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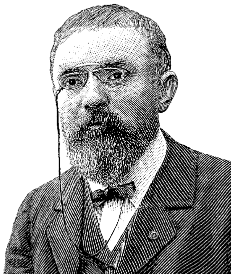

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Geometric influences II: Correlation inequalities and noise sensitivity

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Abstract. In a recent paper, we presented a new definition of influences in product spaces of continuous distributions, and showed that analogues of the most fundamental results on discrete influences, such as the KKL theorem, hold for the new definition in Gaussian space. In this paper we prove Gaussian analogues of two of the central applications of influences: Talagrand's lower bound on the correlation of increasing subsets of the discrete cube, and the Benjamini–Kalai–Schramm (BKS) noise sensitivity theorem. We then use the Gaussian results to obtain analogues of Talagrand's bound for all discrete probability spaces and to reestablish analogues of the BKS theorem for biased two-point product spaces.

Résumé. Dans un papier récent, nous avons présenté une nouvelle définition de l'influence dans des produits d'espaces de fonctions continues et montré que des résultats analogues aux résultats les plus importants sur les influences discrètes, comme le théorème KKL, sont valables pour la nouvelle définition dans des espaces gaussiens. Dans cet article, nous prouvons des analogues gaussiens de deux des applications principales des influences : la borne inférieure de Talagrand sur la corrélation de sous-ensembles croissants du cube discret et le théorème de Benjamini–Kalai–Schramm (BKS) sur la sensibilité au bruit. Ensuite nous utilisons les résultats gaussiens pour obtenir des analogues de la borne de Talagrand pour tous les espaces de probabilités discrets et pour retrouver l'analogue du théorème BKS pour des espaces produits biaisés à deux points.

MSC: 60C05; 05D40

Keywords: Influences; Geometric influences; Noise sensitivity; Correlation between increasing sets; Talagrand's bound; Gaussian measure; Isoperimetric inequality

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Uniform mixing time for random walk on lamplighter graphs

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Abstract. Suppose that $\mathcal{G}$ is a finite, connected graph and X is a lazy random walk on $\mathcal{G}$. The lamplighter chain $X^\diamond$ associated with X is the random walk on the wreath product $\mathcal{G}^\diamond = \mathbf{Z}_2 \wr \mathcal{G}$, the graph whose vertices consist of pairs $(\underline{f}, x)$ where f is a labeling of the vertices of $\mathcal{G}$ by elements of $\mathbf{Z}_2 = \{0, 1\}$ and x is a vertex in $\mathcal{G}$. There is an edge between $(\underline{f}, x)$ and $(\underline{g}, y)$ in $\mathcal{G}^\diamond$ if and only if x is adjacent to y in $\mathcal{G}$ and $f_z = g_z$ for all $z \neq x, y$. In each step, $X^\diamond$ moves from a configuration $(\underline{f}, x)$ by updating x to y using the transition rule of X and then sampling both f_x and f_y according to the uniform distribution on $\mathbf{Z}_2$; f_z for $z \neq x, y$ remains unchanged. We give matching upper and lower bounds on the uniform mixing time of $X^\diamond$ provided $\mathcal{G}$ satisfies mild hypotheses. In particular, when $\mathcal{G}$ is the hypercube $\mathbf{Z}_2^d$, we show that the uniform mixing time of $X^\diamond$ is $\Theta(d2^d)$. More generally, we show that when $\mathcal{G}$ is a torus $\mathbf{Z}_n^d$ for $d \geq 3$, the uniform mixing time of $X^\diamond$ is $\Theta(dn^d)$ uniformly in n and d. A critical ingredient for our proof is a concentration estimate for the local time of the random walk in a subset of vertices.

Résumé. Soit $\mathcal{G}$ un graphe connexe fini et X une marche aléatoire fainéante sur $\mathcal{G}$. La chaîne de l'allumeur de réverbères $X^\diamond$ associée à X est la marche aléatoire sur le groupe produit $\mathcal{G}^\diamond = \mathbf{Z}_2 \wr \mathcal{G}$, le graphe dont les sites sont des paires $(\underline{f}, x)$ où f est un label des sites de $\mathcal{G}$ par des éléments de $\mathbf{Z}_2 = \{0, 1\}$ et x est un site de $\mathcal{G}$. Il existe une arête entre $(\underline{f}, x)$ et $(\underline{g}, y)$ dans $\mathcal{G}^\diamond$ si et seulement si x est adjacent à y dans $\mathcal{G}$ et $f_z = g_z$ pour tout $z \neq x, y$. A chaque pas, $X^\diamond$ se déplace d'une configuration $(\underline{f}, x)$ en mettant à jour x vers y par la règle de translation de X et ensuite en mettant à jour à la fois f_x et f_y selon la distribution uniforme sur $\mathbf{Z}_2$; f_z pour $z \neq x, y$ restant inchangé. Nous prouvons des bornes supérieures et inférieures équivalentes sur le temps de mélange uniforme de $X^\diamond$ sous des hypothèses faibles sur $\mathcal{G}$. En particulier quand $\mathcal{G}$ est l'hypercube $\mathbf{Z}_2^d$, nous montrons que le temps de mélange uniforme de $X^\diamond$ est $\Theta(d2^d)$. Plus généralement, nous montrons que quand $\mathcal{G}$ est le tore $\mathbf{Z}_n^d$ avec $d \geq 3$, le temps de mélange uniforme de $X^\diamond$ est $\Theta(dn^d)$ uniformément en n et d. Un ingrédient crucial de notre preuve est une estimation de concentration pour le temps local d'une marche aléatoire dans un sous ensemble de sites.

MSC: 60J10; 60D05; 37A25

Keywords: Random walk; Uncovered set; Lamplighter walk; Mixing time

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A stationary random graph of no growth rate

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Abstract. We present a random automorphism-invariant subgraph of a Cayley graph such that with probability 1 its exponential growth rate does not exist.

Résumé. Nous construisons un sous-graphe aléatoire invariant par automorphismes d'un groupe de Cayley qui n'a presque sûrement pas un taux de croissance exponentiel bien défini.

MSC: 60C05; 05C80

Keywords: Unimodular graph; Stationary random graph; Growth rate

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Local percolative properties of the vacant set of random interlacements with small intensity

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Abstract. Random interlacements at level u is a one parameter family of connected random subsets of $\mathbb{Z}^d$, $d \geq 3$ (*Ann. Math.* **171** (2010) 2039–2087). Its complement, the vacant set at level u, exhibits a non-trivial percolation phase transition in u (*Comm. Pure Appl. Math.* **62** (2009) 831–858; *Ann. Math.* **171** (2010) 2039–2087), and the infinite connected component, when it exists, is almost surely unique (*Ann. Appl. Probab.* **19** (2009) 454–466).

In this paper we study local percolative properties of the vacant set of random interlacements at level u for all dimensions $d \geq 3$ and small intensity parameter $u > 0$. We give a stretched exponential bound on the probability that a large (hyper)cube contains two distinct macroscopic components of the vacant set at level u. In particular, this implies that finite connected components of the vacant set at level u are unlikely to be large. These results are new for $d \in \{3, 4\}$. The case of $d \geq 5$ was treated in (*Probab. Theory Related Fields* **150** (2011) 529–574) by a method that crucially relies on a certain "sausage decomposition" of the trace of a high-dimensional bi-infinite random walk. Our approach is independent from that of (*Probab. Theory Related Fields* **150** (2011) 529–574). It only exploits basic properties of random walks, such as Green function estimates and Markov property, and, as a result, applies also to the more challenging low-dimensional cases. One of the main ingredients in the proof is a certain conditional independence property of the random interlacements, which is interesting in its own right.

Résumé. Un entrelac aléatoire au niveau u est une famille à un paramètre de sous-ensembles connexes aléatoires de $\mathbb{Z}^d$, $d \geq 3$, introduit dans (*Ann. Math.* **171** (2010) 2039–2087). Son complémentaire, l'ensemble vacant au niveau u, possède une transition de percolation non triviale en u, comme il a été montré dans (*Comm. Pure Appl. Math.* **62** (2009) 831–858) et (*Ann. Math.* **171** (2010) 2039–2087). La composante connexe infinie, lorsqu'elle existe, est presque sûrement unique, voir (*Ann. Appl. Probab.* **19** (2009) 454–466).

Dans ce papier, nous étudions les propriétés percolatives locales de l'ensemble vacant au niveau u en toutes dimensions $d \geq 3$ et pour un petit paramètre d'intensité u. Nous donnons une borne exponentielle tendue sur la probabilité qu'un grand (hyper)cube contienne deux composantes macroscopiques distinctes de l'ensemble vacant au niveau u. Nos résultats impliquent qu'il est peu probable que les composantes connexes finies de l'ensemble vacant au niveau u soient grandes. Ces résultats ont été prouvés dans (*Probab. Theory Related Fields* **150** (2011) 529–574) pour $d \geq 5$. Notre approche est différente (de celle de (*Probab. Theory Related Fields* **150** (2011) 529–574)) et est valide pour $d \geq 3$.

L'un des ingrédients principaux de la preuve est une certaine propriété d'indépendance conditionelle des entrelacs aléatoires, qui est intéressante en elle-même.

MSC: 60K35; 82B43

Keywords: Random interlacement; Random walk; Large finite cluster; Supercriticality; Conditional independence

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Transience, recurrence and speed of diffusions with a non-Markovian two-phase "use it or lose it" drift

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Abstract. We investigate the transience/recurrence of a non-Markovian, one-dimensional diffusion process which consists of a Brownian motion with a non-anticipating drift that has two phases – a transient to $+\infty$ mode which is activated when the diffusion is sufficiently near its running maximum, and a recurrent mode which is activated otherwise. We also consider the speed of a diffusion with a two-phase drift, where the drift is equal to a certain non-negative constant when the diffusion is sufficiently near its running maximum, and is equal to a certain positive constant otherwise.

Résumé. Nous étudions la transience/récurrence d'un processus de diffusion non-Markovien à une dimension, consistant en un mouvement brownien avec une dérive non anticipative qui a deux phases – un mode transitoire à $+\infty$ qui est activé quand la diffusion est suffisamment proche du processus de son maximum, et un mode récurrent qui est activé dans le cas contraire. On considère également la vitesse d'une diffusion avec une dérive à deux phases, où la dérive est égale à une certaine constante positive lorsque la diffusion est suffisamment proche du processus de son maximum, et est égale à une certaine constante strictement positive dans le cas contraire.

MSC: 60J60

Keywords: Diffusion process; Transience; Recurrence; Non-Markovian drift

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Spectral condition, hitting times and Nash inequality

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Abstract. Let X be a μ-symmetric Hunt process on a LCCB space $\mathtt{E}$. For an open set $\mathtt{G} \subseteq \mathtt{E}$, let $\tau_{\mathtt{G}}$ be the exit time of X from $\mathtt{G}$ and $A^{\mathtt{G}}$ be the generator of the process killed when it leaves $\mathtt{G}$. Let $r : [0, \infty[\to [0, \infty[$ and $R(t) = \int_0^t r(s)\,\mathrm{d}s$.

We give necessary and sufficient conditions for $\mathbb{E}_\mu R(\tau_{\mathtt{G}}) < \infty$ in terms of the behavior near the origin of the spectral measure of $-A^{\mathtt{G}}$. When $r(t) = t^l$, $l \geq 0$, by means of this condition we derive the Nash inequality for the killed process.

In the diffusion case this permits to show that the existence of moments of order $l+1$ for $\tau_{\mathtt{G}}$ implies the Nash inequality of order $p = \frac{l+2}{l+1}$ for the whole process. The associated rate of convergence of the semi-group in $\mathbb{L}^2(\mu)$ is bounded by $t^{-(l+1)}$.

Finally, we show for general Hunt processes that the Nash inequality giving rise to a convergence rate of order $t^{-(l+1)}$ of the semi-group implies the existence of moments of order $l + 1 - \varepsilon$ for $\tau_{\mathtt{G}}$, for all $\varepsilon > 0$.

Résumé. Soit X un processus de Hunt μ-symétrique à valeurs dans un espace LCCB $\mathtt{E}$. Pour un ouvert $\mathtt{G} \subseteq \mathtt{E}$, soit $\tau_{\mathtt{G}}$ le temps de sortie de $\mathtt{G}$ par X et $A^{\mathtt{G}}$ le générateur du processus tué lorsqu'il quitte $\mathtt{G}$. Soit $r : [0, \infty[\to [0, \infty[$ et $R(t) = \int_0^t r(s)\,\mathrm{d}s$.

Nous établissons des conditions nécéssaires et suffisantes pour que $\mathbb{E}_\mu R(\tau_{\mathtt{G}}) < \infty$. Ces conditions sont données en termes du comportement au voisinage de zéro de la mesure spectrale de $-A^{\mathtt{G}}$ Dans le cas ou $r(t) = t^l$, $l \geq 0$, en utilisant ces conditions, à partir de $\mathbb{E}_\mu R(\tau_{\mathtt{G}}) < \infty$ nous déduisons l'inégalité de Nash pour le processes tué.

Dans le cas d'un processus de diffusion cela permet de montrer que l'existence des moments d'ordre $l+1$ pour $\tau_{\mathtt{G}}$ implique l'inégalité de Nash d'ordre $p = \frac{l+2}{l+1}$ pour le processus X. La vitesse de convergence du semi-groupe dans $\mathbb{L}^2(\mu)$ est donnée par $t^{-(l+1)}$.

Finalement pour un processus de Hunt μ-symétrique à valeurs dans un espace LCCB nous montrons que l'inégalité de Nash donnant lieu à la convergence du semi-groupe avec la vitesse $t^{-(l+1)}$ implique l'existence des moments d'ordre $l + 1 - \varepsilon$ pour $\tau_{\mathtt{G}}$, pour tout $\varepsilon > 0$.

MSC: 60J25; 60J35; 60J60

Keywords: Recurrence; Hitting times; Dirichlet form; Nash inequality; Weak Poincaré inequality; α-mixing; Continuous time Markov processes

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The parabolic Anderson model in a dynamic random environment: Basic properties of the quenched Lyapunov exponent

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Abstract. In this paper we study the parabolic Anderson equation $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, where the u-field and the ξ-field are $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, and Δ is the discrete Laplacian. The ξ-field plays the role of a *dynamic random environment* that drives the equation. The initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, is taken to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate $2d\kappa$, split into two at rate $\xi \vee 0$, and die at rate $(-\xi) \vee 0$. Our goal is to prove a number of *basic properties* of the solution u under assumptions on ξ that are as weak as possible. These properties will serve as a jump board for later refinements.

Throughout the paper we assume that ξ is stationary and ergodic under translations in space and time, is not constant and satisfies $\mathbb{E}(|\xi(0,0)|) < \infty$, where $\mathbb{E}$ denotes expectation w.r.t. ξ. Under a mild assumption on the tails of the distribution of ξ, we show that the solution to the parabolic Anderson equation exists and is unique for all $\kappa \in [0,\infty)$. Our main object of interest is the *quenched Lyapunov exponent* $\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t}\log u(0,t)$. It was shown in Gärtner, den Hollander and Maillard (In *Probability in Complex Physical Systems. In Honour of Erwin Bolthausen and Jürgen Gärtner* (2012) 159–193 Springer) that this exponent exists and is constant ξ-a.s., satisfies $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ and $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$, and is such that $\kappa \mapsto \lambda_0(\kappa)$ is globally Lipschitz on $(0,\infty)$ outside any neighborhood of 0 where it is finite. Under certain weak space–time mixing assumptions on ξ, we show the following properties: (1) $\lambda_0(\kappa)$ does not depend on the initial condition u_0; (2) $\lambda_0(\kappa) < \infty$ for all $\kappa \in [0,\infty)$; (3) $\kappa \mapsto \lambda_0(\kappa)$ is continuous on $[0,\infty)$ but not Lipschitz at 0. We further conjecture: (4) $\lim_{\kappa\to\infty}[\lambda_p(\kappa) - \lambda_0(\kappa)] = 0$ for all $p \in \mathbb{N}$, where $\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{pt}\log \mathbb{E}([u(0,t)]^p)$ is the pth *annealed Lyapunov exponent*. (In (In *Probability in Complex Physical Systems. In Honour of Erwin Bolthausen and Jürgen Gärtner* (2012) 159–193 Springer) properties (1), (2) and (4) were not addressed, while property (3) was shown under much more restrictive assumptions on ξ.) Finally, we prove that our weak space–time mixing conditions on ξ are satisfied for several classes of interacting particle systems.

Résumé. Dans cet article on étudie l'équation parabolique d'Anderson $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, où les champs u et ξ sont à valeurs dans $\mathbb{R}$, $\kappa \in [0,\infty)$ est la constante de diffusion, et Δ est le laplacien discret. Le champ ξ joue le rôle d'*environnement aléatoire dynamique* et dirige l'équation. La condition initiale $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, est choisie positive et bornée. La solution de l'équation parabolique d'Anderson décrit l'évolution d'un champ de particules effectuant des marches aléatoires simples avec un branchement binaire : les particules sautent au taux $2d\kappa$, se divisent en deux au taux $\xi \vee 0$, et meurent au taux $(-\xi) \vee 0$. Notre but est de prouver un certain nombre de *propriétés basiques* de la solution u sous des conditions sur ξ qui sont aussi faibles que possible. Ces propriétés vont servir d'impulsion pour de futur améliorations.

Tout au long de cet article nous supposons que ξ est stationnaire et ergodique sous les translations en espace et en temps, n'est pas constant et satisfait $\mathbb{E}(|\xi(0,0)|) < \infty$, où $\mathbb{E}$ représente l'espérance par rapport à ξ. Sous une hypothèse très faible sur les queues de la distribution de ξ, nous montrons que la solution de l'équation parabolique d'Anderson existe et est unique pour tout $\kappa \in [0,\infty)$. Notre principal objet d'intérêt est l'*exposant de Lyapunov quenched* $\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t}\log u(0,t)$. Il a été prouvé dans Gärtner, den Hollander et Maillard (In *Probability in Complex Physical Systems. In Honour of Erwin Bolthausen and Jürgen Gärtner* (2012) 159–193 Springer) que cet exposant existe et est constant ξ-a.s., satisfait $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ et

$\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ pour $\kappa \in (0,\infty)$, et est tel que $\kappa \mapsto \lambda_0(\kappa)$ est globalement lipschitzienne sur $(0,\infty)$ à l'extérieur de n'importe quel voisinage de 0 où il est fini. Sous certaines conditions faibles de mélange en espace-temps sur ξ, nous montrons les propriétés suivantes : (1) $\lambda_0(\kappa)$ ne dépend pas de la condition initiale u_0; (2) $\lambda_0(\kappa) < \infty$ pour tout $\kappa \in [0,\infty)$; (3) $\kappa \mapsto \lambda_0(\kappa)$ est continue sur $[0,\infty)$ mais pas lipschitzienne en 0. Nous conjecturons en outre : (4) $\lim_{\kappa\to\infty}[\lambda_p(\kappa) - \lambda_0(\kappa)] = 0$ pour tout $p \in \mathbb{N}$, où $\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{pt} \log \mathbb{E}([u(0,t)]^p)$ est le p-ième *exposant de Lyapunov annealed*. (Dans (In *Probability in Complex Physical Systems. In Honour of Erwin Bolthausen and Jürgen Gärtner* (2012) 159–193 Springer) les propriétés (1), (2) et (4) n'ont pas été abordées, tandis que la propriété (3) a été prouvée sous des hypothèses beaucoup plus restrictives sur ξ.) Finalement, nous prouvons que nos conditions faibles de mélange en espace-temps sur ξ sont satisfaites par plusieurs systèmes de particules en interaction.

MSC: Primary 60H25; 82C44; secondary 60F10; 35B40

Keywords: Parabolic Anderson equation; Percolation; Quenched Lyapunov exponent; Large deviations; Interacting particle systems

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Scaling of a random walk on a supercritical contact process

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Abstract. We prove a strong law of large numbers for a one-dimensional random walk in a dynamic random environment given by a supercritical contact process in equilibrium. The proof uses a coupling argument based on the observation that the random walk eventually gets trapped inside the union of space–time cones contained in the infection clusters generated by single infections. In the case where the local drifts of the random walk are smaller than the speed at which infection clusters grow, the random walk eventually gets trapped inside a single cone. This in turn leads to the existence of regeneration times at which the random walk forgets its past. The latter are used to prove a functional central limit theorem and a large deviation principle under the annealed law.

The qualitative dependence of the asymptotic speed and the volatility on the infection parameter is investigated, and some open problems are mentioned.

Résumé. Nous prouvons une loi forte des grands nombres pour une marche aléatoire dans un milieu aléatoire dynamique donné par un processus de contact sur-critique unidimensionnel en équilibre. La preuve utilise un argument de couplage basé sur l'observation que la marche est finalement confinée dans l'union de cônes spatio-temporels inclus dans les clusters d'infection générés par des infections individuelles. Si les taux locaux de saut de la marche sont plus petits que la vitesse de propagation de l'infection, la marche est finalement confinée dans un seul cône, ce qui entraîne l'existence de temps de régénération en lesquels la marche oublie son passé. Ces temps de régénération sont utilisés pour prouver un théorème central limite fonctionnel et un principe de grandes déviations sous la loi "annealed."

La dépendance de la vitesse et de la variance asymptotiques par rapport au paramètre d'infection est étudiée, et quelques problèmes ouverts sont mentionnés.

MSC: Primary 60F15; 60K35; 60K37; secondary 82B41; 82C22; 82C44

Keywords: Random walk; Dynamic random environment; Contact process; Strong law of large numbers; Functional central limit theorem; Large deviation principle; Space–time cones; Clusters of infections; Coupling; Regeneration times

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From a kinetic equation to a diffusion under an anomalous scaling

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Abstract. A linear Boltzmann equation is interpreted as the forward equation for the probability density of a Markov process $(K(t), i(t), Y(t))$ on $(\mathbb{T}^2 \times \{1, 2\} \times \mathbb{R}^2)$, where $\mathbb{T}^2$ is the two-dimensional torus. Here $(K(t), i(t))$ is an autonomous reversible jump process, with waiting times between two jumps with finite expectation value but infinite variance. $Y(t)$ is an additive functional of K, defined as $\int_0^t v(K(s))\,\mathrm{d}s$, where $|v| \sim 1$ for small k. We prove that the rescaled process $(N \ln N)^{-1/2} Y(Nt)$ converges in distribution to a two-dimensional Brownian motion. As a consequence, the appropriately rescaled solution of the Boltzmann equation converges to the solution of a diffusion equation.

Résumé. Une équation de Boltzmann linéaire est interprétée comme équation de Fokker–Planck associée à la densité de probabilité d'un processus de Markov $(K(t), i(t), Y(t))$ sur $(\mathbb{T}^2 \times \{1, 2\} \times \mathbb{R}^2)$, où $\mathbb{T}^2$ est le tore bidimensionnel. Le processus Markovien $(K(t), i(t))$ est ici un processus de sauts réversible avec des temps d'attente entre deux sauts à moyenne finie mais variance infinie. $Y(t)$ est une fonctionnelle additive de K, définie par $Y(t) = \int_0^t v(K(s))\,\mathrm{d}s$, où $|v| \sim 1$ pour k petit. Nous prouvons que le processus $(N \ln N)^{-1/2} Y(Nt)$ converge en distribution vers un mouvement brownien bidimensionnel. En conséquence, et moyennant un changement d'échelle approprié, la solution de l'équation de Boltzmann converge vers celle d' une équation de diffusion.

MSC: 82C44; 60K35; 60G70

Keywords: Anomalous thermal conductivity; Kinetic limit; Invariance principle

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Three examples of Brownian flows on $\mathbb{R}$

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Abstract. We show that the only flow solving the stochastic differential equation (SDE) on $\mathbb{R}$

$$\mathrm{d}X_t = 1_{\{X_t>0\}} W^+(\mathrm{d}t) + 1_{\{X_t<0\}}\,\mathrm{d}W^-(\mathrm{d}t),$$

where W^+ and W^- are two independent white noises, is a coalescing flow we will denote by $\varphi^{\pm}$. The flow $\varphi^{\pm}$ is a Wiener solution of the SDE. Moreover, $K^+ = \mathsf{E}[\delta_{\varphi^{\pm}}|W^+]$ is the unique solution (it is also a Wiener solution) of the SDE

$$K^+_{s,t} f(x) = f(x) + \int_s^t K_{s,u}\big(1_{\mathbb{R}^+} f'\big)(x) W^+(\mathrm{d}u) + \frac{1}{2}\int_s^t K_{s,u} f''(x)\,\mathrm{d}u$$

for $s < t$, $x \in \mathbb{R}$ and f a twice continuously differentiable function. A third flow φ^+ can be constructed out of the n-point motions of K^+. This flow is coalescing and its n-point motion is given by the n-point motions of K^+ up to the first coalescing time, with the condition that when two points meet, they stay together. We note finally that $K^+ = \mathsf{E}[\delta_{\varphi^+}|W^+]$.

Résumé. Nous montrons que le seul flot solution de l'équation différentielle stochastique (EDS) sur $\mathbb{R}$

$$\mathrm{d}X_t = 1_{\{X_t>0\}} W^+(\mathrm{d}t) + 1_{\{X_t<0\}}\,\mathrm{d}W^-(\mathrm{d}t),$$

où W^+ et W^- sont deux bruits blancs indépendants, est un flot coalescent que nous noterons $\varphi^{\pm}$. Le flot $\varphi^{\pm}$ est une solution Wiener de l'équation. De plus, $K^+ = \mathsf{E}[\delta_{\varphi^{\pm}}|W^+]$ est l'unique solution (c'est aussi une solution Wiener) de l'EDS

$$K^+_{s,t} f(x) = f(x) + \int_s^t K_{s,u}\big(1_{\mathbb{R}^+} f'\big)(x) W^+(\mathrm{d}u) + \frac{1}{2}\int_s^t K_{s,u} f''(x)\,\mathrm{d}u$$

pour tout $s < t$, $x \in \mathbb{R}$ et f une fonction deux fois continûment mesurable. Un troisième flot φ^+ peut être construit à partir des mouvements à n points de K^+. Ce flot est coalescent et ses mouvements à n points sont donnés par les mouvements à n points de K^+ jusqu'au premier temps de coalescence, avec comme condition que lorsque deux points se rencontrent, ils restent confondus. On remarquera finalement que $K^+ = \mathsf{E}[\delta_{\varphi^+}|W^+]$.

MSC: Primary 60H25; secondary 60J60

Keywords: Stochastic flows; Coalescing flow; Arratia flow or Brownian web; Brownian motion with oblique reflection on a wedge

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On smoothing properties of transition semigroups associated to a class of SDEs with jumps

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Abstract. We prove smoothing properties of nonlocal transition semigroups associated to a class of stochastic differential equations (SDE) in $\mathbb{R}^d$ driven by additive pure-jump Lévy noise. In particular, we assume that the Lévy process driving the SDE is the sum of a subordinated Wiener process Y (i.e. $Y = W \circ T$, where T is an increasing pure-jump Lévy process starting at zero and independent of the Wiener process W) and of an arbitrary Lévy process independent of Y, that the drift coefficient is continuous (but not necessarily Lipschitz continuous) and grows not faster than a polynomial, and that the SDE admits a Feller weak solution. By a combination of probabilistic and analytic methods, we provide sufficient conditions for the Markovian semigroup associated to the SDE to be strong Feller and to map $L_p(\mathbb{R}^d)$ to continuous bounded functions. A key intermediate step is the study of regularizing properties of the transition semigroup associated to Y in terms of negative moments of the subordinator T.

Résumé. Nous établissons des propriétés de lissage de semi-groupes de transition non locaux associés à une classe d'équations différentielles stochastiques dans $\mathbb{R}^d$ dirigées par un bruit additif de Lévy sans partie continue. En particulier, nous supposons que le processus de Lévy est la somme d'un processus de Wiener subordonné Y (i.e. $Y = W \circ T$, où T est un processus croissant de Lévy sans partie continue, avec $T_0 = 0$, indépendant du processus de Wiener W) et d'un processus de Lévy arbitraire indépendant de Y; que le coefficient de dérive est continu (mais pas nécessairement lipschitzien) et à croissance polynomiale; et que la EDS admet une solution faible fellerienne. Par une combinaison de méthodes probabilistes et analytiques, nous fournissons des conditions suffisantes pour le semi-groupe markovien associé à l'EDS soit fortement fellérien et envoye $L_p(\mathbb{R}^d)$ dans les fonctions continues bornées. Une étape intermédiaire essentielle est l'étude de certaines propriétés régularisantes du semi-groupe de transition associé à Y qui dépendent de moments négatifs du subordinateur T.

MSC: 60G30; 60G51; 60H07; 60H10; 60J35

Keywords: Lévy processes; Subordination; Transition semigroups; Non-local operators; Malliavin calculus

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Hausdorff dimension of affine random covering sets in torus

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Abstract. We calculate the almost sure Hausdorff dimension of the random covering set $\limsup_{n\to\infty}(g_n + \xi_n)$ in d-dimensional torus $\mathbb{T}^d$, where the sets $g_n \subset \mathbb{T}^d$ are parallelepipeds, or more generally, linear images of a set with nonempty interior, and $\xi_n \in \mathbb{T}^d$ are independent and uniformly distributed random points. The dimension formula, derived from the singular values of the linear mappings, holds provided that the sequences of the singular values are decreasing.

Résumé. Nous calculons presque sûrement la dimension de Hausdorff de l'ensemble de recouvrement aléatoire $\limsup_{n\to\infty}(g_n + \xi_n)$ dans le tore $\mathbb{T}^d$ de dimension d, où $g_n \subset \mathbb{T}^d$ sont des parallélépipèdes, ou plus généralement, des images linéaires d'un ensemble d'intérieur non vide et $\xi_n \in \mathbb{T}^d$ sont des points aléatoires indépendants et uniformément distribués. La formule de dimension, exprimée en fonction des valeurs singulières des applications linéaires, est valable à condition que la suite de ces valeurs singulières soit décroissante.

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Keywords: Random covering set; Hausdorff dimension; Affine Cantor set

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Localization and delocalization for heavy tailed band matrices[1]

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Abstract. We consider some random band matrices with band-width N^μ whose entries are independent random variables with distribution tail in $x^{-\alpha}$. We consider the largest eigenvalues and the associated eigenvectors and prove the following phase transition. On the one hand, when $\alpha < 2(1+\mu^{-1})$, the largest eigenvalues have order $N^{(1+\mu)/\alpha}$, are asymptotically distributed as a Poisson process and their associated eigenvectors are essentially carried by two coordinates (this phenomenon has already been remarked for full matrices by Soshnikov in (*Electron. Comm. Probab*. **9** (2004) 82–91, In *Poisson Statistics for the Largest Eigenvalues in Random Matrix Ensembles* (2006) 351–364) when $\alpha < 2$ and by Auffinger et al. in (*Ann. Inst. H. Poincarè Probab. Statist.* **45** (2005) 589–610) when $\alpha < 4$). On the other hand, when $\alpha > 2(1+\mu^{-1})$, the largest eigenvalues have order $N^{\mu/2}$ and most eigenvectors of the matrix are delocalized, i.e. approximately uniformly distributed on their N coordinates.

Résumé. On considère des matrices aléatoires à structure bande dont la bande a pour largeur N^μ et dont les coefficients sont indépendants à queue de distribution en $x^{-\alpha}$. On s'intéresse aux plus grandes valeurs propres et aux vecteurs propres associés et prouve la transition de phase suivante. D'une part, quand $\alpha < 2(1+\mu^{-1})$, les plus grandes valeurs propres ont pour ordre $N^{(1+\mu)/\alpha}$, sont asymptotiquement distribuées selon un processus de Poisson et les vecteurs propres associés sont essentiellement portés par deux coordonnées (ce phénomène a déjà été remarqué pour des matrices pleines par Soshnikov dans (*Electron. Comm. Probab*. **9** (2004) 82–91, In *Poisson Statistics for the Largest Eigenvalues in Random Matrix Ensembles* (2006) 351–364) quand $\alpha < 2$, et par Auffinger et al. dans (*Ann. Inst. H. Poincarè Probab. Statist.* **45** (2005) 589–610) quand $\alpha < 4$). D'autre part, quand $\alpha > 2(1+\mu^{-1})$, les plus grandes valeurs propres ont pour ordre $N^{\mu/2}$ et la plupart des vecteurs propres de la matrice sont délocalisés, i.e. approximativement uniformément distribués sur leurs N coordonnées.

MSC: 15A52; 60F05

Keywords: Random matrices; Band matrices; Heavy tailed random variables

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A free stochastic partial differential equation[1]

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Abstract. We get stationary solutions of a free stochastic partial differential equation. As an application, we prove equality of non-microstate and microstate free entropy dimensions under a Lipschitz like condition on conjugate variables, assuming also the von Neumann algebra R^{ω} embeddable. This includes an N-tuple of q-Gaussian random variables e.g. for $|q|N \leq 0.13$.

Résumé. Nous construisons des solutions stationnaires de certaines équations différentielles stochastiques libres à coefficients opérateurs non-bornés. Comme application, nous montrons l'égalité des dimensions entropiques libres microcanonique et non-microcanonique sous l'hypothèse d'une variable conjuguée Lipschitz pour les générateurs $X_1, \ldots, X_N$ d'un espace de probabilité non-commutatif inscriptible dans une ultrapuissance R^{ω} du facteur hyperfini. Cette hypothèse de variable conjuguée Lipschitz inclut le cas de N variables aléatories q-Gaussiennes pour de petits q par exemple $|q|N \leq 0.13$.

MSC: 46L54; 60H15

Keywords: Free stochastic partial differential equations; Lower bounds on microstate free entropy dimension; Free probability; q-Gaussian variables

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Two-parameter non-commutative Central Limit Theorem[1]

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Abstract. In 1992, Speicher showed the fundamental fact that the probability measures playing the role of the classical Gaussian in the various non-commutative probability theories (viz. fermionic probability, Voiculescu's free probability, and q-deformed probability of Bożejko and Speicher) all arise as the limits in a generalized Central Limit Theorem. The latter concerns sequences of non-commutative random variables (elements of a $*$-algebra equipped with a state) drawn from an ensemble of pair-wise commuting or anti-commuting elements, with the respective limiting distributions determined by the average value of the commutation coefficients.

In this paper, we derive a more general form of the Central Limit Theorem in which the pair-wise commutation coefficients are arbitrary real numbers. The classical Gaussian statistics now undergo a second-parameter refinement as a result of controlling for the first *and the second* moments of the commutation coefficients. An application yields the random matrix models for the (q,t)-Gaussian statistics, which were recently shown to have rich connections to operator algebras, special functions, orthogonal polynomials, mathematical physics, and random matrix theory.

Résumé. En 1992, Speicher a montré que les mesures de probabilités jouant le rôle des lois gaussiennes dans les différentes théories des probabilités non-commutatives (probabilités fermioniques, probabilités libres à la Voiculescu, probabilités q-déformées à la Bożejko et Speicher) apparaissent toutes comme limites d'un Théorème de la limite centrale généralisé. Ceci concerne des suites de variables aléatoires non-commutatives (éléments d'une $*$-algèbre munie d'un état) choisies dans un ensemble d'éléments qui commutent ou anti-commutent deux-à-deux, avec les distributions limites respectives déterminées par la valeur moyenne des coefficients de commutation.

Dans ce papier, nous dérivons une forme plus générale du Théorème de la limite centrale où les coefficients de commutation deux-à-deux sont des nombres réels arbitraires. Les statistiques gaussiennes classiques dépendent maintenant d'un second paramètre comme résultat du contrôle du premier *et du second* moment des coefficients de commutation. Une application donne le modèle de matrices aléatoires pour les statistiques (q,t)-gaussiennes, pour lesquelles il a été montré récemment qu'elles ont des profondes connexions avec les algèbres d'opérateurs, les fonctions spéciales, les polynômes orthogonaux, la physique mathématique et la théorie des matrices aléatoires.

MSC: Primary 60F05; 46L50; secondary 60B20; 81S05

Keywords: Central Limit Theorem; Free probability; Random matrices; q-Gaussians

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Universality for random tensors

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Abstract. We prove two universality results for random tensors of arbitrary rank D. We first prove that a random tensor whose entries are N^D independent, identically distributed, complex random variables converges in distribution in the large N limit to the same limit as the distributional limit of a Gaussian tensor model. This generalizes the universality of random matrices to random tensors.

We then prove a second, stronger, universality result. Under the weaker assumption that the joint probability distribution of tensor entries is invariant, assuming that the cumulants of this invariant distribution are uniformly bounded, we prove that in the large N limit the tensor again converges in distribution to the distributional limit of a Gaussian tensor model. We emphasize that the covariance of the large N Gaussian is *not* universal, but depends strongly on the details of the joint distribution.

Résumé. Nous démontrons deux théorèmes d'universalité pour les tenseurs aléatoires de rang D quelconque. Nous montrons d'abord qu'un tenseur aléatoire dont les entrées sont N^D variables complexes indépendantes identiquement distribuées converge en distribution dans la limite N grand vers la même limite que la limite en distribution d'un modèle de tenseurs Gaussien. Cela généralise l'universalité des matrices aléatoires aux tenseurs aléatoires.

Nous démontrons ensuite un deuxième théorème d'universalité, plus fort. Sous l'hypothèse que la distribution de probabilité jointe des entrées du tenseur est invariante, et en supposant que les cumulants de cette distribution invariante sont uniformément bornés, nous prouvons que dans la limite N grand le tenseur converge à nouveau en distribution vers la même limite que la limite en distribution d'un modèle de tenseurs Gaussien. La covariance de la distribution Gaussienne à N grand *n'est pas* universelle, mais dépend des détails de la distribution jointe.

MSC: 60B99; 60F99

Keywords: Random tensors; Large N limit

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