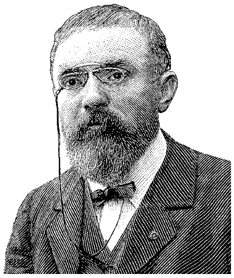

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Matrix models for multilevel Heckman–Opdam and multivariate Bessel measures

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Abstract. We study multilevel matrix ensembles at general β by identifying them with a class of processes defined via the branching rules for multivariate Bessel and Heckman–Opdam hypergeometric functions. For $\beta = 1, 2$, we express the joint multilevel density of the eigenvalues of a generalized β-Wishart matrix as a multivariate Bessel ensemble, generalizing a result of Dieker–Warren in (*ALEA Lat. Am. J. Probab. Math. Stat.* **6** (2009) 369–376). In the null case, we prove the conjecture of Borodin–Gorin in (*Comm. Pure Appl. Math.* **68** (2015) 1774–1844) that the joint multilevel density of the β-Jacobi ensemble is given by a principally specialized Heckman–Opdam measure.

Résumé. Nous étudions les ensembles de matrices multi-niveaux pour un β général, en les identifiant à une classe de processus définis via les règles de branchement pour les fonctions de Bessel multivariées et les fonctions hypergéométriques de Heckman–Opdam. Pour $\beta = 1, 2$, nous exprimons la densité multi-niveau conjointe des valeurs propres d'une matrice de type β-Wishart généralisé comme un ensemble de Bessel multivarié, généralisant un résultat de Dieker–Warren dans (*ALEA Lat. Am. J. Probab. Math. Stat.* **6** (2009) 369–376). Dans le cas nul, nous prouvons la conjecture de Borodin–Gorin dans (*Comm. Pure Appl. Math.* **68** (2015) 1774–1844), selon laquelle la densité multi-niveau conjointe de l'ensemble β-Jacobi est donnée par une mesure de Heckman–Opdam principalement spécialisée.

MSC2020 subject classifications: Primary 15B52; secondary 60B20; 33C67

Keywords: Random matrix theory; Hypergeometric function; Wishart ensemble; Jacobi ensemble

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Universality of the least singular value and singular vector delocalisation for Lévy non-symmetric random matrices

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Abstract. In this paper we consider $N \times N$ matrices D_N with i.i.d. entries all following an a-stable law divided by $N^{1/a}$. We prove that the least singular value of D_N, multiplied by N, tends to the same law as in the Gaussian case, for almost all $a \in (0, 2)$. This is proven by considering the symmetrization of the matrix D_N and using a version of the three step strategy, a well known strategy in the random matrix theory literature. In order to apply the three step strategy, we also prove an isotropic local law for the symmetrization of matrices after slightly perturbing them by a Gaussian matrix with a similar structure. The isotropic local law is proven for a general class of matrices that satisfy some regularity assumption. We also prove the complete delocalization for the left and right singular vectors of D_N at small energy, i.e., for energies at a small interval around 0.

Résumé. Dans cet article, nous considérons des $N \times N$ matrices D_N, avec des entrées i.i.d. de loi a-stable normalisés par $N^{1/a}$. Nous démontrons que la plus petite valeur singulière de D_N, multipliée par N, tend vers la même loi que dans le cas gaussien, pour presque tout $a \in (0, 2)$. Ceci est établi en considérant la symétrisation de la matrice D_N et en utilisant une version de la stratégie en trois étapes, une stratégie bien connue en théorie des matrices aléatoires. Pour utiliser cette approche, nous établissons également une loi locale isotrope pour la matrice symétrisée et légèrement perturbée par une matrice gaussienne de structure similaire. Cette loi locale isotrope est établie pour une classe générale de matrices qui satisfont des hypothèses de régularité. Nous démontrons également la délocalisation complète des vecteurs propres à gauche et à droite de D_N, associés aux énergies dans un petit intervalle autour de 0.

MSC2020 subject classifications: Primary 60B20; 60F15; 60E07; secondary 15B52

Keywords: Least singular value; Heavy tailed random matrices; Lévy random matrices; Eigenvector delocalization; Isotropic local law

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An edge CLT for the log determinant of Laguerre beta ensembles

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Abstract. We obtain a CLT for $\log|\det(M_n - s_n)|$ where M_n is a scaled Laguerre beta ensemble and $s_n = d_+ + \sigma_n n^{-2/3}$ with d_+ denoting the upper edge of the limiting spectrum of M_n and σ_n a slowly growing function ($\log\log^2 n \ll \sigma_n \ll \log^2 n$). In the special cases of LUE and LOE, we prove that the CLT also holds for σ_n of constant order. A similar result was proved for Wigner matrices by Johnstone, Klochkov, Onatski, and Pavlyshyn. Obtaining this type of CLT of Laguerre matrices is of interest for statistical testing of critically spiked sample covariance matrices as well as free energy of bipartite spherical spin glasses at critical temperature.

Résumé. Nous obtenons un TCL pour $\log|\det(M_n - s_n)|$, où M_n est un ensemble beta Laguerre mis à l'échelle et où $s_n = d_+ + \sigma_n n^{-2/3}$, avec d_+ désignant la borne supérieure du spectre limite de M_n et σ_n désignant une fonction à croissance lente ($\log\log^2 n \ll \sigma_n \ll \log^2 n$). Dans les cas particuliers du LUE et du LOE, nous prouvons que le TCL reste valide lorsque σ_n est d'ordre constant. Un résultat similaire a été prouvé pour les matrices de Wigner par Johnstone, Klochkov, Onatski et Pavlyshyn. Ce type de TCL pour des matrices de Laguerre est intéressant pour les tests statistiques de matrices de covariance à perturbations critiques ainsi que d'énergie libre de verres de spin sphériques bipartites à température critique.

MSC2020 subject classifications: 60B20; 60F05

Keywords: Laguerre beta ensemble; Edge statistics

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Mesoscopic eigenvalue statistics for Wigner-type matrices

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Abstract. We prove a universal mesoscopic central limit theorem for linear eigenvalue statistics of a Wigner-type matrix inside the bulk of the spectrum with compactly supported twice continuously differentiable test functions. The main novel ingredient is an optimal local law for the two-point function $T(z, \zeta)$ and a general class of related quantities involving two resolvents at nearby spectral parameters.

Résumé. On établit un théorème limite central universel pour les statistiques linéaires mésoscopiques des valeurs propres d'une matrice de type Wigner au milieu du spectre, avec des fonctions de classe C^2 et à support compact. La principale nouveauté de cette approche est qu'elle repose sur une loi locale optimale pour la fonction à deux points $T(z, \zeta)$, ainsi que pour une classe plus générale d'observables impliquant deux résolvantes évaluées en des paramètres proches.

MSC2020 subject classifications: 60B20; 15B52

Keywords: Wigner-type matrix; Mesoscopic eigenvalue statistics; Central limit theorem

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Extreme eigenvalues of log-concave ensemble

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Abstract. In this paper, we consider the log-concave ensemble of random matrices, a class of covariance-type matrices XX^* with isotropic log-concave X-columns. A main example is the covariance estimator of the uniform measure on isotropic convex body. Non-asymptotic estimates and first order asymptotic limits for the extreme eigenvalues have been obtained in the literature. In this paper, with the recent advancements on log-concave measures (*Geom. Funct. Anal.* **31** (2021) 34–61; *Geom. Funct. Anal.* **32** (2022) 1134–1159), we take a step further to locate the eigenvalues with a nearly optimal precision, namely, the spectral rigidity of this ensemble is derived. Based on the spectral rigidity and an additional "unconditional" assumption, we further derive the Tracy–Widom law for the extreme eigenvalues of XX^*, and the Gaussian law for the extreme eigenvalues in case strong spikes are present.

Résumé. Dans cet article, nous considérons l'ensemble log-concave de matrices aléatoires, une classe de matrices de type covariance XX^* avec des colonnes log-concaves isotropes X. Un exemple important est l'estimateur de covariance de la mesure uniforme sur un corps convexe isotrope. Des estimations non asymptotiques et des limites asymptotiques du premier ordre pour les valeurs propres extrêmes ont été obtenues dans la littêrature. Dans cet article, avec les progrès récents sur les mesures log-concaves (*Geom. Funct. Anal.* **31** (2021) 34–61 ; *Geom. Funct. Anal.* **32** (2022) 1134–1159), nous allons plus loin pour localiser les valeurs propres avec une précision presque optimale, à savoir, nous dérivons la rigidité spectrale de cet ensemble. Sur la base de cette rigidité spectrale et d'une hypothèse "inconditionnelle" supplémentaire, nous dérivons également la loi de Tracy–Widom pour les valeurs propres extrêmes de XX^*, ainsi que la loi gaussienne pour les valeurs propres extrêmes en présence de fortes hétérogénéités de variance.

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Keywords: Log-concave distribution; Sample covariance matrix; Local law; Spectral rigidity; Tracy–Widom law; Spiked model

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Stochastic difference equation with diagonal matrices

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Abstract. We consider the stochastic equation $\mathbf{X} \stackrel{d}{=} \mathbf{AX} + \mathbf{B}$ where $\mathbf{A}$ is a random diagonal matrix and $\mathbf{X}$, $\mathbf{B}$ are random vectors, $\mathbf{X}$, $\mathbf{A}$ are independent and the equation is meant in law. We prove that $\mathbf{X}$ is regularly varying in a multivariate nonstandard sense. The results are applicable to stochastic recursions with diagonal matrices, in particular, to multivariate autoregressive models like CCC-GARCH or BEKK-ARCH as well as to Gaussian multiplicative chaos.

Résumé. On considère l'équation stochastique $\mathbf{X} \stackrel{d}{=} \mathbf{AX} + \mathbf{B}$ où $\mathbf{A}$ est une matrice diagonale aléatoire, $\mathbf{X}$, $\mathbf{B}$ sont des vecteurs aléatoires et $\mathbf{X}$, $\mathbf{A}$ sont indépendants. Nous prouvons que la queue de distribution de $\mathbf{X}$ varie régulièrement. Les résultats sont applicables aux récursions stochastiques avec des matrices diagonales, en particulier, aux modèles autorégressifs multivariés comme CCC-GARCH ou BEKK-ARCH ainsi qu'au chaos multiplicatif gaussien.

MSC2020 subject classifications: Primary 60G70; 60G10; 60H25; secondary 62M10; 91B84

Keywords: Stochastic difference equation; Multivariate regular variation; Non-standard regular variation; Autoregressive models

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Scaling limit of random plane quadrangulations with a simple boundary, via restriction

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Abstract. We prove that quadrangulations with a simple boundary converge to the Brownian disk. More precisely, we fix a sequence (p_n) of even positive integers with $p_n \sim 2\alpha\sqrt{2n}$ for some $\alpha \in (0, \infty)$. Then, for the Gromov–Hausdorff topology, a quadrangulation with a simple boundary uniformly sampled among those with n inner faces and boundary length p_n weakly converges, in the usual scaling $n^{-1/4}$, toward the Brownian disk of perimeter 3α.

Our method consists in seeing a uniform quadrangulation with a simple boundary as a conditioned version of a model of maps for which the Gromov–Hausdorff scaling limit is known. We then explain how classical techniques of unconditionning can be used in this setting of random maps.

Résumé. Nous prouvons que les quadrangulations à bord simple convergent vers le disque brownien. Plus précisément, nous fixons une suite (p_n) d'entiers pairs strictement positifs tels que $p_n \sim 2\alpha\sqrt{2n}$ pour un certain $\alpha \in (0, \infty)$. Alors, pour la topologie de Gromov–Hausdorff, une quadrangulation à bord simple, choisie uniformément au hasard parmi celles ayant n faces internes et périmètre p_n, converge faiblement, dans l'échelle usuelle $n^{-1/4}$, vers le disque brownien de périmètre 3α.

Notre méthode consiste à considérer une quadrangulation à bord simple uniforme comme une version conditionnée d'un modèle de cartes pour lequel la limite d'échelle au sens de Gromov–Hausdorff est déjà connue. Nous expliquons ensuite comment utiliser les techniques classiques de déconditionnement dans ce contexte de cartes aléatoires.

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Keywords: Plane maps; Brownian disk; Quadrangulation; Scaling limit; Simple boundary

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Hydrodynamic limit of an exclusion process with vorticity

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Abstract. We construct a non reversible exclusion process with Bernoulli product invariant measure and having, in the diffusive hydrodynamic scaling, a non symmetric diffusion matrix, that can be explicitly computed. The antisymmetric part does not affect the evolution of the density but it is relevant for the evolution of the current. Switching on a weak external field we obtain a symmetric mobility matrix that is related just to the symmetric part of the diffusion matrix by an Einstein relation. We argue that this fact is typical within a class of generalized gradient models. We consider for simplicity the model in dimension $d = 2$, but a similar behavior can be also obtained in higher dimensions.

Résumé. Nous construisons un processus d'exclusion non réversible avec pour mesure invariante un produit de Bernoulli et ayant, dans l'échelle hydrodynamique diffusive, une matrice de diffusion non symétrique, qui peut être calculée explicitement. La partie antisymétrique n'affecte pas l'évolution de la densité mais elle est pertinente pour l'évolution du courant. En ajoutant un champ externe faible, nous obtenons une matrice de mobilité symétrique qui est liée uniquement à la partie symétrique de la matrice de diffusion par une relation d'Einstein. Nous soutenons que ce fait est typique dans une classe de modèles de gradient généralisés. Nous considérons pour simplifier le modèle en dimension $d = 2$, mais un comportement similaire peut également être obtenu en dimension supérieure.

MSC2020 subject classifications: 82C22; 82C70

Keywords: Lattice gases; Hydrodynamic scaling limits; Exclusion process; Discrete Hodge decomposition

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Tail bounds for detection times in mobile hyperbolic graphs

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Abstract. Motivated by Krioukov et al.'s model of random hyperbolic graphs Krioukov et al. (*Phys. Rev. E* **82** (2010) 036106) for real-world networks, and inspired by the analysis of a dynamic model of graphs in Euclidean space by Peres et al. (*Probab. Theory Related Fields* **156** (2013) 273–305), we introduce a dynamic model of hyperbolic graphs in which vertices are allowed to move according to a Brownian motion maintaining the distribution of vertices in hyperbolic space invariant. For different parameters of the speed of angular and radial motion, we analyze tail bounds for detection times of a fixed target and obtain a complete picture, for very different regimes, of how and when the target is detected: as a function of the time passed, we characterize the subset of the hyperbolic space where particles typically detecting the target are initially located. Our analysis shows that our dynamic model exhibits a phase transition as a function of the relation of angular and radial speed. We overcome several substantial technical difficulties not present in Euclidean space, and provide a complete picture on tail bounds. On the way, moreover, we obtain results for a class of one-dimensional continuous processes with drift and reflecting barrier, concerning the time they spend within a certain interval. We also derive improved bounds for the tail of independent sums of Pareto random variables.

Résumé. Motivés par le modèle des graphes aléatoires hyperboliques de Krioukov et al. (*Phys. Rev. E* **82** (2010) 036106) et inspirés par l'analyse du modèle dynamique de graphes en espace Euclidéen de Peres et al. (*Probab. Theory Related Fields* **156** (2013) 273–305), nous présentons un modèle dynamique des graphes aléatoires hyperboliques dans lequel les sommets se déplacent selon un mouvement Brownien qui maintient la distribution des sommets dans l'espace hyperbolique invariante. Pour différentes valeurs de vitesse du mouvement angulaire et radial, nous obtenons des bornes précises sur la queue de distribution pour le temps de détection d'une cible fixée, et nous décrivons comment et quand cette cible est détectée : en fonction du temps écoulé, nous caractérisons le sous-ensemble de l'espace hyperbolique où les particules détectant (typiquement) la cible sont initialement localisées. Notre analyse montre que notre modèle subit une transition de phase en fonction de la relation entre la vitesse angulaire et la vitesse radiale. Nous surmontons des difficultés techniques substantielles qui n'existent pas dans le cas Euclidien, et, en passant, nous obtenons des résultats concernant le temps passé dans un certain intervalle de l'espace pour une classe de processus continus unidimensionnels avec dérive et barrière réflectante. De plus, nous obtenons des limites améliorées pour la queue de distribution des sommes indépendantes des variables aléatoires suivant une loi de Pareto.

MSC2020 subject classifications: 60J70; 05C80

Keywords: Random hyperbolic graphs; Brownian motion

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Uniqueness of Markov random fields with higher-order dependencies

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Abstract. Markov random fields on a countable set V are studied. They are canonically set by a specification γ, for which the dependence structure is defined by a pre-modification $(h_e)_{e\in\mathsf{E}}$ – a consistent family of functions $h_e : S^e \to [0, +\infty)$, where S is a standard Borel space and E is an infinite collection of finite $e \subset \mathsf{V}$. Different e may contain distinct number of elements, which, in particular, means that the dependence graph $\mathsf{H} = (\mathsf{V}, \mathsf{E})$ is a hypergraph. Given $e \in \mathsf{E}$, let $\delta(e)$ be the logarithmic oscillation of h_e. The principal result of this work is the assertion that the set of all fields $\mathcal{G}(\gamma)$ is a singleton whenever $\delta(e)$ and the hypergraph H satisfy a condition, a particular version of which can be $\delta(e) \leq \varkappa g(n_{\mathsf{L}}(e))$, holding for all e and some H-specific $\varkappa \in (0, 1)$. Here g is an increasing function, e.g., $g(n) = a + \log n$, and $n_{\mathsf{L}}(e)$ is the degree of e in the line-graph $\mathsf{L}(\mathsf{H})$, which may grow ad infinitum. This uniqueness condition is essentially less restrictive than those based on classical Dobrushin's methods, according to which either of $|e|$, $n_{\mathsf{L}}(e)$ and $\delta(e)$ should be globally bounded. We also prove that its fulfilment implies that the unique element of $\mathcal{G}(\gamma)$ is globally Markov. Two illustrating examples are provided.

Résumé. Les champs aléatoires de Markov sur un ensemble dénombrable V sont étudiés. Ils sont définis canoniquement par une spécification γ, pour laquelle la structure de dépendance est définie par un pré-modification $(h_e)_{e\in\mathsf{E}}$ – une famille cohérente de fonctions $h_e : S^e \to [0, +\infty)$, où S est un espace Borel standard et E est un collection infinie de $e \subset \mathsf{V}$. Différent e peut contenir un nombre distinct d'éléments, ce qui signifie, en particulier, que le graphe de dépendance $\mathsf{H} = (\mathsf{V}, \mathsf{E})$ est un hypergraphe. Étant donné $e \in \mathsf{E}$, soit $\delta(e)$ l'oscillation logarithmique de h_e. Le résultat principal de ce travail est l'assertion selon laquelle l'ensemble de tous les champs $\mathcal{G}(\gamma)$ est un singleton chaque fois que $\delta(e)$ et l'hypergraphe H satisfont une condition, dont une version particuliére peut être la suivante : $\delta(e) \leq \varkappa g(n_{\mathsf{L}}(e))$, valable pour tous les e et une certaine constante H-spécifique $\varkappa \in (0, 1)$. Ici g est une fonction croissante, par exemple, $g(n) = a + \log n$, et $n_{\mathsf{L}}(e)$ est le degré de e dans la graphe-ligne $\mathsf{L}(\mathsf{H})$, qui peut croître à l'infini. Cette condition d'unicité est essentiellement moins restrictive que celles basées sur les méthodes de Dobrushin, selon laquelle soit $|e|$, $n_{\mathsf{L}}(e)$ ou $\delta(e)$ devrait être globalement bornés. Nous prouvons également que cette condition implique que l'élément unique de $\mathcal{G}(\gamma)$ est globalement Markovien. Deux exemples illustratifs sont fournis.

MSC2020 subject classifications: Primary 60G60; secondary 60C05; 60K35; 82B20

Keywords: Specification; Hypergraph; Global Markov property; Dobrushin condition; Tempered degree growth; Disordered system

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Denseness of biadapted Monge mappings

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Abstract. Adapted or causal transport theory aims to extend classical optimal transport from probability measures to stochastic processes. On a technical level, the novelty is to restrict to couplings which are *bicausal*, i.e. satisfy a property which reflects the temporal evolution of information in stochastic processes. We show that in the case of absolutely continuous marginals, the set of bicausal couplings is obtained precisely as the closure of the set of (bi-) adapted processes. That is, we obtain an analogue of the classical result on denseness of Monge couplings in the set of Kantorovich transport plans: bicausal transport plans represent the relaxation of adapted mappings in the same manner as Kantorovich transport plans are the appropriate relaxation of Monge maps.

Résumé. La théorie du transport adapté ou causal vise à étendre le transport optimal classique des mesures de probabilité aux processus stochastiques. Sur le plan technique, la nouveauté consiste à restreindre aux couplages qui sont bicausaux, c'est-à-dire qui satisfont une propriété qui reflète l'évolution temporelle de l'information dans les processus stochastiques. Nous montrons que dans le cas de marginales absolument continues, l'ensemble des couplages bicausaux est obtenu précisément comme la fermeture de l'ensemble des processus (bi-) adaptés. C'est-à-dire que nous obtenons un analogue du résultat classique sur la densité des couplages de Monge dans l'ensemble des plans de transport de Kantorovich : les plans de transport bicausaux représentent la relaxation des applications adaptées de la même manière que les plans de transport de Kantorovich sont la relaxation appropriée des applications de Monge.

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Keywords: Causal optimal transport; Adapted Wasserstein distance; Monge maps; Kantorovich transport plans

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Exponential ergodicity of branching processes with immigration and competition

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Abstract. We study the ergodic property of a continuous-state branching process with immigration and competition. The exponential ergodicity in a weighted total variation distance is proved under natural assumptions. The main theorem applies to subcritical, critical and supercritical branching mechanisms, including all those of stable types. The proof is based on the construction of a Markov coupling process and the choice of a nonsymmetric control function for the distance. Those are designed to identify and to take the advantage of the dominating factor from the branching, immigration and competition mechanisms in different parts of the state space. The approach provides a way of finding a lower bound of the ergodicity rate.

Résumé. Nous étudions la propriété ergodique d'un processus de branchement en temps et espace continu avec immigration et compétition. L'ergodicité exponentielle dans une distance de variation totale pondérée est prouvée sous des hypothèses naturelles. Le théorème principal s'applique aux mécanismes de branchement sous-critiques, critiques et sur-critiques, y compris tous les types stables. La démonstration est basée sur la construction d'un processus Markovien de couplage et le choix d'une fonction de contrôle non symétrique pour la distance. Ces éléments sont conçus pour identifier et profiter du facteur dominant des mécanismes de branchement, d'immigration et de compétition dans différentes parties de l'espace d'états. Cette approche permet de trouver une borne inférieure pour le taux d'ergodicité.

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Uniform convergence of Dyson Ferrari–Spohn diffusions to the Airy line ensemble

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Abstract. We consider the Dyson Ferrari–Spohn diffusion $\mathcal{X}^N = (\mathcal{X}_1^N, \dots, \mathcal{X}_N^N)$, consisting of N non-intersecting Ferrari–Spohn diffusions $\mathcal{X}_1^N > \cdots > \mathcal{X}_N^N > 0$ on $\mathbb{R}$. This object was introduced by Ioffe, Velenik, and Wachtel (*Probab. Theory Related Fields* **170** (2018) 11–47) as a scaling limit for line ensembles of N non-intersecting random walks above a hard wall with area tilts, which model certain three-dimensional interfaces in statistical physics. It was shown by Ferrari and Shlosman (*J. Phys. A: Math. Theor.* **56** (2023) 1–15) that as $N \to \infty$, after a spatial shift of order $N^{2/3}$ and constant rescaling in time, the top curve $\mathcal{X}_1^N$ converges to the Airy$_2$ process in the sense of finite-dimensional distributions. We extend this result by showing that the full ensemble $\mathcal{X}^N$ converges with the same shift and time scaling to the Airy line ensemble in the topology of uniform convergence on compact sets. In our argument we formulate a Brownian Gibbs property with area tilts for $\mathcal{X}^N$, which we show is equivalent after a global parabolic shift to the usual Brownian Gibbs property introduced by Corwin and Hammond (*Invent. Math.* **195** (2014) 441–508).

Résumé. Nous considérons la diffusion de Dyson Ferrari–Spohn $\mathcal{X}^N = (\mathcal{X}_1^N, \dots \mathcal{X}_N^N)$, constituée de N diffusions de Ferrari–Spohn non-intersectantes $\mathcal{X}_1^N > \cdots > \mathcal{X}_N^N > 0$ sur $\mathbb{R}$. Cet objet a été introduit par Ioffe, Velenik et Wachtel (*Probab. Theory Related Fields* **170** (2018) 11–47) en tant que limite d'échelle pour l'ensemble de lignes de N marches aléatoires non-intersectantes au-dessus d'un mur dur avec pondération des aires, modélisant certaines interfaces tri-dimensionnelles en physique statistique. Ferrari et Shlosman (*J. Phys. A : Math. Theor.* **56** (2023) 1–15) ont montré que quand $N \to \infty$, après un décalage spatial d'ordre $N^{2/3}$ et un changement d'échelle constant dans le temps, la courbe supérieure $\mathcal{X}_1^N$ converge vers le processus d'Airy$_2$ au sens des lois finies dimensionnelles. Nous étendons ce résultat en montrant que l'ensemble complet $\mathcal{X}^N$ converge avec le même décalage et la même échelle de temps vers l'ensemble de lignes d'Airy sous la topologie de convergence uniforme sur les compacts. Dans notre démonstration, nous formulons une propriété de Gibbs brownienne avec pondération des aires pour $\mathcal{X}^N$, que nous montrons être équivalente après un décalage parabolique global, à la propriété de Gibbs brownienne usuelle introduite par Corwin et Hammond (*Invent. Math.* **195** (2014) 441–508).

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Conditioned local limit theorems for random walks on the real line

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Abstract. Consider a random walk $S_n = \sum_{i=1}^n X_i$ with independent and identically distributed real-valued increments X_i of zero mean and finite variance. Assume that X_i is non-lattice and has a moment of order $2+\delta$. For any starting point $x \geq 0$, let $\tau_x = \inf\{k \geq 1 : x + S_k < 0\}$ be the first time when the random walk $x + S_n$ leaves the half-line $[0, \infty)$. We study the asymptotic behavior of the probability $\mathbb{P}(\tau_x > n)$ and that of the expectation $\mathbb{E}(f(x + S_n - y), \tau_x > n)$ for a large class of target functions f and various values of x, y possibly depending on n. This general setting implies limit theorems for the joint distribution $\mathbb{P}(x + S_n \in y + [0, \Delta], \tau_x > n)$, where $\Delta > 0$ may also depend on n. In particular, the case of moderate deviations $y = \sigma\sqrt{qn \log n}$ is considered. We also deduce some new asymptotics for random walks with drift and give explicit constants in the asymptotic of the probability $\mathbb{P}(\tau_x = n)$. For the proofs we establish new conditioned integral limit theorems with precise error terms.

Résumé. Considérons une marche aléatoire $S_n = \sum_{i=1}^n X_i$ avec des incréments X_i à valeur réelles indépendants et identiquement distribués de moyenne nulle et de variance finie. Supposons que X_i ne soit pas contenu dans un réseau et ait un moment d'ordre $2+\delta$. Pour tout point de départ $x \geq 0$, soit $\tau_x = \inf\{k \geq 1 : x + S_k < 0\}$ le premier instant où la marche aléatoire $x + S_n$ quitte la demi-ligne $[0, \infty)$. Nous étudions le comportement asymptotique de la probabilité $\mathbb{P}(\tau_x > n)$ et celui de l'espérance $\mathbb{E}(f(x + S_n - y), \tau_x > n)$ pour une grande classe de fonctions cible f et diverses valeurs de x, y qui peuvent éventuellement dépendre de n. Ce cadre général conduit à des théorémes limites pour la distribution conjointe $\mathbb{P}(x + S_n \in y + [0, \Delta], \tau_x > n)$, où $\Delta > 0$ peut également dépendre de n. En particulier, le cas des écarts modérés $y = \sigma\sqrt{qn \log n}$ est considéré. Nous en déduisons également de nouvelles estimations asymptotiques pour les marches aléatoires avec dérive et donnons des constantes explicites dans l'asymptotique de la probabilité $\mathbb{P}(\tau_x = n)$. Pour les preuves, nous établissons de nouveaux théorèmes limites intégraux conditionnés avec des termes d'erreur précis.

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Non-uniqueness in law of the two-dimensional surface quasi-geostrophic equations forced by random noise

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Abstract. Via probabilistic convex integration, we prove non-uniqueness in law of the two-dimensional surface quasi-geostrophic equations forced by random noise of additive type. In its proof we work on the equation of the momentum rather than the temperature, which is new in the study of the stochastic surface quasi-geostrophic equations. We also generalize the classical Calderón commutator estimate to the case of fractional Laplacians.

Résumé. Par le biais de l'intégration convexe probabiliste, nous prouvons la non-unicité en loi des équations quasi-géostrophiques de surfaces de deux dimensions forcées par un bruit aléatoire de type additif. Dans la démonstration nous travaillons sur l'équation de la quantité de mouvement plutôt que sur la température, ce qui est nouveau dans l'étude des équations quasi-géostrophiques stochastiques de surface. Nous généralisons également l'estimation du commutateur classique de Calderón au cas des Laplaciens fractionnaires.

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Keywords: Non-uniqueness; Convex integration

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Annealed limit for a diffusive disordered mean-field model with random jumps

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Abstract. We study a sequence of N-particle mean-field systems, each driven by N simple point processes $Z^{N,i}$ in a random environment. Each $Z^{N,i}$ has the same intensity $(f(X^N_{t-}))_t$ and at every jump time of $Z^{N,i}$, the process X^N does a jump of height $U_i/\sqrt{N}$ where the U_i are disordered centered random variables attached to each particle. We prove the convergence in distribution of X^N to some limit process $\bar{X}$ that is solution to an SDE with a random environment given by a Gaussian variable, with a convergence speed for the finite-dimensional distributions. This Gaussian variable is created by a CLT as the limit of the patial sums of the U_i. To prove this result, we use a coupling for the classical CLT relying on the result of (*Z. Wahrsch. Verw. Gebiete* **34** (1976) 33–58), that allows to compare the conditional distributions of X^N and $\bar{X}$ given the environment variables, with the same Markovian technics as the ones used in (*Bernoulli* **28** (2022) 125–149).

Résumé. Nous étudions une suite de systèmes de N particules en champ-moyen, où chaque système est dirigé par N processus ponctuels simples $Z^{N,i}$ dans un environnement aléatoire. Chaque $Z^{N,i}$ a la même intensité $f(X^N_{t-})_t$ et à chaque instant de saut de $Z^{N,i,}$, le processus X^N fait un saut d'amplitude $U_i/\sqrt{N}$ où U_i sont des variables aléatoires de désordre centrées attachées à chaque particule. Nous prouvons la convergence en loi de X^N vers un processus limite $\bar{X}$ qui est solution d'une EDS dans un environnement aléatoire donné par une variable gaussienne, avec une vitesse de convergence pour les lois fini-dimensionelles. Cette variable gaussienne est créé par un TCL comme la limite des sommes partielles des U_i. Pour montrer ce résultat, nous utilisons un couplage pour le TCL classique qui repose sur le résultat de (*Z. Wahrsch. Verw. Gebiete* **34** (1976) 33–58), qui permet de comparer les lois conditionnelles de X^N et de $\bar{X}$ étant donné les variables d'environnement, avec les mêmes techniques markoviennes que celles utilisées dans (*Bernoulli* **28** (2022) 125–149).

MSC2020 subject classifications: Primary 60K37; 60J35; 60J25; 60J60; secondary 60F05; 60G50; 60G55

Keywords: Annealed limit in random environment; Central limit theorem coupling; Piecewise deterministic Markov processes; Mean-field model

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Superdiffusion transition for a phonon Boltzmann equation

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Abstract. We consider an infinite harmonic chain of charged particles submitted to the action of a magnetic field of intensity B and subject to the action of a stochastic noise conserving the energy. In (*Ann. Appl. Probab.* **9** (2009) 2270–2300) it has been proved that if $B = 0$ the transport of energy is described by a $3/4$-fractional diffusion while it has been proved in (*Comm. Math. Phys.* **372** (2019) 151–182) that if $B \neq 0$ it is described by a $5/6$-fractional diffusion. In (*Ann. Appl. Probab.* **9** (2009) 2270–2300; *Comm. Math. Phys.* **372** (2019) 151–182) the authors used a two step argument, i.e. they first proved that the kinetic limit of the Wigner distribution is the solution of a phonon Boltzmann equation and then proved that this solution converges to the solution of a fractional diffusion equation with exponent $3/4$ if $B = 0$ (see (*Ann. Appl. Probab.* **9** (2009) 2270–2300)) and exponent $5/6$ if $B \neq 0$ (see (*Comm. Math. Phys.* **372** (2019) 151–182)). In this paper we quantify the intensity of the magnetic field required to switch from one macroscopic regime to the other one from the phonon Boltzmann equation. We also describe the transition mechanism to cross the two different phases.

Résumé. On considère une chaîne harmonique infinie de particules chargées, soumise à l'action d'un champ magnétique d'intensité B et à un bruit stochastique conservant l'énergie du système. Dans (*Ann. Appl. Probab.* **9** (2009) 2270–2300) il a été prouvé que si $B = 0$, le transport d'énergie est décrit par une équation de diffusion fractionnaire d'exposant $3/4$ tandis que les auteurs de (*Comm. Math. Phys.* **372** (2019) 151–182) on prouvé que si $B \neq 0$ alors le comportement est cette fois décrit par une équation de diffusion fractionnaire d'exposant $5/6$. Dans (*Ann. Appl. Probab.* **9** (2009) 2270–2300 ; *Comm. Math. Phys.* **372** (2019) 151–182) les auteurs ont utilisé un argument en deux étapes, ils ont d'abord prouvé que la limite cinétique de la distribution de Wigner convergeait vers la solution d'une équation de Boltzmann phononique puis ils ont prouvé que cette solution converge vers la solution d'une équation de diffusion fractionnaire dont l'exposant est $3/4$ si $B = 0$ (voir (*Ann. Appl. Probab.* **9** (2009) 2270–2300) et d'exposant $5/6$ si $B \neq 0$ (voir (*Comm. Math. Phys.* **372** (2019) 151–182)). Dans ce papier nous quantifions l'intensité du champ magnétique nécessaire pour passer d'un régime macroscopique à l'autre à partir de l'équation de Boltzmann. Nous décrivons également le mécanisme de transition pour traverser les deux phases.

MSC2020 subject classifications: Primary 60K50; 60G52; 60F17; 60J76; secondary 34A08

Keywords: Harmonic chain; Hydrodynamic limit; Lévy stable processes; Fractional diffusion

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Focusing Gibbs measures with harmonic potential

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Abstract. In this paper, we study the Gibbs measures associated to the focusing nonlinear Schrödinger equation with harmonic potential on Euclidean spaces. We establish a dichotomy for normalizability vs non-normalizability in the one dimensional case, and under radial assumption in the higher dimensional cases. In particular, we complete the programs of constructing Gibbs measures in the presence of a harmonic potential initiated by Burq–Thomann–Tzvetkov (2013) in dimension one and Deng (2012) in dimension two with radial assumption.

Résumé. Dans cet article, nous étudions la construction des mesures de Gibbs associées à l'équation de Schrödinger non-linéaire focalisante avec potentiel harmonique sur les espaces Euclidiens. En dimension une, nous établissons une dichotomie entre normalisabilité ou non de la mesure selon la croissance de la non-linéarité. Nous obtenons un résultat similaire en dimension supérieure en se restreignant à la mesure supportée sur les fonctions radiales. En particulier ces résultats terminent l'étude de la construction des mesures de Gibbs en présence d'un potentiel harmonique initiée par Burq–Thomann–Tzvetkov (2013) en dimension une, et par Deng (2012) en dimension deux dans le cas radial.

MSC2020 subject classifications: Primary 60H30; 81T08; secondary 35Q55

Keywords: Focusing Gibbs measure; Nonlinear Schrödinger equation; Harmonic potential; Normalizability; Variational approach

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Refined regularity of SLE

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Abstract. We prove refined (variation and Hölder-type) regularity statements for the SLE trace (under capacity parametrisation). More precisely, we show that the trace has finite ψ-variation for $\psi(x) = x^d(\log 1/x)^{-d-\varepsilon}$ and Hölder-type modulus $\varphi(t) = t^\alpha(\log 1/t)^\beta$ where d and α are the optimal p-variation and Hölder exponents of SLE_κ which have been previously identified by Viklund, Lawler (*Duke Math. J.* **159** (2011) 351–383) and Friz, Tran (*Forum Math. Sigma* **5** (2017) e19). For SLE_8, we simplify a step in the proof by Kavvadias, Miller, and Schoug (2021), and get the modulus $\varphi(t) = (\log 1/t)^{-1/4}(\log\log 1/t)^{2+\varepsilon}$.

Finally, for $\kappa \ge 8$, we prove regularity estimates for the uniformising maps that hold uniformly in time, namely $\sup_t |\hat{f}_t'(u+iv)| \lesssim v^{2\alpha-1}(\log 1/v)^\beta$ in case $\kappa > 8$ and $v^{-1}(\log 1/v)^{-1/4}(\log\log 1/v)^{1+\varepsilon}$ in case $\kappa = 8$.

Our results are obtained from analysing the forward Loewner differential equation (in contrast to the other mentioned works which analyse the backward equation).

Résumé. Nous démontrons des résultats de régularité affinés (variation et de type Hölder) pour la trace du SLE (sous la paramétrisation de capacité). Plus précisément, nous montrons que la trace a une ψ-variation finie pour $\psi(x) = x^d(\log 1/x)^{-d-\varepsilon}$ et un module de type Hölder $\varphi(t) = t^\alpha(\log 1/t)^\beta$ où d et α sont les exposants optimaux de p-variation et de Hölder de SLE_κ qui ont été identifiés précédemment par Viklund, Lawler (*Duke Math. J.* **159** (2011) 351–383) et Friz, Tran (*Forum Math. Sigma* **5** (2017) e19). Pour SLE_8, nous simplifions une étape de la preuve de Kavvadias, Miller et Schoug (2021) et obtenons le module $\varphi(t) = (\log 1/t)^{-1/4}(\log\log 1/t)^{2+\varepsilon}$.

Enfin, pour $\kappa \ge 8$, nous démontrons des estimations de régularité pour les applications d'uniformisation qui restent valables uniformément en temps, à savoir $\sup_t |\hat{f}_t'(u+iv)| \lesssim v^{2\alpha-1}(\log 1/v)^\beta$ dans le cas $\kappa > 8$ et $v^{-1}(\log 1/v)^{-1/4}(\log\log 1/v)^{1+\varepsilon}$ dans le cas $\kappa = 8$.

Nos résultats sont obtenus en analysant l'équation différentielle de Loewner progressive (contrairement aux autres travaux mentionnés qui analysent l'équation rétrograde).

MSC2020 subject classifications: Primary 60J67; secondary 60G17; 30C20

Keywords: Schramm–Loewner evolution; Generalised variation; Modulus of continuity; Radial Bessel process

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Uniform minorization condition and convergence bounds for discretizations of kinetic Langevin dynamics

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Abstract. We study the convergence in total variation and V-norm of discretization schemes of the underdamped Langevin dynamics. Such algorithms are very popular and commonly used in molecular dynamics and computational statistics to approximatively sample from a target distribution of interest. We show first that, for a very large class of schemes, a minorization condition uniform in the stepsize holds. This class encompasses popular methods such as the Euler-Maruyama scheme and the schemes based on splitting strategies. Second, we provide mild conditions ensuring that the class of schemes that we consider satisfies a geometric Foster–Lyapunov drift condition, again uniform in the stepsize. This allows us to derive geometric convergence bounds, with a convergence rate scaling linearly with the stepsize. This kind of result is of prime interest to obtain estimates on norms of solutions to Poisson equations associated with a given numerical method.

Résumé. Nous étudions la convergence en variation totale et en norme-V des schémas de discrétisation de la dynamique de Langevin sous-amortie. De tels algorithmes sont très populaires et communément utilisés en dynamique moléculaire et en statistique computationnelle pour échantillonner approximativement une loi cible d'intérêt. Nous montrons premièrement que, pour une très large classe de schémas, une condition de minorisation uniforme par rapport au pas de discrétisation est vérifiée. Cette classe englobe des méthodes populaires comme le schéma d'Euler-Maruyama et les schémas basés sur les stratégies de fractionnement. Deuxièmement, nous donnons de faibles conditions pour que la classe de schémas que nous considérons vérifie une condition de dérive de Foster–Lyapunov géométrique, toujours uniforme par rapport au pas de discrétisation. Cela nous permet d'en déduire une borne de convergence géométrique, avec un taux de convergence linéaire par rapport au pas de discrétisation. Ce type de résultat est primordial pour obtenir une estimation de la norme des solutions de l'équation de Poisson associées à une méthode numérique donnée.

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Keywords: Underdamped Langevin diffusion; Discretization; Minorization conditions; Geometric ergodicity

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Estimation of smooth functionals of covariance operators: Jackknife bias reduction and bounds in terms of effective rank

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Abstract. Let E be a separable Banach space and let $X, X_1, \dots, X_n, \dots$ be i.i.d. Gaussian random variables taking values in E with mean zero and unknown covariance operator $\Sigma : E^* \mapsto E$. The complexity of estimation of Σ based on the observations $X_1, \dots, X_n$ is naturally characterized by the so called effective rank of Σ: $\mathbf{r}(\Sigma) := \frac{\mathbb{E}_\Sigma \|X\|^2}{\|\Sigma\|}$, where $\|\Sigma\|$ is the operator norm of Σ. Given a smooth real valued functional f defined on the space $L(E^*, E)$ of symmetric linear operators from E^* into E (equipped with the operator norm), our goal is to study the problem of estimation of $f(\Sigma)$ based on $X_1, \dots, X_n$. A jackknife type bias reduction method will be considered for this problem and the dependence of the Orlicz norm error rates of the resulting estimators of $f(\Sigma)$ on the effective rank $\mathbf{r}(\Sigma)$, the sample size n and the degree of Hölder smoothness s of functional f will be studied. In particular, it will be shown that, if $\mathbf{r}(\Sigma) \lesssim n^\alpha$ for some $\alpha \in (0, 1)$ and $s \geq \frac{1}{1-\alpha}$, then the classical $\sqrt{n}$-rate is attainable and, if $s > \frac{1}{1-\alpha}$, then asymptotic normality and asymptotic efficiency of the resulting estimators hold. Previously, the results of this type (for different estimators) were obtained only in the case of finite dimensional Euclidean space $E = \mathbb{R}^d$ and for covariance operators Σ whose spectrum is bounded away from zero (in which case, $\mathbf{r}(\Sigma) \asymp d$).

Résumé. Soit E un espace de Banach séparable et soient $X, X_1, \dots, X_n, \dots$ des variables aléatoires i.i.d. gaussiennes à valeurs dans E de moyenne nulle et un opérateur de covariance inconnu $\Sigma : E^* \mapsto E$. La complexité de l'estimation de Σ à partir des observations $X_1, \dots, X_n$ est naturellement caractérisée par la notion bien connue de rang effectif de Σ : $\mathbf{r}(\Sigma) := \frac{\mathbb{E}_\Sigma \|X\|^2}{\|\Sigma\|}$, oú $\|\Sigma\|$ est la norme d'opérateur de Σ. Étant donné f une fonctionnelle á valeurs réelles différentiable définie sur l'espace $L(E^*, E)$ des opérateurs linéaires symétriques de E^* dans E (équipé de la norme d'opérateur), notre objectif est d'étudier le problème d'estimation de $f(\Sigma)$ basé sur $X_1, \dots, X_n$. Pour cela, nous proposons une nouvelle méthode de réduction du bias de type jackknife. Puis nous explicitons les vitesses d'estimations en norme d'Orlicz de nos nouvelles procédures d'estimation de $f(\Sigma)$ en fonction du rang effectif $\mathbf{r}(\Sigma)$, du nombre d'observations n et du degré de régularité de Hölder s de la fonctionnelle f. En particulier, nous démontrons que, si $\mathbf{r}(\Sigma) \lesssim n^\alpha$ pour un certain $\alpha \in (0, 1)$ et $s \geq \frac{1}{1-\alpha}$, alors la vitesse paramétrique classique en $\sqrt{n}$ est atteignable et, si $s > \frac{1}{1-\alpha}$, alors nos estimateurs sont asymptotiquement normaux et asymptotiquement efficaces. Les résultats précédents de ce type (pour différents estimateurs) n'étaient obtenus que dans le cas d'un espace euclidien de dimension finie $E = \mathbb{R}^d$ et pour les opérateurs de covariance Σ dont le spectre est bien séparé de zéro (dans ce cas, $\mathbf{r}(\Sigma) \asymp d$).

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A covariance formula for the number of excursion set components of Gaussian fields and applications

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Abstract. We derive a covariance formula for the number of excursion or level set components of a smooth stationary Gaussian field on $\mathbb{R}^d$ contained in compact domains. We also present two applications of this formula: (1) for fields whose correlations are integrable we prove that the variance of the component count in large domains is of volume order and give an expression for the leading constant, and (2) for fields with slower decay of correlation we give an upper bound on the variance which is of optimal order if correlations are regularly varying, and improves on best-known bounds if correlations are oscillating (e.g. monochromatic random waves).

Résumé. Nous démontrons une formule de covariance pour le nombre d'excursions ou de composantes d'ensemble de niveau d'un champ Gaussien stationnaire lisse sur $\mathbb{R}^d$ sur un domaine compact. Nous présentons également deux applications de cette formule : (1) pour les champs dont la corrélation est intégrable nous démontrons que la variance du nombre de composantes dans des grands domaines est de l'ordre de volume du domaine et donnons une expression pour le coefficient constant du terme principale, et (2) pour les champs dont la corrélation décroit lentement nous donnons une majoration de la variance qui est d'ordre optimal si la corrélation varie régulièrement, et qui améliore les meilleures majorations connues si la corrélation oscille (par exemple dans le cas des ondes monochromatiques aléatoires).

MSC2020 subject classifications: 60G15; 60G60

Keywords: Gaussian fields; Excursion set; Level set; Component count; Covariance formula

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Critical and near-critical level-set percolation of the Gaussian free field on regular trees

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Abstract. For the Gaussian free field on a $(d+1)$-regular tree with $d \geq 2$, we study the percolative properties of its level sets in the critical and the near-critical regime. In particular, we show the continuity of the percolation probability, derive an exact asymptotic tail estimate for the cardinality of the connected component of the critical level set, and describe the asymptotic behaviour of the percolation probability in the near-critical regime.

Résumé. Nous étudions les propriétés percolatives des ensembles de niveau d'un champ libre gaussien sur un arbre régulier de degré $(d+1)$, avec $d \geq 2$, dans le régime critique et près de ce régime. En particulier, nous démontrons la continuité de la probabilité de percolation, obtenons une estimation asymptotique exacte de la queue de la distribution de la cardinalité de la composante connexe de l'ensemble de niveau critique, et décrivons le comportement asymptotique de la probabilité de percolation près du point critique.

MSC2020 subject classifications: 60K35; 60G15; 60J80

Keywords: Gaussian free field; Level-set percolation; Critical behaviour; Regular trees

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