global well-posedness for a class of dissipative semilinear stochastic evolution equations with singular drift and multiplicative Wiener noise. In particular, the nonlinear term in the drift is the superposition operator associated to a maximal monotone graph everywhere defined on the real line, on which neither continuity nor growth assumptions are imposed. The hypotheses on the diffusion coefficient are also very general, in the sense that the noise does not need to take values in spaces of continuous, or bounded, functions in space and time. Our approach combines variational techniques with a priori estimates, both pathwise and in expectation, on solutions to regularized equations.

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STRONG SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS WITH ROUGH COEFFICIENTS

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We study strong existence and pathwise uniqueness for stochastic differential equations in $\mathbb{R}^d$ with rough coefficients, and without assuming uniform ellipticity for the diffusion matrix. Our approach relies on direct quantitative estimates on solutions to the SDE, assuming Sobolev bounds on the drift and diffusion coefficients, and L^p bounds for the solution of the corresponding Fokker–Planck PDE, which can be proved separately. This allows a great flexibility regarding the method employed to obtain these last bounds. Hence we are able to obtain general criteria in various cases, including the uniformly elliptic case in any dimension, the one-dimensional case and the Langevin (kinetic) case.

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DIMENSIONS OF RANDOM COVERING SETS IN RIEMANN MANIFOLDS

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Let **M**, **N** and **K** be d-dimensional Riemann manifolds. Assume that $\mathbf{A} := (A_n)_{n\in\mathbb{N}}$ is a sequence of Lebesgue measurable subsets of **M** satisfying a necessary density condition and $\mathbf{x} := (x_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables, which are distributed on **K** according to a measure, which is not purely singular with respect to the Riemann volume. We give a formula for the almost sure value of the Hausdorff dimension of random covering sets $\mathbf{E}(\mathbf{x}, \mathbf{A}) := \limsup_{n\to\infty} A_n(x_n) \subset \mathbf{N}$. Here, $A_n(x_n)$ is a diffeomorphic image of A_n depending on x_n. We also verify that the packing dimensions of $\mathbf{E}(\mathbf{x}, \mathbf{A})$ equal d almost surely.

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MSC2010 subject classifications. 60D05, 28A80.
Key words and phrases. Random covering set, Hausdorff dimension, packing dimension.

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OPTIMAL SURVIVING STRATEGY FOR DRIFTED BROWNIAN MOTIONS WITH ABSORPTION

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We study the “Up the River” problem formulated by Aldous (2002), where a unit drift is distributed among a finite collection of Brownian particles on $\mathbb{R}_+$, which are annihilated once they reach the origin. Starting K particles at $x = 1$, we prove Aldous’ conjecture [Aldous (2002)] that the “push-the-laggard” strategy of distributing the drift asymptotically (as $K \to \infty$) maximizes the total number of surviving particles, with approximately $\frac{4}{\sqrt{\pi}}\sqrt{K}$ surviving particles. We further establish the hydrodynamic limit of the particle density, in terms of a two-phase partial differential equation (PDE) with a moving boundary, by utilizing certain integral identities and coupling techniques.

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Key words and phrases. Atlas model, competing Brownian particles, hydrodynamic limit, Stefan problems, moving boundary.

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DISCRETISATIONS OF ROUGH STOCHASTIC PDES

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We develop a general framework for spatial discretisations of parabolic stochastic PDEs whose solutions are provided in the framework of the theory of regularity structures and which are functions in time. As an application, we show that the dynamical Φ^4_3 model on the dyadic grid converges after renormalisation to its continuous counterpart. This result in particular implies that, as expected, the Φ^4_3 measure with a sufficiently small coupling constant is invariant for this equation and that the lifetime of its solutions is almost surely infinite for almost every initial condition.

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MULTIDIMENSIONAL SDES WITH SINGULAR DRIFT AND UNIVERSAL CONSTRUCTION OF THE POLYMER MEASURE WITH WHITE NOISE POTENTIAL

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We study the existence and uniqueness of solution for stochastic differential equations with distributional drift by giving a meaning to the Stroock–Varadhan martingale problem associated to such equations. The approach we exploit is the one of paracontrolled distributions introduced in (*Forum Math. Pi* **3** (2015) e6). As a result, we make sense of the three-dimensional polymer measure with white noise potential.

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CHAINING, INTERPOLATION AND CONVEXITY II: THE CONTRACTION PRINCIPLE[1]

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The generic chaining method provides a sharp description of the suprema of many random processes in terms of the geometry of their index sets. The chaining functionals that arise in this theory are however notoriously difficult to control in any given situation. In the first paper in this series, we introduced a particularly simple method for producing the requisite multiscale geometry by means of real interpolation. This method is easy to use, but does not always yield sharp bounds on chaining functionals. In the present paper, we show that a refinement of the interpolation method provides a canonical mechanism for controlling chaining functionals. The key innovation is a simple but powerful contraction principle that makes it possible to efficiently exploit interpolation. We illustrate the utility of this approach by developing new dimension-free bounds on the norms of random matrices and on chaining functionals in Banach lattices. As another application, we give a remarkably short interpolation proof of the majorizing measure theorem that entirely avoids the greedy construction that lies at the heart of earlier proofs.

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