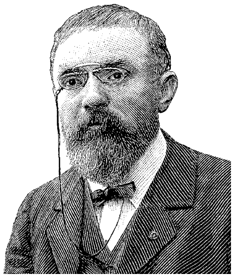

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Hydrodynamic behavior of long-range symmetric exclusion with a slow barrier: Diffusive regime

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Abstract. In this article we analyse the hydrodynamical behavior of the symmetric exclusion process with long jumps and in the presence of a slow barrier. The jump rates for fast bonds are given by a transition probability $p(\cdot)$ which is symmetric and has finite variance, while for slow bonds the jump rates are given by $p(\cdot)\alpha n^{-\beta}$ (with $\alpha > 0$ and $\beta \geq 0$), and correspond to some of the jumps between $\mathbb{Z}_{-}^{*}$ and $\mathbb{N}$. We prove that: if there is a fast bond from $\mathbb{Z}_{-}^{*}$ and $\mathbb{N}$, then the hydrodynamic limit is given by the heat equation with no boundary conditions; otherwise, it is given by the previous equation if $0 \leq \beta < 1$, but for $\beta \geq 1$ boundary conditions appear, namely, we get Robin (linear) boundary conditions if $\beta = 1$ and Neumann boundary conditions if $\beta > 1$.

Résumé. Dans cet article, nous analysons le comportement hydrodynamique du processus d'exclusion symétrique avec des longs sauts et en présence d'une barrière lente. Les taux de saut pour les liens rapides sont donnés par une probabilité de transition $p(\cdot)$ qui est symétrique et a une variance finie, tandis que pour les liens lents, les taux de saut sont donnés par $p(\cdot)\alpha n^{-\beta}$ (avec $\alpha > 0$ et $\beta \geq 0$), et correspondent à certains des sauts entre $\mathbb{Z}_{-}^{*}$ et $\mathbb{N}$. Nous prouvons que : s'il existe un lien rapide entre $\mathbb{Z}_{-}^{*}$ et $\mathbb{N}$, alors la limite hydrodynamique est donnée par l'équation de la chaleur sans conditions aux limites ; sinon, elle est donnée par l'équation précédente si $0 \leq \beta < 1$, mais pour $\beta \geq 1$ des conditions aux limites apparaissent, à savoir, on obtient des conditions aux limites de Robin (linéaires) si $\beta = 1$ et des conditions aux limites de Neumann si $\beta > 1$.

MSC2020 subject classifications: 60k35; 35R11; 35S15

Keywords: Long-range exclusion; Slow barrier; Hydrodynamic limit; Diffusive regime

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Hydrodynamic limit for asymmetric simple exclusion with accelerated boundaries

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Abstract. We consider the asymmetric simple exclusion process (ASEP) on the one-dimensional finite lattice $\{1, 2, \dots, N\}$. The particles can be created/annihilated at the boundaries with given rates. These rates are L^∞ functions of time and are independent of the jump rates in the bulk (cf. *Comm. Math. Phys.* **310** (2012) 1–24). The boundary dynamics is modified by a factor N^θ with $\theta > 0$. We study the hydrodynamic limit for the particle density profile under the hyperbolic space-time scale. The macroscopic equation is given by (inviscid) Burgers equation with boundary conditions which are characterized by the boundary entropy (*C. R. Acad. Sci. Paris* **322** (1996) 729–734). A grading scheme is developed to control the formulation of boundary layers on the microscopic level.

Résumé. Nous considérons le processus d'exclusion simple asymétrique (ASEP) sur le réseau fini unidimensionnel $\{1, 2, \dots, N\}$. Les particules peuvent être créées/annihilées sur les points de frontière avec des taux qui sont des fonctions L^∞ du temps et sont indépendants des taux de saut à l'intérieur du système (cf. *Comm. Math. Phys.* **310** (2012) 1–24). La dynamique des bords est modifiée d'un facteur N^θ avec $\theta > 0$. Nous étudions la limite hydrodynamique du profil de densité des particules sous l'échelle espace-temps hyperbolique. L'équation macroscopique est donnée par l'équation (non visqueuse) de Burgers avec des conditions aux limites caractérisées par l'entropie du processus sur les points de frontière (*C. R. Acad. Sci. Paris* **322** (1996) 729–734). Un schéma adapté est développé pour contrôler la formulation des couches limites au niveau microscopique.

MSC2020 subject classifications: 82C22; 82C70; 60K35

Keywords: Asymmetric simple exclusion process; Open boundary; Hydrodynamic limit; Entropy solution; Boundary layer

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KPZ on torus: Gaussian fluctuations

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Abstract. We study the KPZ equation on a torus and derive Gaussian fluctuations in large time.

Résumé. Nous étudions l'équation KPZ sur un tore et dérivons les fluctuations gaussiennes en temps long.

MSC2020 subject classifications: 60H15; 35Q82; 37A25

Keywords: KPZ equation; Directed polymer; Invariant measure

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Linear and superlinear spread for stochastic combustion growth process

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Abstract. Consider a stochastic growth model on $\mathbb{Z}^d$. Start with some active particle at the origin and sleeping particles elsewhere. The initial number of particles at $x \in \mathbb{Z}^d$ is $\eta(x)$, where $(\eta(x))$ are independent random variables distributed according to μ. Active particles perform a simple continuous-time random walk while sleeping particles stay put until the first arrival of an active particle to their location. Upon the arrival all sleeping particles at the site activate at once and start moving according to their own simple random walks. The aim of this paper is to give conditions on μ under which the spread of the process is linear or faster than linear. The proofs rely on comparison to various percolation models.

Résumé. Considérons un modèle de croissance stochastique sur $\mathbb{Z}^d$. Dans la configuration initiale, il y a une particule active à l'origine et des particules dormantes sur les autres sites. Le nombre initial de particules en $x \in \mathbb{Z}^d$ est $\eta(x)$, où $(\eta(x))$ sont des variables aléatoires indépendantes distribuées selon μ. Les particules actives effectuent une marche aléatoire simple en temps continu tandis que les particules dormantes restent en place jusqu'à la première arrivée d'une particule active à leur emplacement. À l'arrivée, toutes les particules dormantes sur le site s'activent en même temps et commencent à se déplacer selon leurs propres marches aléatoires simples. Le but de cet article est de donner des conditions sur μ sous lesquelles la propagation du processus est linéaire ou plus rapide que linéaire. Les démonstrations reposent sur la comparaison avec divers modèles de percolation.

MSC2020 subject classifications: Primary 60K35; secondary 60J10

Keywords: Infection spread; Stochastic growth model; Random walk; Frog model; Percolation

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Estimation of statistics of transitions and Hill relation for Langevin dynamics

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Abstract. In molecular dynamics, statistics of transitions, such as the mean transition time, are macroscopic observables which provide important dynamical information on the underlying microscopic stochastic process. A direct estimation using simulations of microscopic trajectories over long time scales is typically computationally intractable in metastable situations. To overcome this issue, several numerical methods rely on a potential-theoretic identity, sometimes attributed to Hill in the computational statistical physics literature, which expresses statistics of transitions in terms of the invariant measure of the sequence of configurations by which the underlying process enters metastable sets. The use of this identity then allows to replace the long time simulation problem with a rare event sampling problem, for which efficient algorithms are available.

In this article, we rigorously analyse such a method for molecular systems modelled by the Langevin dynamics. Our main contributions are twofold. First, we prove the Hill relation in the fairly general context of positive Harris recurrent chains, and show that this formula applies to the Langevin dynamics. Second, we provide an explicit expression of the invariant measure involved in the Hill relation, and describe an elementary exact simulation procedure. Overall, this yields a simple and complete numerical method to estimate statistics of transitions.

Résumé. En dynamique moléculaire, les statistiques de transition, telles que le temps moyen de transition, sont des observables macroscopiques qui fournissent des informations dynamiques importantes sur le processus stochastique microscopique sous-jacent. Dans les situations métastables, l'estimation directe de ces quantités à partir de la simulation de longues trajectoires microscopiques est typiquement intractable. Pour résoudre ce probème, plusieurs méthodes numériques reposent sur une identité provenant de la théorie du potentiel, parfois attribuée à Hill dans la littérature de physique statistique numérique, qui exprime les statistiques de transition en fonction de la mesure invariante de la suite des configurations par lesquelles le processus sous-jacent entre dans les états métastables. L'utilisation de cette identité permet alors de remplacer le problème de simulation d'une trajectoire sur un temps long par un problème d'échantillonnage d'événement rare, pour lequel existent des algorithmes efficaces.

Dans cet article, nous analysons rigoureusement une telle méthode pour des systèmes moléculaires modélisés par la dynamique de Langevin. Nous proposons deux contributions principales. D'abord, nous prouvons la relation de Hill dans le contexte général des chaînes de Markov positivement récurrentes au sens de Harris, et nous montrons que cette formule s'applique à la dynamique de Langevin. Ensuite, nous présentons une expression explicite de la mesure invariante employée dans la relation de Hill, et décrivons une procédure de simulation exacte élémentaire. La combinaison de ces résultats fournit une méthode numérique simple et complète pour estimer les statistiques de transition.

MSC2020 subject classifications: 60J20; 60J70

Keywords: Langevin dynamics; Transition path; Invariant measure; Hill relation

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Uniqueness and non-uniqueness of the Gaussian free field evolution under the two-dimensional Wick ordered cubic wave equation

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Abstract. We study the nonlinear wave equation (NLW) on the two-dimensional torus $\mathbb{T}^2$ with Gaussian random initial data on $H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$, $s < 0$, distributed according to the base Gaussian free field μ associated with the invariant Gibbs measure studied by Thomann and the first author (2020). In particular, we investigate the approximation property of the corresponding solution by smooth (random) solutions. Our main results in this paper are two-fold. (i) We show that the solution map for the renormalized cubic NLW defined on the Gaussian free field μ is the unique extension of the solution map defined for smoothed Gaussian initial data obtained by mollification, independent of mollification kernels. (ii) We also show that there is a regularization of the Gaussian initial data so that the corresponding smooth solutions almost surely have no limit in the natural topology. This second result in particular states that one can not use arbitrary smooth approximation for the renormalized cubic NLW dynamics.

As a preliminary step for proving (ii), we establish a (deterministic) norm inflation result at general initial data for the (unrenormalized) cubic NLW on $\mathbb{T}^d$ and $\mathbb{R}^d$ in negative Sobolev spaces, extending the norm inflation result by Christ, Colliander, and Tao (2003).

Résumé. On considère l'équation des ondes (NLW) posée sur le tore de dimension deux $\mathbb{T}^2$ avec une condition initiale aléatoire dans $H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$, $s < 0$, distribuée selon le champ libre gaussien μ associé à la mesure invariante de Gibbs étudiée par Thomann et le premier auteur (2020). En particulier, nous essayons de comprendre si on peut approximer les solutions avec condition initiale typique par des solutions lisses aléatoires. Nous obtenons deux résultats complémentaires : (i) Nous démontrons que le flot du NLW cubique renormalisé défini sur le champ libre gaussien est l'unique extension du flot défini sur des données gaussiennes régularisées par convolution (et cela indépendamment du noyau de convolution). (ii) Nous démontrons également qu'il existe une régularisation des données initiales gaussiennes telle que les solutions régulières correspondantes n'ont pas de limite presque sûrement dans la topologie naturelle. Par conséquent, nous ne pouvons pas utiliser une approximation arbitraire pour construire la dynamique du NLW cubique renormalisé. Une étape préliminaire dans la preuve de (ii) consiste en une élaboration significative sur un résultat d'inflation de norme dû à Christ, Colliander, et Tao (2003).

MSC2020 subject classifications: 35L71; 60H30

Keywords: Nonlinear wave equation; Gaussian free field; Ill-posedness; Norm inflation; Almost sure norm inflation

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Gibbs partitions: A comprehensive phase diagram

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Abstract. We study Gibbs partition models, also known as composition schemes. Our main results comprehensively describe their phase diagram, including a phase transition from the convergent case described in Stufler (*Random Structures Algorithms* **53** (2018) 537–558) to a new dense regime characterized by a linear number of components with fluctuations of smaller order quantified by an α-stable law for $1 < \alpha \leq 2$. We prove a functional scaling limit for a process whose jumps correspond to the component sizes and discuss applications to extremal component sizes. At the transition we observe a mixture of the two asymptotic shapes. We also treat extended composition schemes and prove a local limit theorem in a dilute regime with the limiting law being related to an α-stable law for $0 < \alpha < 1$. We describe the asymptotic size of the largest components via a point process limit.

Résumé. Nous étudions les modèles de partition de Gibbs, également connus sous le nom de schémas de composition. Nos principaux résultats décrivent de manière exhaustive leur diagramme de phase, y compris une transition de phase du cas convergent décrit dans Stufler (*Random Structures Algorithms* **53** (2018) 537–558) à un nouveau régime dense caractérisé par un nombre linéaire de composants avec des fluctuations d'ordre inférieur quantifiées par une loi α-stable pour $1 < \alpha \leq 2$. Nous prouvons une limite d'échelle fonctionnelle pour un processus dont les sauts correspondent aux tailles des composants et discutons des applications aux tailles extrêmes des composants. À la transition, nous observons un mélange des deux formes asymptotiques. Nous traitons également des schémas de composition étendus et prouvons un théorème de limite locale dans un régime dilué, la loi limite étant liée à une loi α-stable pour $0 < \alpha < 1$. Nous décrivons la taille asymptotique des plus grandes composantes via un processus ponctuel limite.

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Multi-colour competition with reinforcement

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Abstract. We study a system of interacting urns where balls of different colour/type compete for their survival, and annihilate upon contact. For competition between two types, the underlying graph (finite and connected), determining the interaction between the urns, is known to be irrelevant for the possibility of coexistence, whereas for $K \geq 3$ types the structure of the graph does affect the possibility of coexistence. We show that when the underlying graph is a cycle, competition between $K \geq 3$ types almost surely has a single survivor, thus establishing a conjecture of Griffiths, Janson, Morris and the first author. Along the way, we give a detailed description of an auto-annihilative process on the cycle, which can be perceived as an expression of the geometry of a Möbius strip in a discrete setting.

Résumé. Nous étudions un système d'urnes en interaction où des boules de différentes couleurs/types sont en compétition pour leur survie et s'annihilent lorsqu'elles entrent en contact. Dans le cas d'une compétition entre deux types, il est connu que le graphe sous-jacent (fini et connexe) qui détermine l'interaction entre les urnes n'influe pas sur la possibilité de coexistence, tandis que pour $K \geq 3$ types, la structure du graphe a un effet sur la possibilité de coexistence. Nous montrons que lorsque le graphe sous-jacent est un cycle, la compétition entre $K \geq 3$ types a presque sûrement un seul survivant, établissant ainsi une conjecture de Griffiths, Janson, Morris et du premier auteur. Au passage, nous donnons une description détaillée d'un processus d'auto-annihilation sur le cycle, qui peut être perçu comme une expression de la géométrie d'une bande de Möbius dans un cadre discret.

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Keywords: Urn model; Reinforcement process; Coexistence; Spatial growth

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Convergence of the dynamical discrete web to the dynamical Brownian web

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Abstract. In this paper we study the convergence of dynamical discrete web (DyDW) to the dynamical Brownian web (DyBW) in a path space topology. We show that almost surely the DyBW has RCLL paths taking values in an appropriate metric space, and as a sequence of RCLL paths, the rescaled DyDW converges to the DyBW. This proves weak convergence of the DyDW process to the DyBW process.

Résumé. Dans cet article, nous étudions la convergence du Dynamical Discrete Web (DyDW) vers le Dynamical Brownian Web (DyBW) pour la topologie associée à l'espace des trajectoires. Nous démontrons que presque sûrement le DyBW possède des trajectoires càdlàg à valeurs dans un espace métrique approprié et, en tant que suite de trajectoires càdlàg, le DyDW re-échelonné converge vers le DyBW. Ceci établit la convergence faible du processus DyDW vers le DyBW.

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Keywords: Brownian web; Brownian net; Dynamic Brownian web; Local time

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Backbone scaling limits for random walks on random critical trees

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Abstract. We prove the existence of scaling limits for the projection on the backbone of the random walks on the Incipient Infinite Cluster and the Invasion Percolation Cluster on a regular tree. We treat these projected random walks as Randomly trapped random walks (as defined in (*Ann. Probab.* **43** (2015) 2405–2457)) and thus describe these scaling limits as spatially subordinated Brownian motions.

Résumé. Nous prouvons l'existence de la limite d'échelle pour la projection sur la lignée infinie de la marche aléatoire sur l'amas de percolation critique infini conditionné (IIC). Nous considérons aussi le cas de l'amas de percolation d'invasion d'un arbre régulier. Nous étudions ces marches projetées comme des marches aléatoires piégées de manière aléatoire (comme définies dans (*Ann. Probab.* **43** (2015) 2405–2457)). Nous pouvons décrire ces limites d'échelle comme des mouvements Browniens subordonnés spatialement.

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Keywords: Percolation; Random walk

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Random walks on decorated Galton–Watson trees

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Abstract. In this article we study a simple random walk on a decorated Galton–Watson tree, obtained from a Galton–Watson tree by replacing each vertex of degree n with an independent copy of a graph G_n and gluing the inserted graphs along the tree structure. We assume that there exist constants $d, R \geq 1$, $v < \infty$ such that the diameter, effective resistance across and volume of G_n respectively grow like $n^{\frac{1}{d}}$, $n^{\frac{1}{R}}$, n^v as $n \to \infty$. We also assume that the underlying Galton–Watson tree is critical with offspring tails $\xi(x)$ decaying like $cx^{-\alpha-1}$ as $x \to \infty$ for some constant c and some $\alpha \in (1, 2)$. We establish the fractal dimension, spectral dimension, walk dimension and simple random walk displacement exponent for the resulting metric space as functions of α, d, R and v, along with bounds on the fluctuations of these quantities.

Résumé. Dans cet article, nous étudions une marche aléatoire simple sur un arbre de Galton–Watson décoré, obtenu à partir d'un arbre de Galton–Watson en remplaçant chaque sommet de degré n par une copie indépendante d'un graphe G_n et en collant les graphes insérés le long de la structure de l'arbre. Nous supposons qu'il existe des constantes $d, R \geq 1$, $v < \infty$ telles que le diamètre, la résistance effective et le volume de G_n croissent respectivement comme $n^{\frac{1}{d}}$, $n^{\frac{1}{R}}$, n^v lorsque $n \to \infty$. Nous supposons également que l'arbre de Galton–Watson sous-jacent est critique avec des queues de la loi de reproduction $\xi(x)$ qui décroit comme $cx^{-\alpha-1}$ lorsque $x \to \infty$, pour une certaine constante c et $\alpha \in (1, 2)$. Nous établissons la dimension fractale, la dimension spectrale, la dimension de la marche et l'exposant de déplacement de la marche aléatoire simple pour l'espace métrique obtenu en fonction de α, d, R et v, ainsi que des bornes sur les fluctuations de ces quantités.

MSC2020 subject classifications: Primary 60K37; secondary 60J80; 60J35; 60J10

Keywords: Galton–Watson tree; Spectral dimension; Simple random walk

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Scaling limits of random looptrees and bipartite plane maps with prescribed large faces

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Abstract. We first rephrase and unify known bijections between bipartite plane maps and labelled trees with the formalism of looptrees, which we argue to be both more relevant and technically simpler since the geometry of a looptree is explicitly encoded by the depth-first walk (or Łukasiewicz path) of the tree, as opposed to the height or contour process for the tree. We then construct continuum analogues associated with any càdlàg path with no negative jump and derive several invariance principles. We especially focus on uniformly random looptrees and maps with prescribed face degrees and study their scaling limits in the presence of macroscopic faces, which complements a previous work in the case of no large faces. The limits (along subsequences for maps) form new families of random metric measured spaces related to processes with exchangeable increments with no negative jumps and our results generalise previous works which concerned the Brownian and stable Lévy bridges.

Résumé. Nous proposons une reformulation qui unifie des bijections connues entre cartes planes biparties et arbres étiquetés à travers le formalisme des *looptrees*, pour laquelle nous justifions qu'elle est à la fois plus pertinente mais aussi plus simple d'un point de vue technique puisqu'un looptree est codé explicitement par la marche en profondeur (ou marche de Łukasiewicz) de l'arbre, sans passer par sa fonction de hauteur ou de contour qui est plus difficile à étudier. Nous construisons ensuite des analogues continus à partir de n'importe quelle fonction càdlàg et sans saut négatif et montrons des résultats de limites d'échelle. Nous nous concentrons en particulier sur les looptrees et cartes tirés uniformément au hasard avec des degrés de faces donnés et leurs limites en présence de faces macroscopiques, ce qui vient compléter un travail précédent. Les limites (le long de sous-suite pour les cartes) forment de nouveaux espaces métriques mesurés aléatoires reliés à des processus à accroissements échangeables et nos résultats généralisent de précédents qui portaient sur le cas brownien ou des ponts de processus de Lévy stables.

MSC2020 subject classifications: Primary 05C80; 60F17; secondary 60B05; 60D05

Keywords: Random maps; Looptrees; Scaling limits; Exchangeable increments

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Height of weighted recursive trees with sub-polynomially growing total weight

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Abstract. Weighted recursive trees are built by adding successively vertices with predetermined weights to a tree: each new vertex is attached to a parent chosen at random with probability proportional to its weight. In the case where the total weight of the tree at step n grows polynomially in n, we obtained in (*Ann. Appl. Probab.* **32** (2022) 3027–3059) an asymptotic expansion for the height of the tree, which falls into the university class of the maximum of branching random walks. In this paper, we consider the case of a total weight growing sub-polynomially in n and obtain asymptotics for the height of the tree in several regimes, showing that universality is broken and that the model exhibits new behavior.

Résumé. Les arbres récursifs pondérés sont contruits en ajoutant successivement des sommets aux poids prédéterminés : chaque nouveau sommet est attaché à un parent choisi au hasard avec une probabilité proportionnelle à son poids. Dans le cas où le poids total des n premiers sommets croît polynomialement en n, nous avons obtenu dans (*Ann. Appl. Probab.* **32** (2022) 3027–3059) un développement asymptotique pour la hauteur de l'arbre, qui tombe dans la classe d'universalité du maximum d'une marche aléatoire branchante. Dans le présent article, nous considérons le cas où le poids total croît de manière sous-polynomiale en n et nous décrivons le comportement asympotique de la hauteur de l'arbre dans plusieurs régimes. Nous décrivons dans ce cas de nouveaux comportements qui sortent de la classe d'universalité mentionnée au-dessus.

MSC2020 subject classifications: Primary 60J80; secondary 05C05; 60G70

Keywords: Random trees; Weighted recursive trees; Asymptotic height

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The critical tree of a renormalization model as a growth-fragmentation process

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Abstract. We study a branching system which describes the evolution indexed by a continuous time parameter ranging in $[0, 1)$ of a population of cells; the size of each cell increases deterministically and linearly except when the cell splits into two daughter cells. The system appears as the scaling limit of the critical tree in the family of hierarchical renormalization models studied in (*J. Stat. Phys.* **156** (2014) 268–290), conditioned on survival; it is also a growth-fragmentation process in the sense of Bertoin (*Bernoulli* **23** (2017) 1082–1101). We are interested in the empirical measure of the process representing the sizes of the cells that are alive at time $t \in [0, 1)$, and establish a general result, called the master formula, for exponential functionals of the empirical measure. The formula allows to determine the joint distribution of the sum of cell sizes and the number of cells at time t, which improves a previous result by Hu, Mallein and Pain (*Comm. Math. Phys.* **375** (2020) 605–651) who proved joint weak convergence of these two quantities when $t \to 1^-$. The main result in our paper, established also relying on the master formula, is a law of large numbers for the empirical measure when $t \to 1^-$, the limiting distribution explicitly identified. Our system can be viewed as an exactly solvable example of a growth-fragmentation process.

Résumé. Nous étudions un système de branchement qui décrit l'évolution d'une population de cellules ; le paramètre de temps est à valeurs dans $[0, 1)$; la taille de chaque cellule augmente de façon déterministe et linéaire sauf lorsque la cellule se divise en deux cellules filles. Le système apparaît comme la limite d'échelle de l'arbre critique conditionné par la survie d'une famille de modèles de renormalisation hiérarchiques étudiés dans (*J. Stat. Phys.* **156** (2014) 268–290) ; c'est aussi un processus de croissance-fragmentation au sens de Bertoin (*Bernoulli* **23** (2017) 1082–1101). Nous nous intéressons à la mesure empirique du processus ponctuel représentant les tailles des cellules vivant au temps $t \in [0, 1)$ et nous établissons un résultat général, appelé formule maîtresse, pour les fonctionnelles exponentielles de la mesure empirique. La formule permet de déterminer la distribution conjointe de la somme des tailles des cellules et du nombre de cellules au temps t, ce qui améliore un résultat précédent de Hu, Mallein et Pain (*Comm. Math. Phys.* **375** (2020) 605–651) ayant prouvé la convergence en loi jointe de ces deux quantités lorsque $t \to 1^-$. Le résultat principal de notre article, qui est établi via la formule maîtresse, est une loi des grands nombres pour la mesure empirique lorsque $t \to 1^-$, la loi limite étant explicitement identifiée. Notre système peut être considéré comme un exemple exactement soluble d'un processus de croissance-fragmentation.

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Keywords: Hierarchical renormalization model; Growth-fragmentation process

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On moments of multiplicative coalescents

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Abstract. We prove existence of all moments of the multiplicative coalescent at all times. We obtain as byproducts a number of related results which could be of general interest. In particular, we show the finiteness of the second moment of the l^2 norm for any extremal eternal version of multiplicative coalescent. Our techniques are in part inspired by percolation, and in part are based on tools from stochastic analysis, notably the semi-martingale and the excursion theory.

Résumé. Nous démontrons l'existence de tous les moments de la coalescence multiplicative en tout temps. Nous obtenons ainsi un certain nombre de résultats additionnels qui pourraient être d'intérêt général. En particulier, nous montrons la finitude du second moment de la norme l^2 pour toute version extrémale éternelle de la coalescence multiplicative. Nos techniques sont en partie inspirées de la percolation, et en partie fondées sur des outils de l'analyse stochastique, notamment des semi-martingales et la théorie des excursions.

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Keywords: Multiplicative coalescent; Random graph; Excursion; Lévy process; Moment estimates

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Cramér's moderate deviations for martingales with applications

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Abstract. Let $(\xi_i, \mathcal{F}_i)_{i\geq 1}$ be a sequence of martingale differences. Set $X_n = \sum_{i=1}^n \xi_i$ and $\langle X\rangle_n = \sum_{i=1}^n \mathbf{E}(\xi_i^2|\mathcal{F}_{i-1})$. We prove Cramér's moderate deviation expansions for $\mathbf{P}(X_n/\sqrt{\langle X\rangle_n} \geq x)$ and $\mathbf{P}(X_n/\sqrt{\mathbf{E}X_n^2} \geq x)$ as $n \to \infty$. Our results extend the classical Cramér result to the cases of normalized martingales $X_n/\sqrt{\langle X\rangle_n}$ and standardized martingales $X_n/\sqrt{\mathbf{E}X_n^2}$, with martingale differences satisfying the conditional Bernstein condition. Applications to elephant random walks and autoregressive processes are also discussed.

Résumé. Soit $(\xi_i, \mathcal{F}_i)_{i\geq 1}$ une suite de différences de martingale. Soient $X_n = \sum_{i=1}^n \xi_i$ et $\langle X\rangle_n = \sum_{i=1}^n \mathbf{E}(\xi_i^2|\mathcal{F}_{i-1})$. Nous prouvons les développements de déviation modérée de Cramér pour $\mathbf{P}(X_n/\sqrt{\langle X\rangle_n} \geq x)$ et $\mathbf{P}(X_n/\sqrt{\mathbf{E}X_n^2} \geq x)$ lorsque $n \to \infty$. Nos résultats étendent le résultat classique de Cramér aux cas des martingales normalisées $X_n/\sqrt{\langle X\rangle_n}$ et des martingales standardisées $X_n/\sqrt{\mathbf{E}X_n^2}$, où les différences de martingale vérifient la condition de Bernstein conditionnelle. Des applications aux marches aléatoires des éléphants et aux processus autorégressifs sont également discutées.

MSC2020 subject classifications: Primary 60G42; 60F10; 60E20; secondary 62M10

Keywords: Martingales; Cramér's moderate deviations; Berry–Esseen's bounds; Elephant random walks

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About an extension of the Matsumoto–Yor property

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Abstract. We prove that if $\alpha, \beta > 0$ are distinct and if A and B are independent non-degenerate positive random variables then

$$S = \frac{1}{B}\frac{\beta A + B}{\alpha A + B} \quad \text{and} \quad T = \frac{1}{A}\frac{\beta A + B}{\alpha A + B}$$

are independent if and only if A and B have generalized inverse Gaussian distributions (GIG) with suitable parameters. Essentially, this has already been proved in Bao and Noack (2021) with a supplementary hypothesis on the existence of smooth densities.

The motivation of this work comes from an observation about independence properties of the exponential Brownian motion due to Matsumoto and Yor (*Nagoya Math. J.* **162** (2001) 65–86) and a recent work of Croydon and Sasada (2020) on random recursion models rooted in the discrete Korteweg–de Vries equation, where the above result was conjectured.

We also extend the result to random matrices, proving that a matrix-variate analogue of the above independence property is satisfied by independent matrix-variate GIG variables. The question of characterization of GIG random matrices through this independence property remains open.

Résumé. Soit $\alpha \neq \beta$ deux nombres strictement positifs et soit A et B deux variables aléatoires positives non dégénérées telles que

$$S = \frac{1}{B}\frac{\beta A + B}{\alpha A + B} \quad \text{and} \quad T = \frac{1}{A}\frac{\beta A + B}{\alpha A + B}$$

soient indépendantes. Nous montrons que cela entraine que A et B suivent des lois gaussiennes inverses généralisées avec des paramètres convenables. Ce fait a déjà été montré par Bao et Noack (2021), mais avec de fortes hypothèses d'existence de densités différentiables.

Les sources de ces questions sont d'abord l'observation d'une propriété d'indépendance du mouvement brownien exponentiel faite par Matsumoto et Yor (*Nagoya Math. J.* **162** (2001) 65–86), et ensuite un récent travail de Croydon et Sasada (2020) sur des modèles récursifs issus de l'équation de Korteweg–de Vries discrète, où les auteurs conjecturent le résultat ci-dessus.

Le résultat direct est aussi étendu aux matrices définies positives en montrant qu'une généralisation de la propriété ci-dessus d'indépendance est satisfaite par des matrices suivant des lois GIG. La question de la caractérisation des lois GIG pour des matrices par cette propriété est toutefois ouverte.

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Probabilistic limit theorems via the operator perturbation method, under optimal moment assumptions

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Abstract. The Nagaev-Guivarc'h operator perturbation method is well known to provide various probabilistic limit theorems for Markov random walks. A natural conjecture is that this method should provide these limit theorems under the same moment assumptions as the optimal ones in the case of sums of independent and identically distributed random variables. In the past decades, assumptions have been weakened, without achieving fully this purpose (achieving it either with the help of an extra proof of the central limit theorem, or with an additional ε in the moment assumptions). The aim of this article is to give a positive answer to this conjecture via the Keller-Liverani theorem. We present here an approach allowing the establishment of limit theorems (including higher order ones) under optimal moment assumptions. Our method is based on Taylor expansions obtained via the perturbation operator method, combined with a new weak compactness argument without the use of any other extra tool (such as Martingale decomposition method, etc.).

Résumé. La méthode de perturbation d'opérateur de Nagaev-Guivarc'h est célèbre pour son efficacité dans l'établissement de divers théorèmes limites probabilistes pour des marches aléatoires markoviennes. Une conjecture naturelle est que cette méthode devrait permettre d'établir ces théorèmes limites sous les hypothèses de moment optimales dans le cas de sommes de variables aléatoires indépendantes et de même loi. Au cours des dernières décennies, les hypothèses ont été affaiblies sans atteindre pleinement cet objectif (soit à l'aide d'un théorème central limite établi par un autre argument, soit avec un ε supplémentaire dans les hypothèses de moment). Le but de cet article est de donner une réponse positive à cette conjecture via le théorème de Keller-Liverani. Notre méthode est basée sur des développements de Taylor obtenus par la méthode de perturbation d'opérateur, combinée à un nouvel argument de compacité faible sans faire appel à un argument d'une autre nature (tel que la méthode de décomposition de Martingale, etc.).

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Sharp high-dimensional central limit theorems for log-concave distributions

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Abstract. Let $X_1, \dots, X_n$ be i.i.d. log-concave random vectors in $\mathbb{R}^d$ with mean 0 and covariance matrix Σ. We study the problem of quantifying the normal approximation error for $W = n^{-1/2}\sum_{i=1}^n X_i$ with explicit dependence on the dimension d. Specifically, without any restriction on Σ, we show that the approximation error over rectangles in $\mathbb{R}^d$ is bounded by $C(\log^{13}(dn)/n)^{1/2}$ for some universal constant C. Moreover, if the Kannan–Lovász–Simonovits (KLS) spectral gap conjecture is true, this bound can be improved to $C(\log^3(dn)/n)^{1/2}$. This improved bound is optimal in terms of both n and d in the regime $\log n = O(\log d)$. We also give p-Wasserstein bounds with all $p \geq 1$ and a Cramér type moderate deviation result for this normal approximation error, and they are all optimal under the KLS conjecture. To prove these bounds, we develop a new Gaussian coupling inequality that gives almost dimension-free bounds for projected versions of p-Wasserstein distance for every $p \geq 1$. We prove this coupling inequality by combining Stein's method and Eldan's stochastic localization procedure.

Résumé. Soient $X_1, \dots, X_n$ des vecteurs aléatoires log-concaves i.i.d. à valeurs dans $\mathbb{R}^d$, centrées et de matrice de covariance Σ. Nous étudions le problème de quantification de l'erreur d'approximation normale pour $W = n^{-1/2}\sum_{i=1}^n X_i$ avec une dépendance explicite de la dimension d. Plus précisément, sans aucune restriction sur Σ, nous montrons que l'erreur d'approximation sur des rectangles dans $\mathbb{R}^d$ est bornée par $C(\log^{13}(dn)/n)^{1/2}$ pour une constante universelle C. De plus, si la conjecture du trou spectral de Kannan–Lovász–Simonovits (KLS) est vraie, cette borne peut être améliorée à $C(\log^3(dn)/n)^{1/2}$. Cette borne améliorée est optimale en termes de n et de d dans le régime $\log n = O(\log d)$. Nous donnons également des bornes p-Wasserstein pour tout $p \geq 1$ ainsi qu'un résultat de déviation modérée de type Cramér pour cette erreur d'approximation normale, et tous sont optimaux sous la conjecture KLS. Pour prouver ces bornes, nous développons une nouvelle inégalité de couplage gaussienne qui donne des bornes presque indépendantes de la dimension pour les versions projetées de la distance p-Wasserstein pour tout $p \geq 1$. Nous prouvons cette inégalité de couplage en combinant la méthode de Stein et la procédure de localisation stochastique d'Eldan.

MSC2020 subject classifications: Primary 60F05; 62E17; secondary 60J60

Keywords: Coupling; Cramér type moderate deviations; Föllmer process; p-Wasserstein distance; Stein's method; Stochastic localization

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Phase transition for extremes of a family of stationary multiple-stable processes

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Abstract. We investigate a family of stationary processes that may exhibit either long-range or short-range dependence, depending on the parameters. The processes can be represented as multiple stable integrals, and there are two parameters for the processes, the memory parameter $\beta \in (0,1)$ and the multiplicity parameter $p \in \mathbb{N}$. We investigate the macroscopic limit of extremes of the process, in terms of convergence of random sup-measures, for the full range of parameters. Our results show that (i) the extremes of the process exhibit long-range dependence when $\beta_p := p\beta - p + 1 \in (0,1)$, with a new family of random sup-measures arising in the limit, (ii) the extremes are of short-range dependence when $\beta_p < 0$, with independently scattered random sup-measures arising in the limit, and (iii) there is a delicate phase transition at the critical regime $\beta_p = 0$.

Résumé. Nous étudions une famille de processus stationnaires qui peuvent présenter, en fonction des paramètres, soit une dépendance à longue portée, soit à courte portée. Ces processus peuvent s'écrire comme intégrales stables multiples reposant sur deux paramètres : le paramètre de mémoire $\beta \in (0,1)$ et le paramètre de multiplicité $p \in \mathbb{N}$. Nous étudions la limite macroscopique pour les valeurs extrêmes de ces processus en termes de convergence des sup-mesures aléatoires associées et pour l'ensemble des paramètres possibles. Notre résultat montre que : (i) Les valeurs extrêmes ont un comportement de dépendance à longue portée quand $\beta_p := p\beta - p + 1 \in (0,1)$ et une nouvelle famille de sup-mesures aléatoires apparaît alors à la limite ; (ii) Les valeurs extrêmes ont un comportement de dépendance à courte portée quand $\beta_p < 0$ et la limite est une sup-mesure aléatoire dispersée indépendamment ; (iii) Il y a une transition de phase délicate à appréhender au régime critique $\beta_p = 0$.

MSC2020 subject classifications: Primary 60F17; 60G70; secondary 60G52; 60K05

Keywords: Stable regenerative set; Random sup-measure; Long-range dependence; Infinite ergodic theory; Phase transition; Regular variation; Multiple integral; Renewal process

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Roughness of geodesics in Liouville quantum gravity

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Abstract. The metric associated with the Liouville quantum gravity (LQG) surface has been constructed through a series of recent works and several properties of its associated geodesics have been studied. In the current article we confirm the folklore conjecture that the Euclidean Hausdorff dimension of LQG geodesics is strictly greater than 1 for all values of the so-called Liouville first passage percolation (LFPP) parameter ξ. We deduce this from a general criterion due to Aizenman and Burchard (*Duke Math. J.* **99** (1999), 419–453) which in our case amounts to *near*-geometric bounds on the probabilities of certain crossing events for LQG geodesics in the number of crossings. We obtain such bounds using the axiomatic characterization of the LQG metric after proving a special regularity property for the Gaussian free field (GFF). We also prove an analogous result for the LFPP geodesics.

Résumé. La métrique associée à la surface de gravité quantique de Liouville (LQG) a été construite grâce à une série de travaux récents et plusieurs propriétés de ses géodésiques associées ont été étudiées. Dans cet article nous confirmons la conjecture rentrée dans le folklore selon laquelle la dimension de Hausdorff euclidienne des géodésiques LQG est strictement supérieure à 1 pour toutes les valeurs du paramètre ξ de percolation de premier passage de Liouville (LFPP). Nous déduisons cela d'un critère général dû à Aizenman et Burchard (*Duke Math. J.* **99** (1999), 419–453) qui dans notre cas revient à des bornes exponentielles étirées sur les probabilités de certains événements de croisement pour les géodésiques LQG, par rapport au nombre de croisements. Nous obtenons de telles bornes en utilisant la caractérisation axiomatique de la métrique LQG après avoir prouvé une propriété spéciale de régularité pour le champ libre gaussien (GFF). Nous prouvons également un résultat analogue pour les géodésiques LFPP.

MSC2020 subject classifications: 60G60; 60G15

Keywords: Liouville quantum gravity (LQG); Liouville first passage percolation (LFPP); Gaussian free field (GFF); Random metrics; Random curves; Hausdorff dimension

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Central Limit Theorem for the number of real roots of random orthogonal polynomials

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Abstract. We study the number of real roots of a wide class of random linear combinations of orthogonal polynomials with Gaussian coefficients. The orthogonal polynomials in our model are defined by a deterministic measure with compact support on the real line. Using the method of Wiener Chaos, we show that the fluctuation for the number of real roots in the bulk is asymptotically Gaussian, by proving that this number of roots in the intervals inside the support of the orthogonality measure obeys the standard Central Limit Theorem. Wiener Chaos expansions were previously used to prove the CLT for classical ensembles of random trigonometric polynomials, and that approach is generalized in our paper via careful analysis of the correlations by using asymptotics for the reproducing kernels of orthogonal polynomials. A new interesting feature found on this path is that the local correlations for the number of real roots of our random orthogonal polynomials are different. In fact, our local correlations depend on the potential theoretic equilibrium measure for the support of the orthogonality measure.

Résumé. Nous étudions le nombre de racines réelles d'une large classe de combinaisons linéaires aléatoires de polynômes orthogonaux à coefficients gaussiens. Les polynômes orthogonaux de notre modèle sont définis par une mesure déterministe à support compact sur la droite réelle. En utilisant la méthode du chaos de Wiener, nous montrons que la fluctuation du nombre de racines réelles dans le volume est asymptotiquement gaussienne, en prouvant que ce nombre de racines dans les intervalles à l'intérieur du support de la mesure d'orthogonalité obéit au théorème central limite standard. Les expansions en chaos de Wiener étaient auparavant utilisées pour prouver le TCL pour des ensembles classiques de polynômes trigonométriques aléatoires, et cette approche est généralisée dans notre article via une analyse minutieuse des corrélations en utilisant des asymptotiques pour les noyaux reproduisants de polynômes orthogonaux. Une nouvelle caractéristique intéressante trouvée sur ce chemin est que les corrélations locales pour le nombre de racines réelles de nos polynômes orthogonaux aléatoires sont différentes. En fait, nos corrélations locales dépendent de la mesure d'équilibre de la théorie du potentielle pour le support de la mesure d'orthogonalité.

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Keywords: Random polynomials; CLT; Orthogonal polynomials

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Quadratic variations for Gaussian isotropic random fields on the sphere

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Abstract. In this paper we define (empirical) quadratic variations for a Gaussian isotropic random field f on the unit sphere as sums over equidistant increments on one single geodesic line on the surface of the sphere. We prove a noncentral limit theorem for a fixed Fourier component of such a field as well as quantitative central limit theorems in the increasing frequency regime. Based on these results we propose estimators of the angular power spectrum and study their properties. Moreover, we show a quantitative central limit theorem for quadratic variations over the field f and construct an estimator for the Hurst parameter of a $L^2(\mathbb{S}^2)$-valued fractional Brownian motion.

Résumé. Dans cet article, nous définissons les variations quadratiques (empiriques) pour un champ aléatoire isotrope gaussien f sur la sphère unitaire comme des sommes sur des incréments équidistants au carré sur une seule ligne géodésique sur la surface de la sphère. Nous prouvons un théorème limite non central pour une composante de Fourier fixe d'un tel champ ainsi que des théorèmes limites centraux quantitatifs dans le régime de fréquence croissante. Sur la base de ces résultats, nous proposons des estimateurs du spectre de puissance angulaire et étudions leurs propriétés. De plus, nous montrons un théorème central limite quantitatif pour les variations quadratiques sur le champ f et construisons un estimateur pour le paramètre de Hurst d'un mouvement Brownien fractionnaire à valeurs dans $L^2(\mathbb{S}^2)$.

MSC2020 subject classifications: Primary 60G60; 60G22; 60F10; 62M15; secondary 42C10

Keywords: Spherical random fields; Malliavin–Stein approximations; Quadratic variations; Angular power spectrum; Fractional Brownian motion

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Erratum: Rates of convergence in the central limit theorem for martingales in the non stationary setting

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