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A REGULARIZED KELLERER THEOREM IN ARBITRARY DIMENSION

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We present a multidimensional extension of Kellerer's theorem on the existence of mimicking Markov martingales for peacocks, a term derived from the French for stochastic processes increasing in convex order. For a continuous-time peacock in arbitrary dimension, after Gaussian regularization, we show that there exists a strongly Markovian mimicking martingale Itô diffusion. A novel compactness result for martingale diffusions is a key tool in our proof. Moreover, we provide counterexamples to show, in dimension $d \geq 2$, that uniqueness may not hold, and that some regularization is necessary to guarantee existence of a mimicking Markov martingale.

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NONLINEAR SEMIGROUPS AND LIMIT THEOREMS FOR CONVEX EXPECTATIONS

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Based on the Chernoff approximation, we provide a general approximation result for convex monotone semigroups which are continuous w.r.t. the mixed topology on suitable spaces of continuous functions. Starting with a family $(I(t))_{t\geq 0}$ of operators, the semigroup is constructed as the limit $S(t)f := \lim_{n\to\infty} I(t/n)^n f$ and is uniquely determined by the time derivative $I'(0)f$ for smooth functions. We identify explicit conditions for the generating family $(I(t))_{t\geq 0}$ that are transferred to the semigroup $(S(t))_{t\geq 0}$ and can easily be verified in applications. Furthermore, there is a structural link between Chernoff type approximations for nonlinear semigroups and law of large numbers and central limit theorem type results for convex expectations. The framework also includes large deviation results.

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STRUCTURAL RESULTS FOR THE TREE BUILDER RANDOM WALK

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We study the tree builder random walk: a randomly growing tree, built by a walker as she is walking around the tree. Namely, at each time n, she adds a leaf to her current vertex with probability $p_n = n^{-\gamma}$, $\gamma \in (2/3, 1]$, then moves to a uniform random neighbor on the possibly modified tree. We show that the tree process at its growth times, after a random finite number of steps, can be coupled to be identical to the Barabási–Albert preferential attachment tree model.

Thus, our TBRW-model is a local dynamics giving rise to the BA-model. The coupling also implies that many properties known for the BA-model, such as diameter and degree distribution, can be directly transferred to our TBRW-model, extending previous results.

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SOLVING MCKEAN–VLASOV SDES VIA RELATIVE ENTROPY

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In this paper we explore the merit of relative entropy in proving weak well-posedness of McKean–Vlasov SDEs and SPDEs, extending the technique introduced in Lacker (*Probab. Math. Phys.* **4** (2023) 377–432). In the SDE setting, we prove weak existence and uniqueness when the interaction is path dependent and only assumed to have linear growth. Meanwhile, we recover and extend the current results when the interaction has Krylov's $L_t^q - L_x^p$ type singularity for $\frac{d}{p} + \frac{2}{q} < 1$, where d is the dimension of space. We connect the aforementioned two cases which are traditionally disparate, and form a solution theory that is sufficiently robust to allow perturbations of sublinear growth at the presence of singularity, giving rise to the well-posedness of a new family of McKean–Vlasov SDEs. Our strategy naturally extends to the cases of a fractional Brownian driving noise B^H for all $H \in (0,1)$, obtaining new results in each separate case $H \in (0, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$. In the SPDE setting, we construct McKean–Vlasov-type SPDEs with bounded measurable coefficients from the prototype of stochastic heat equation in spatial dimension one, and we do the same construction for the stochastic wave equation and a SPDE with white noise acting only on the boundary.

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CONVERGENCE OF DIRICHLET FORMS FOR MCMC OPTIMAL SCALING WITH DEPENDENT TARGET DISTRIBUTIONS ON LARGE GRAPHS

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Markov chain Monte Carlo (MCMC) algorithms have played a significant role in statistics, physics, machine learning and others, and they are the only known general and efficient approach for some high-dimensional problems. The random walk Metropolis (RWM) algorithm as the most classical MCMC algorithm, has had a great influence on the development and practice of science and engineering. The behavior of the RWM algorithm in high-dimensional problems is typically investigated through a weak convergence result of diffusion processes. In this paper, we utilize the Mosco convergence of Dirichlet forms in analyzing the RWM algorithm on large graphs, whose target distribution is the Gibbs measure that includes any probability measure satisfying a Markov property. The abstract and powerful theory of Dirichlet forms allows us to work directly and naturally on the infinite-dimensional space, and our notion of Mosco convergence allows Dirichlet forms associated with the RWM chains to lie on changing Hilbert spaces. Through the optimal scaling problem, we demonstrate the impressive strengths of the Dirichlet form approach over the standard diffusion approach.

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THE ODE METHOD FOR ASYMPTOTIC STATISTICS IN STOCHASTIC APPROXIMATION AND REINFORCEMENT LEARNING

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The paper concerns the stochastic approximation recursion,

$$\theta_{n+1} = \theta_n + \alpha_{n+1} f(\theta_n, \Phi_{n+1}), \quad n \geq 0,$$

where the *estimates* $\{\theta_n\}$ evolve on $\mathbb{R}^d$, and $\boldsymbol{\Phi} := \{\Phi_n\}$ is a stochastic process on a general state space, satisfying a conditional Markov property that allows for parameter-dependent noise. In addition to standard Lipschitz assumptions and conditions on the vanishing step-size sequence, it is assumed that the associated *mean flow* $\frac{d}{dt}\vartheta_t = \overline{f}(\vartheta_t)$ is globally asymptotically stable, with stationary point denoted θ^*. The main results are established under additional conditions on the mean flow and an extension of the Donsker–Varadhan Lyapunov drift condition known as (DV3):

(i) A Lyapunov function is constructed for the joint process $\{\theta_n, \Phi_n\}$ that implies convergence of the estimates in L_4.

(ii) A functional central limit theorem (CLT) is established, as well as the usual one-dimensional CLT for the normalized error. Moment bounds combined with the CLT imply convergence of the normalized covariance $\mathsf{E}[z_n z_n^{\intercal}]$ to the asymptotic covariance Σ_Θ in the CLT, where $z_n := (\theta_n - \theta^*)/\sqrt{\alpha_n}$.

(iii) The CLT holds for the normalized averaged parameters $z_n^{\mathsf{PR}} := \sqrt{n}(\theta_n^{\mathsf{PR}} - \theta^*)$, with $\theta_n^{\mathsf{PR}} := n^{-1}\sum_{k=1}^n \theta_k$, subject to standard assumptions on the step-size. Moreover, the covariance of z_n^{PR} converges to $\Sigma_\Theta^{\mathsf{PR}}$, the minimal covariance of Polyak and Ruppert.

(iv) An example is given where f and $\overline{f}$ are linear in θ, and $\boldsymbol{\Phi}$ is a geometrically ergodic Markov chain but does not satisfy (DV3). While the algorithm is convergent, the second moment of θ_n is unbounded and in fact diverges.

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THE CRITICAL KARP–SIPSER CORE OF RANDOM GRAPHS

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We study the Karp–Sipser core of a random graph made of a configuration model with vertices of degree 1, 2 and 3. This core is obtained by recursively removing the leaves as well as their unique neighbors in the graph. We settle a conjecture of Bauer and Golinelli (*Eur. Phys. J. B* **24** (2001) 339–352) and prove that at criticality, the Karp–Sipser core has size $\approx \mathrm{Cst} \cdot \vartheta^{-2} \cdot n^{3/5}$ where ϑ is the hitting time of the curve $t \mapsto \frac{1}{t^2}$ by a linear Brownian motion started at 0. Our proof relies on a detailed multi-scale analysis of the Markov chain associated to the Karp–Sipser leaf-removal algorithm close to its extinction time.

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UNIFORM CONVERGENCE OF THE FLEMING–VIOT PROCESS IN A HARD KILLING METASTABLE CASE

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We study the long-time convergence of a Fleming–Viot process, in the case where the underlying process is a metastable diffusion killed when it reaches some level set. Through a coupling argument, we establish the long-time convergence of the Fleming–Viot process toward some stationary measure at an exponential rate independent of N, the size of the system, as well as uniform in time propagation of chaos estimates.

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THE ROUGHNESS EXPONENT AND ITS MODEL-FREE ESTIMATION

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Motivated by pathwise stochastic calculus, we say that continuous real-valued function x admits the roughness exponent R if the pth variation of x converges to zero for $p > 1/R$ and to infinity for $p < 1/R$. In our main result, we provide a mild condition on the Faber–Schauder coefficients of x under which the roughness exponent exists and is given as the limit of the classical Gladyshev estimates $\widehat{R}_n(x)$. This result can be viewed as a strong consistency result for the Gladyshev estimators in an entirely model-free setting, because it works strictly trajectory-wise and requires no probabilistic assumptions. Nonetheless, our proof is probabilistic and relies on a martingale hidden in the Faber–Schauder expansion of x. We show that the condition of our main result is satisfied for the typical sample paths of fractional Brownian motion with drift, and we provide almost sure convergence rates for the corresponding Gladyshev estimates. We also discuss the connections between the roughness exponent and the related concepts of Besov regularity and weighted quadratic variation. Since the Gladyshev estimators are not scale-invariant, we construct several scale-invariant estimators. Finally, we extend our results to the case in which the pth variation of x is defined over a sequence of unequally spaced partitions.

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INSIGHT FROM THE KULLBACK–LEIBLER DIVERGENCE INTO ADAPTIVE IMPORTANCE SAMPLING SCHEMES FOR RARE EVENT ANALYSIS IN HIGH DIMENSION

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We study two adaptive importance sampling schemes for estimating the probability of a rare event in the high-dimensional regime $d \to \infty$ with d the dimension. The first scheme is the prominent cross-entropy (CE) method, and the second scheme, motivated by recent results, uses as auxiliary distribution a projection of the optimal auxiliary distribution on a lower-dimensional subspace. In these schemes, two samples are used: the first one to learn the auxiliary distribution and the second one, drawn according to the learned distribution, to perform the final probability estimation. Contrary to the common belief that the sample size needs to grow exponentially in the dimension to make the estimator consistent and avoid the weight degeneracy phenomenon, we find that a polynomial sample size in the first learning step is enough. We prove this result assuming that the sought probability is bounded away from 0. For CE, insight is provided on the polynomial growth rate which remains implicit. In contrast, we study the second scheme in a simple computational framework assuming that samples from the conditional distribution are available. This makes it possible to show that the sample size only needs to grow like rd with r the effective dimension of the projection, which highlights the potential benefits of these projection methods.

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MACROSCOPIC DIFFUSIVE FLUCTUATIONS FOR GENERALIZED HARD RODS DYNAMICS

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We study the fluctuations in equilibrium for a dynamics of rods with random length. This includes the classical hard rod elastic collisions, when rod lengths are constant and equal to a positive value. We prove that in the diffusive space-time scaling, an initial fluctuation of density of particles of velocity v, after recentering on its Euler evolution, evolve randomly shifted by a Brownian motion of variance $\mathcal{D}(v)$.

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SHARP ANALYSIS ON THE JOINT DISTRIBUTION OF THE NUMBER OF DESCENTS AND INVERSE DESCENTS IN A RANDOM PERMUTATION

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Chatterjee and Diaconis have recently shown the asymptotic normality for the joint distribution of the number of descents and inverse descents in a random permutation. A noteworthy point of their results is that the asymptotic variance of the normal distribution is diagonal, which means that the number of descents and inverse descents are asymptotically uncorrelated. The goal of this paper is to go further in this analysis by proving a large deviation principle for the joint distribution. We shall show that the rate function of the joint distribution is the sum of the rate functions of the marginal distributions, which also means that the number of descents and inverse descents are asymptotically independent at the large deviation level. However, we are going to prove that they are finely dependent at the sharp large deviation level.

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CUTOFF PHENOMENON IN NONLINEAR RECOMBINATIONS

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We investigate a quadratic dynamical system known as nonlinear recombinations. This system models the evolution of a probability measure over the Boolean cube, converging to the stationary state obtained as the product of the initial marginals. Our main result reveals a cutoff phenomenon for the total variation distance in both discrete and continuous time. Additionally, we derive the explicit cutoff profiles in the case of monochromatic initial distributions. These profiles are different in the discrete and continuous time settings. The proof leverages a pathwise representation of the solution in terms of a fragmentation process associated to a binary tree. In continuous time, the underlying binary tree is given by a branching random process, thus requiring a more elaborate probabilistic analysis.

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FLUCTUATIONS OF STOCHASTIC PDES WITH LONG-RANGE CORRELATIONS

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We study the large-scale dynamics of the solution to a nonlinear stochastic heat equation (SHE) in dimensions $d \geq 3$ with long-range dependence. This equation is driven by multiplicative Gaussian noise, which is white in time and coloured in space with nonintegrable spatial covariance that decays at the rate of $|x|^{-\kappa}$ at infinity, where $\kappa \in (2, d)$. Inspired by recent studies on SHE and KPZ equations driven by noise with compactly supported spatial correlation, we demonstrate that the correlations persist in the large-scale limit. The fluctuations of the diffusively scaled solution converge to the solution of a stochastic heat equation with additive noise whose correlation is the Riesz kernel of degree $-\kappa$. Moreover, the fluctuations converge as a distribution-valued process in the optimal Hölder topologies.

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A QUASI-STATIONARY APPROACH TO THE LONG-TERM ASYMPTOTICS OF THE GROWTH-FRAGMENTATION EQUATION

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In a growth-fragmentation system, cells grow in size slowly and split apart at random. Typically, the number of cells in the system grows exponentially and the distribution of the sizes of cells settles into an equilibrium "asymptotic profile". In this work we introduce a new method to prove this asymptotic behaviour for the growth-fragmentation equation, and show that the convergence to the asymptotic profile occurs at exponential rate. We do this by identifying an associated sub-Markov process and studying its quasi-stationary behaviour via a Lyapunov function condition. By doing so, we are able to simplify and generalise results in a number of common cases and offer a unified framework for their study. In the course of this work we are also able to prove the existence and uniqueness of solutions to the growth-fragmentation equation in a wide range of situations.

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CONCENTRATION OF CONTRACTIVE STOCHASTIC APPROXIMATION: ADDITIVE AND MULTIPLICATIVE NOISE

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In this paper, we establish maximal concentration bounds for the iterates generated by a stochastic approximation (SA) algorithm under a contractive operator with respect to some arbitrary norm (e.g., the ℓ_∞-norm). We consider two settings where the iterates are potentially unbounded: SA with bounded multiplicative noise and SA with sub-Gaussian additive noise. Our maximal concentration inequalities state that the convergence error has a sub-Gaussian tail in the additive noise setting and a Weibull tail (which is faster than polynomial decay but could be slower than exponential decay) in the multiplicative noise setting. In addition, we provide an impossibility result showing that it is generally impossible to have sub-exponential tails under multiplicative noise. To establish the maximal concentration bounds, we develop a novel bootstrapping argument that involves bounding the moment-generating function of a modified version of the generalized Moreau envelope of the convergence error and constructing an exponential supermartingale to enable using Ville's maximal inequality. We demonstrate the applicability of our theoretical results in the context of linear SA and reinforcement learning.

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REINFORCED GALTON–WATSON PROCESSES II: LARGE TIME BEHAVIORS

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Reinforced Galton–Watson processes have been introduced in (*Random Structures Algorithms* **65** (2024) 387–410) as population models with nonoverlapping generations, such that reproduction events along genealogical lines can be repeated at random. We investigate here some of their sample path properties such as asymptotic growth rates and survival, for which the effects of reinforcement on the evolution appear quite strikingly.

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AVERAGE-CASE AND SMOOTHED ANALYSIS OF GRAPH ISOMORPHISM

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We propose a simple and efficient local algorithm for graph isomorphism which succeeds for a large class of sparse graphs. This algorithm produces a low-depth canonical labeling, which is a labeling of the vertices of the graph that identifies its isomorphism class using vertices' local neighborhoods.

Prior work by Czajka and Pandurangan showed that in the Erdős–Rényi model $G(n, p_n)$, the degree profile of a vertex (i.e., the sorted list of the degrees of its neighbors) gives a canonical labeling with high probability when $np_n = \omega(\log^4(n)/\log\log n)$ (and $p_n \le 1/2$); subsequently, Mossel and Ross showed that the same holds when $np_n = \omega(\log^2(n))$. We first show that their analysis essentially cannot be improved: we prove that when $np_n = o(\log^2(n)/(\log\log n)^3)$, with high probability there exist distinct vertices with isomorphic 2-neighborhoods. Our first main result is a positive counterpart to this, showing that 3-neighborhoods give a canonical labeling when $np_n \ge (1+\delta)\log n$ (and $p_n \le 1/2$); this improves a recent result of Ding, Ma, Wu and Xu, completing the picture above the connectivity threshold.

Our second main result is a smoothed analysis of graph isomorphism, showing that for a large class of deterministic graphs, a small random perturbation of the edge set yields a graph which admits a canonical labeling from 3-neighborhoods, with high probability. While the worst-case complexity of graph isomorphism is still unknown, this shows that graph isomorphism has polynomial smoothed complexity.

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CENTRAL LIMIT THEOREM FOR GRAM-SCHMIDT RANDOM WALK DESIGN

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We prove a central limit theorem for the Horvitz–Thompson estimator based on the Gram–Schmidt walk (GSW) design, recently developed in Harshaw et al. (*J. Amer. Statist. Assoc.* **119** (2024) 2934–2946). In particular, we consider the version of GSW design, which uses a *randomized pivot order*. We deduce our result under very mild assumptions involving *only* the problem parameters, such as the (sum) potential outcome vector and the covariate matrix. As a very important consequence of our analysis, we obtain the precise limiting variance of the estimator in terms of these parameters, which is *smaller* than the previously known upper bound. The main ingredients are a simplified *skeletal* process approximating the GSW design and concentration phenomena for random matrices obtained from random sampling using Stein's method for exchangeable pairs.

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WEAK WELL-POSEDNESS OF STOCHASTIC VOLTERRA EQUATIONS WITH COMPLETELY MONOTONE KERNELS AND NONDEGENERATE NOISE

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We establish weak existence and uniqueness in law for stochastic Volterra equations (SVEs for short) with completely monotone kernels and nondegenerate noise under mild regularity assumptions. In particular, our results reveal the regularization-by-noise effect for SVEs with singular kernels, allowing for multiplicative noise with Hölder diffusion coefficients. In order to prove our results, we reformulate the SVE into an equivalent stochastic evolution equation (SEE for short) defined on a Gelfand triplet of Hilbert spaces. We prove weak well-posedness of the SEE using stochastic control arguments, and then translate it into the original SVE.

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