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Articles

The landscape of the planted clique problem: Dense subgraphs and the overlap gap property . . . . DAVID GAMARNIK AND ILIAS ZADIK 3375
Asymptotic bias of inexact Markov chain Monte Carlo methods in high dimension
ALAIN DURMUS AND ANDREAS EBERLE 3435
The maximal degree in random recursive graphs with random weights
BAS LODEWIJKS AND MARCEL ORTGIESE 3469
Cylinders' percolation: Decoupling and applications
CAIO ALVES AND AUGUSTO TEIXEIRA 3541
Supercritical spatial SIR epidemics: Spreading speed and herd immunity
XINGHUA ZHENG AND QINGSAN ZHU 3584
Coupling from the past for the null recurrent Markov chain
FRANÇOIS BACCELLI, MIR-OMID HAJI-MIRSADEGHI AND SAYEH KHANIHA 3631
A reverse ergodic theorem for inhomogeneous killed Markov chains and application to a new uniqueness result for reflecting diffusions
CRISTINA COSTANTINI AND THOMAS G. KURTZ 3665
Statistical limits of correlation detection in trees
LUCA GANASSALI, LAURENT MASSOULIÉ AND GUILHEM SEMERJIAN 3701
A phase transition in Arrow's theorem with three alternatives
FREDERIC KOEHLER AND ELCHANAN MOSSEL 3735
Repeated averages on graphs
RAMIS MOVASSAGH, MARIO SZEGEDY AND GUANYANG WANG 3781
Metastable Γ-expansion of finite state Markov chains level two large deviations rate functions . . . . L. BERTINI, D. GABRIELLI AND C. LANDIM 3820
Ratio convergence rates for Euclidean first-passage percolation: Applications to the graph infinity Laplacian . . . . LEON BUNGERT, JEFF CALDER AND TIM ROITH 3870
The frog model on Galton–Watson trees . . MARCUS MICHELEN AND JOSH ROSENBERG 3911
Universality of approximate message passing algorithms and tensor networks
TIANHAO WANG, XINYI ZHONG AND ZHOU FAN 3943
Asymptotic probability of energy increasing solutions to the homogeneous Boltzmann equation . . . . GIADA BASILE, DARIO BENEDETTO, LORENZO BERTINI AND EMANUELE CAGLIOTI 3995
Explicit convergence bounds for Metropolis Markov chains: Isoperimetry, spectral gaps and profiles . . . . CHRISTOPHE ANDRIEU, ANTHONY LEE, SAM POWER AND ANDI Q. WANG 4022
On the number of cycles in commutators of random permutations
GUILLAUME DUBACH 4072
Control on Hilbert spaces and application to some mean field type control problems
ALAIN BENSOUSSAN, P. JAMESON GRABER AND SHEUNG CHI PHILLIP YAM 4085
Localization of a one-dimensional simple random walk among power-law renewal obstacles . . . . JULIEN POISAT AND FRANÇOIS SIMENHAUS 4137

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THE LANDSCAPE OF THE PLANTED CLIQUE PROBLEM: DENSE SUBGRAPHS AND THE OVERLAP GAP PROPERTY

BY DAVID GAMARNIK[1,a] AND ILIAS ZADIK[2,b]

[1]*Operations Research Center, Massachusetts Institute of Technology,* [a]*gamarnik@mit.edu*
[2]*Center for Data Science, New York University,* [b]*zadik@nyu.edu*

We study the computational-statistical gap of the planted clique problem, where a clique of size k is planted in an Erdős–Rényi graph $G(n, \frac{1}{2})$. The goal is to recover the planted clique vertices by observing the graph. It is known that the clique can be recovered as long as $k \geq (2+\epsilon)\log n$ for any $\epsilon > 0$, but no polynomial-time algorithm is known for this task unless $k = \Omega(\sqrt{n})$. Following a statistical-physics inspired point of view, as a way to understand the nature of this computational-statistical gap, we study the landscape of the "sufficiently dense" subgraphs of G and their overlap with the planted clique.

Using the first moment method, we present evidence of a phase transition for the presence of the overlap gap property (OGP) at $k = \Theta(\sqrt{n})$. OGP is a concept originating in spin glass theory and known to suggest algorithmic hardness when it appears. We further prove the presence of the OGP when k is a small positive power of n, and therefore, for an exponential-in-n part of the gap, by using a conditional second moment method. As our main technical tool, we establish the first, to the best of our knowledge, concentration results for the K-densest subgraph problem for the Erdős–Rényi model $G(n, \frac{1}{2})$ when $K = n^{0.5-\epsilon}$ for arbitrary $\epsilon > 0$. Our methodology throughout the paper, is based on a certain form of overparametrization, which is conceptually aligned with a large body of recent work in learning theory and optimization.

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ASYMPTOTIC BIAS OF INEXACT MARKOV CHAIN MONTE CARLO METHODS IN HIGH DIMENSION

BY ALAIN DURMUS[1,a] AND ANDREAS EBERLE[2,b]

[1]*Centre de Mathématiques Appliquées CMAP, CNRS, Ecole Polytechnique, Institut Polytechnique de Paris,* [a]*alain.durmus@polytechnique.edu*

[2]*Institut für Angewandte Mathematik, Universität Bonn,* [b]*eberle@uni-bonn.de*

Inexact Markov chain Monte Carlo methods rely on Markov chains that do not exactly preserve the target distribution. Examples include the unadjusted Langevin algorithm (ULA) and unadjusted Hamiltonian Monte Carlo (uHMC). This paper establishes bounds on Wasserstein distances between the invariant probability measures of inexact MCMC methods and their target distributions with a focus on understanding the precise dependence of this asymptotic bias on both dimension and discretization step size. Assuming Wasserstein bounds on the convergence to equilibrium of either the exact or the approximate dynamics, we show that for both ULA and uHMC, the asymptotic bias depends on key quantities related to the target distribution or the stationary probability measure of the scheme. As a corollary, we conclude that for models with a limited amount of interactions such as mean-field models, finite range graphical models, and perturbations thereof, the asymptotic bias has a similar dependence on the step size and the dimension as for product measures.

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THE MAXIMAL DEGREE IN RANDOM RECURSIVE GRAPHS WITH RANDOM WEIGHTS

BY BAS LODEWIJKS[1,a] AND MARCEL ORTGIESE[2,b]

[1]*Department of Mathematics, University of Augsburg,* [a]*bas.lodewijks@uni-a.de*
[2]*Department of Mathematical Sciences, University of Bath,* [b]*m.ortgiese@bath.ac.uk*

We study a generalisation of the random recursive tree (RRT) model and its multigraph counterpart, the uniform directed acyclic graph (DAG). Here, vertices are equipped with a random vertex-weight representing initial inhomogeneities in the network, so that a new vertex connects to one of the old vertices with a probability that is proportional to their vertex-weight. We first identify the asymptotic degree distribution of a uniformly chosen vertex for a general vertex-weight distribution. For the maximal degree, we distinguish several classes that lead to different behaviour: For bounded vertex-weights we obtain results for the maximal degree that are similar to those observed for RRTs and DAGs. If the vertex-weights have unbounded support, then the maximal degree has to satisfy the right balance between having a high vertex-weight and being born early.

For vertex-weights in the Fréchet maximum domain of attraction the first-order behaviour of the maximal degree is random, while for those in the Gumbel maximum domain of attraction the leading order is deterministic. Surprisingly, in the latter case, the second order is random when considering vertices in a compact window in the optimal region, while it becomes deterministic when considering all vertices.

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CYLINDERS' PERCOLATION: DECOUPLING AND APPLICATIONS

BY CAIO ALVES[1,a] AND AUGUSTO TEIXEIRA[2,b]

[1]*Mathematics in Computation Section, Oak Ridge National Laboratory,* [a]*demagalhaesc@ornl.gov*
[2]*IMPA,* [b]*augusto@impa.br*

In this paper we establish a strong decoupling inequality for the cylinder's percolation process introduced by Tykesson and Windisch (*Probab. Theory Related Fields* **154** (2012) 165–191). This model features a very strong dependency structure, making it difficult to study, and this is why such decoupling inequalities are desirable. It is important to notice that the type of dependencies featured by cylinder's percolation is particularly intricate, given that the cylinders have infinite range (unlike some models like Boolean percolation) while at the same time being rigid bodies (unlike processes such as random interlacements). Our work introduces a new notion of fast decoupling, proves that it holds for the model in question and finishes with an application. More precisely, we prove that for a small enough density of cylinders, a random walk on a connected component of the vacant set is transient for all dimensions $d \geq 3$.

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SUPERCRITICAL SPATIAL SIR EPIDEMICS: SPREADING SPEED AND HERD IMMUNITY

BY XINGHUA ZHENG[1,a] AND QINGSAN ZHU[2,b]

[1]*Department of ISOM, Hong Kong University of Science and Technology,* [a]*xhzheng@ust.hk*
[2]*HKUST Jockey Club Institute for Advanced Study, Hong Kong University of Science and Technology,* [b]*iaszhuqs@ust.hk*

We study supercritical spatial SIR epidemics on $\mathbb{Z}^2 \times \{1, 2, \dots, N\}$, where each site in $\mathbb{Z}^2$ represents a village and N stands for the village size. We establish several asymptotic results as $N \to \infty$. In particular, we derive the probability that the epidemic will last forever if the epidemic is started by one infected individual. Moreover, we show that, conditional on that the epidemic lasts forever, the epidemic spreads out linearly in all directions and derive an explicit formula for the spreading speed. Furthermore, we prove that the accumulated proportion of infection converges to a number that is constant over space and find its explicit value. An important message is that if there is no vaccination, then the accumulated proportion of infection can be *much higher* than the vaccination proportion required to prevent sustained spread of infection.

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COUPLING FROM THE PAST FOR THE NULL RECURRENT MARKOV CHAIN

BY FRANÇOIS BACCELLI[1,a], MIR-OMID HAJI-MIRSADEGHI[2,b] AND SAYEH KHANIHA[1,c]

[1]*INRIA/ENS Paris,* [a]*francois.baccelli@ens.fr*
[2]*Department of Mathematics, Sharif University of Technology,* [b]*mirsadeghi@sharif.ir,* [c]*sayeh.khaniha@inria.fr*

The Doeblin graph of a countable state space Markov chain describes the joint pathwise evolutions of the Markov dynamics starting from all possible initial conditions, with two paths coalescing when they reach the same point of the state space at the same time. Its bridge Doeblin subgraph only contains the paths starting from a tagged point of the state space at all possible times. In the irreducible, aperiodic, and positive recurrent case, the following results are known: the bridge Doeblin graph is an infinite tree that is unimodularizable. Moreover, it contains a single bi-infinite path which allows one to build a perfect sample of the stationary state of the Markov chain. The present paper is focused on the null recurrent case. It is shown that when assuming irreducibility and aperiodicity again, the bridge Doeblin graph is either a single infinite tree or a forest made of a countable collection of infinite trees. In the first case, the infinite tree in question has a single end, is not unimodularizable in general, but is always locally unimodular. These key properties are used to study the stationary regime of several measure-valued random dynamics on this bridge Doeblin Tree, which can be seen as pathwise extensions of classical distributional dynamics associated to the Markov chain. This includes the taboo random dynamics, which admits as steady state a random measure with mean measure equal to the invariant measure of the Markov chain, and the potential random dynamics which admits as steady state a locally finite random measure, with a mean measure equal to infinity at every point of the state space. The practical interest of these two random measures is discussed in the context of perfect sampling.

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A REVERSE ERGODIC THEOREM FOR INHOMOGENEOUS KILLED MARKOV CHAINS AND APPLICATION TO A NEW UNIQUENESS RESULT FOR REFLECTING DIFFUSIONS

BY CRISTINA COSTANTINI[1,a] AND THOMAS G. KURTZ[2,b]

[1]*Department of Economic Studies and INdAM Local Unit, University of Chieti–Pescara,* [a]*c.costantini@unich.it*
[2]*Department of Mathematics and Department of Statistics, University of Wisconsin-Madison,* [b]*kurtz@math.wisc.edu*

Bass and Pardoux (*Probab. Theory Related Fields* (1987) **76** 557–572) deduce from the Krein–Rutman theorem a reverse ergodic theorem for a subprobability transition function, which turns out to be a key tool in proving uniqueness of reflecting Brownian motion in cones in Kwon and Williams (*Trans. Amer. Math. Soc* (1991) **32** 739–780) and Taylor and Williams (*Probab. Theory Related Fields* (1993) **96** 283–317). By a different approach, we are able to prove an analogous reverse ergodic theorem for a family of inhomogeneous subprobability transition functions.

This allows us to prove existence and uniqueness for a semimartingale diffusion process with varying, oblique direction of reflection, in a domain with one singular point that can be approximated, near the singular point, by a smooth cone, under natural, easily verifiable geometric conditions.

Along the way we also show that, under our conditions, the parameter α of Kwon and Williams (1991) is strictly less than 1, thus partially extending the results of Williams (*Z. Warsch. Verw. Gebiete* (1985) **69** 161–176) to higher dimension.

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STATISTICAL LIMITS OF CORRELATION DETECTION IN TREES

BY LUCA GANASSALI[1,a], LAURENT MASSOULIÉ[1,b] AND GUILHEM SEMERJIAN[2,c]

[1]*Inria, DI/ENS, PSL Research University,* [a]*luca.ganassali@inria.fr,* [b]*laurent.massoulie@inria.fr*
[2]*Laboratoire de Physique de l'École normale supérieure, ENS, Université PSL, CNRS, Sorbonne Université, Université Paris Cité,* [c]*guilhem.semerjian@phys.ens.fr*

In this paper we address the problem of testing whether two observed trees (t, t') are sampled either independently or from a joint distribution under which they are correlated. This problem, which we refer to as *correlation detection in trees*, plays a key role in the study of graph alignment for two correlated random graphs. Motivated by graph alignment, we investigate the conditions of existence of one-sided tests, that is, tests which have vanishing type I error and nonvanishing power in the limit of large tree depth.

For the correlated Galton–Watson model with Poisson offspring of mean $\lambda > 0$ and correlation parameter $s \in (0, 1)$, we identify a phase transition in the limit of large degrees at $s = \sqrt{\alpha}$, where $\alpha \sim 0.3383$ is Otter's constant. Namely, we prove that no such test exists for $s \leq \sqrt{\alpha}$, and that such a test exists whenever $s > \sqrt{\alpha}$, for λ large enough.

This result sheds new light on the graph alignment problem in the sparse regime (with $O(1)$ average node degrees) and on the performance of the `MPAlign` method studied in (Ganassali, Massoulié and Lelarge (2022), *J. Stat. Mech. Theory Exp.* **2022** (2022)), proving in particular the conjecture of (*J. Stat. Mech. Theory Exp.* **2022** (2022)) that `MPAlign` succeeds in the partial recovery task for correlation parameter $s > \sqrt{\alpha}$ provided the average node degree λ is large enough.

As a byproduct, we identify a new family of orthogonal polynomials for the Poisson–Galton–Watson measure which enjoy remarkable properties. These polynomials may be of independent interest for a variety of problems involving graphs, trees or branching processes, beyond the scope of graph alignment.

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A PHASE TRANSITION IN ARROW'S THEOREM WITH THREE ALTERNATIVES

BY FREDERIC KOEHLER[1,a] AND ELCHANAN MOSSEL[2,b]

[1]*Department of Statistics and Data Science Institute, University of Chicago,* [a]*fkoehler@uchicago.edu*
[2]*Department of Mathematics, Massachusetts Institute of Technology,* [b]*elmos@mit.edu*

Arrow's theorem concerns a fundamental problem in social choice theory: given the individual preferences of members of a group, how can they be aggregated to form rational group preferences? Arrow showed that in an election between three or more candidates, there are situations where any voting rule satisfying a small list of natural "fairness" axioms must produce an apparently irrational intransitive outcome. Furthermore, quantitative versions of Arrow's theorem in the literature show that when voters choose rankings in an i.i.d. fashion, the outcome is intransitive with nonnegligible probability.

It is natural to ask if such a quantitative version of Arrow's theorem holds for non-i.i.d. models. To answer this question, we study Arrow's theorem under a natural non-i.i.d. model of voters inspired by canonical models in statistical physics; indeed, a version of this model was previously introduced by Raffaelli and Marsili in the physics literature. This model has a parameter, temperature, that prescribes the correlation between different voters. We show that the behavior of Arrow's theorem in this model of an election with three alternatives undergoes a striking phase transition: in the entire high temperature regime of the model, a quantitative Arrow's theorem holds showing that the probability of paradox for any voting rule satisfying the axioms is nonnegligible; this is tight because the probability of paradox under pairwise majority goes to zero when approaching the critical temperature, and becomes exponentially small in the number of voters beyond it. We prove this occurs in another natural model of correlated voters and conjecture this phenomena is quite general.

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REPEATED AVERAGES ON GRAPHS

BY RAMIS MOVASSAGH[1,a], MARIO SZEGEDY[2,b] AND GUANYANG WANG[3,c]

[1] *IBM Quantum, MIT-IBM Watson AI Lab,* [a] *ramis@us.ibm.com*
[2] *Department of Computer Science, Rutgers University,* [b] *szegedy@cs.rutgers.edu*
[3] *Department of Statistics, Rutgers University,* [c] *guanyang.wang@rutgers.edu*

Sourav Chatterjee, Persi Diaconis, Allan Sly, and Lingfu Zhang (*Ann. Probab.* **50** (2022) 1–17), prompted by a question of Ramis Movassagh, renewed the study of a process proposed in the early 1980s by Jean Bourgain. A state vector $v \in \mathbb{R}^n$, labeled with the vertices of a connected graph, G, changes in discrete time steps following the simple rule that at each step a random edge (i, j) is picked and v_i and v_j are both replaced by their average $(v_i + v_j)/2$. It is easy to see that the value associated with each vertex converges to $\sum_{i=1}^n v_i/n$. The question focused on understanding the time denoted as $t_{\epsilon,1}$, which represents how quickly will v be ϵ-close to uniform in the L^1 norm in the case of the complete graph, K_n, when v is initialized as a standard basis vector that takes the value 1 on one coordinate, and zeros everywhere else. They have established a sharp cutoff of $\frac{1}{2\log 2} n \log n + O(n\sqrt{\log n})$. Our main result is to prove, that $\frac{(1-\epsilon)}{2\log 2} n \log n - O(n)$ is a general lower bound for all connected graphs on n nodes. We also get sharp magnitude of $t_{\epsilon,1}$ for several important families of graphs, including star, expander, dumbbell, and cycle. In order to establish our results we make several observations about the process, such as the worst case initialization is always a standard basis vector. Our results add to the body of work of (*J. Theoret. Probab.* **2** (1989) 91–100; *Probab. Surv.* **9** (2012) 90–102; *Ann. Appl. Probab.* **33** (2023) 936–971; *Math. Methods Appl. Sci.* **46** (2023) 3583–3596; *SIAM J. Control Optim.* **48** (2009) 33–55), and others. The renewed interest is partly due to an analogy to a question related to the Google's supremacy circuit. For the proof of our main theorem we employ a concept that we call *augmented entropy function* which may find independent interest in the probability theory and computer science communities.

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METASTABLE Γ-EXPANSION OF FINITE STATE MARKOV CHAINS LEVEL TWO LARGE DEVIATIONS RATE FUNCTIONS

BY L. BERTINI[1,a], D. GABRIELLI[2,b] AND C. LANDIM[3,c]

[1]*Dipartimento di Matematica, Università di Roma 'La Sapienza',* [a]*bertini@mat.uniroma1.it*
[2]*DISIM, Università dell'Aquila,* [b]*gabriell@univaq.it*
[3]*IMPA and CNRS UMR 6085,* [c]*landim@impa.br*

We examine two analytical characterisation of the metastable behavior of a sequence of Markov chains. The first one expressed in terms of its transition probabilities, and the second one in terms of its large deviations rate functional.

Consider a sequence of continuous-time Markov chains $(X_t^{(n)} : t \geq 0)$ evolving on a fixed finite state space V. Under a hypothesis on the jump rates, we prove the existence of time-scales $\theta_n^{(p)}$ and probability measures with disjoint supports $\pi_j^{(p)}$, $j \in S_p$, $1 \leq p \leq \mathfrak{q}$, such that (a) $\theta_n^{(1)} \to \infty$, $\theta_n^{(k+1)}/\theta_n^{(k)} \to \infty$, (b) for all $p, x \in V, t > 0$, starting from x, the distribution of $X^{(n)}_{t\theta_n^{(p)}}$ converges, as $n \to \infty$, to a convex combination of the probability measures $\pi_j^{(p)}$. The weights of the convex combination naturally depend on x and t.

Let $\mathscr{I}_n$ be the level two large deviations rate functional for $X_t^{(n)}$, as $t \to \infty$. Under the same hypothesis on the jump rates and assuming, furthermore, that the process is reversible, we prove that $\mathscr{I}_n$ can be written as $\mathscr{I}_n = \mathscr{I}^{(0)} + \sum_{1 \leq p \leq \mathfrak{q}} (1/\theta_n^{(p)}) \mathscr{I}^{(p)}$ for some rate functionals $\mathscr{I}^{(p)}$ which take finite values only at convex combinations of the measures $\pi_j^{(p)}$: $\mathscr{I}^{(p)}(\mu) < \infty$ if, and only if, $\mu = \sum_{j \in S_p} \omega_j \pi_j^{(p)}$ for some probability measure ω in S_p.

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RATIO CONVERGENCE RATES FOR EUCLIDEAN FIRST-PASSAGE PERCOLATION: APPLICATIONS TO THE GRAPH INFINITY LAPLACIAN

BY LEON BUNGERT[1,a], JEFF CALDER[2,b] AND TIM ROITH[3,c]

[1]*Institute of Mathematics, University of Würzburg,* [a]*leon.bungert@uni-wuerzburg.de*
[2]*School of Mathematics, University of Minnesota,* [b]*jwcalder@umn.edu*
[3]*Helmholtz Imaging, Deutsches Elektronen-Synchrotron DESY,* [c]*tim.roith@desy.de*

In this paper we prove the first quantitative convergence rates for the graph infinity Laplace equation for length scales at the connectivity threshold. In the graph-based semisupervised learning community this equation is also known as Lipschitz learning. The graph infinity Laplace equation is characterized by the metric on the underlying space, and convergence rates follow from convergence rates for graph distances. At the connectivity threshold, this problem is related to Euclidean first passage percolation, which is concerned with the Euclidean distance function $d_h(x, y)$ on a homogeneous Poisson point process on $\mathbb{R}^d$, where admissible paths have step size at most $h > 0$. Using a suitable regularization of the distance function and subadditivity we prove that $d_{h_s}(0, se_1)/s \to \sigma$ as $s \to \infty$ almost surely where $\sigma \geq 1$ is a dimensional constant and $h_s \gtrsim \log(s)^{1/d}$. A convergence rate is not available due to a lack of approximate superadditivity when $h_s \to \infty$. Instead, we prove convergence rates for the *ratio* $\frac{d_h(0,se_1)}{d_h(0,2se_1)} \to \frac{1}{2}$ when h is frozen and does not depend on s. Combining this with the techniques that we developed in (*IMA J. Numer. Anal.* **43** (2023) 2445–2495), we show that this notion of ratio convergence is sufficient to establish uniform convergence rates for solutions of the graph infinity Laplace equation at percolation length scales.

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THE FROG MODEL ON GALTON–WATSON TREES

BY MARCUS MICHELEN[a] AND JOSH ROSENBERG[b]

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, [a]*michelen.math@gmail.com,* [b]*joshrsix@hotmail.com*

We consider an interacting particle system on trees known as the frog model: initially, a single active particle begins at the root and i.i.d. Poiss(λ) many inactive particles are placed at each nonroot vertex. Active particles perform discrete time simple random walk and activate the inactive particles they encounter. We show that for Galton–Watson trees with offspring distributions Z satisfying $\mathbf{P}(Z \geq 2) = 1$ and $\mathbf{E}[Z^{4+\varepsilon}] < \infty$ for some $\varepsilon > 0$, there is a critical value $\lambda_c \in (0, \infty)$ separating recurrent and transient regimes for almost surely every tree, thereby answering a question of Hoffman–Johnson–Junge. In addition, we also establish that this critical parameter depends on the entire offspring distribution, not just the maximum value of Z, answering another question of Hoffman–Johnson–Junge and showing that the frog model and contact process behave differently on Galton–Watson trees.

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UNIVERSALITY OF APPROXIMATE MESSAGE PASSING ALGORITHMS AND TENSOR NETWORKS

BY TIANHAO WANG[a], XINYI ZHONG[b] AND ZHOU FAN[c]

Department of Statistics and Data Science, Yale University, [a]*tianhao.wang@yale.edu,* [b]*xinyi.zhong@yale.edu,* [c]*zhou.fan@yale.edu*

Approximate message passing (AMP) algorithms provide a valuable tool for studying mean-field approximations and dynamics in a variety of applications. Although these algorithms are often first derived for matrices having independent Gaussian entries or satisfying rotational invariance in law, their state evolution characterizations are expected to hold over larger universality classes of random matrix ensembles.

We develop several new results on AMP universality. For AMP algorithms tailored to independent Gaussian entries, we show that their state evolutions hold over broadly defined generalized Wigner and white noise ensembles, including matrices with heavy-tailed entries and heterogeneous entrywise variances that may arise in data applications. For AMP algorithms tailored to rotational invariance in law, we show that their state evolutions hold over delocalized sign-and-permutation-invariant matrix ensembles that have a limit distribution over the diagonal, including sensing matrices composed of subsampled Hadamard or Fourier transforms and diagonal operators.

We establish these results via a simplified moment-method proof, reducing AMP universality to the study of products of random matrices and diagonal tensors along a tensor network. As a by-product of our analyses, we show that the aforementioned matrix ensembles satisfy a notion of asymptotic freeness with respect to such tensor networks, which parallels usual definitions of freeness for traces of matrix products.

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ASYMPTOTIC PROBABILITY OF ENERGY INCREASING SOLUTIONS TO THE HOMOGENEOUS BOLTZMANN EQUATION

BY GIADA BASILE[a], DARIO BENEDETTO[b], LORENZO BERTINI[c] AND EMANUELE CAGLIOTI[d]

Dipartimento di Matematica, Università di Roma 'La Sapienza', [a]*basile@mat.uniroma1.it,* [b]*benedetto@mat.uniroma1.it,* [c]*bertini@mat.uniroma1.it,* [d]*caglioti@mat.uniroma1.it*

Weak solutions to the homogeneous Boltzmann equation with increasing energy have been constructed by Lu and Wennberg. We consider an underlying microscopic stochastic model with binary collisions (Kac's model) and show that these solutions are atypical. More precisely, we prove that the probability of observing these paths is exponentially small in the number of particles and compute the exponential rate. This result is obtained by improving the established large deviation estimates in the canonical setting. Key ingredients are the extension of Sanov's theorem to the microcanonical ensemble and large deviations for the Kac's model in the microcanonical setting.

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EXPLICIT CONVERGENCE BOUNDS FOR METROPOLIS MARKOV CHAINS: ISOPERIMETRY, SPECTRAL GAPS AND PROFILES

BY CHRISTOPHE ANDRIEU[1,a], ANTHONY LEE[1,b], SAM POWER[1,c] AND ANDI Q. WANG[2,d]

[1]*School of Mathematics, University of Bristol,* [a]*c.andrieu@bristol.ac.uk,* [b]*anthony.lee@bristol.ac.uk,* [c]*sam.power@bristol.ac.uk*

[2]*Department of Statistics, University of Warwick,* [d]*andi.wang@warwick.ac.uk*

We derive the first explicit bounds for the spectral gap of a random walk Metropolis algorithm on $\mathbb{R}^d$ for any value of the proposal variance, which when scaled appropriately recovers the correct d^{-1} dependence on dimension for suitably regular invariant distributions. We also obtain explicit bounds on the L^2-mixing time for a broad class of models. In obtaining these results, we refine the use of isoperimetric profile inequalities to obtain conductance profile bounds, which also enable the derivation of explicit bounds in a much broader class of models. We also obtain similar results for the preconditioned Crank–Nicolson Markov chain, obtaining dimension-independent bounds under suitable assumptions.

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ON THE NUMBER OF CYCLES IN COMMUTATORS OF RANDOM PERMUTATIONS

BY GUILLAUME DUBACH[a]

CMLS, École Polytechnique, [a]*guillaume.dubach@polytechnique.edu*

We present general links between statistics of non-Hermitian random matrices and the distribution of the number of cycles of some specific random permutations. In particular, we derive explicit formulas for the generating functions of the number of cycles in the commutator $[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1}$ where σ is uniformly distributed, and τ is either one cycle, the product of many transpositions, or the product of two cycles of same size, the latter case being a new result.

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CONTROL ON HILBERT SPACES AND APPLICATION TO SOME MEAN FIELD TYPE CONTROL PROBLEMS

BY ALAIN BENSOUSSAN[1,a], P. JAMESON GRABER[2,b] AND SHEUNG CHI PHILLIP YAM[3,c]

[1]*International Center for Decision and Risk Analysis, Jindal School of Management, University of Texas at Dallas,* [a]*axb046100@utdallas.edu*

[2]*Department of Mathematics, Baylor University,* [b]*jameson_graber@baylor.edu*

[3]*Department of Statistics, Chinese University of Hong Kong,* [c]*scpyam@cuhk.edu.hk*

We propose a new approach to studying classical solutions of the second order Bellman equation and master equation for mean field type control problems, using a novel form of the "lifting" idea introduced by P.-L. Lions. Rather than studying the usual system of Hamilton–Jacobi/Fokker–Planck PDEs using analytic techniques, we instead study a stochastic control problem on a specially constructed Hilbert space, which is reminiscent of a tangent space on the Wasserstein space in optimal transport. On this Hilbert space we can use classical control theory techniques, despite the fact that it is infinite-dimensional. A consequence of our construction is that the mean field type control problem appears as a special case. Thus we preserve the advantages of the lifting procedure, while removing some of the difficulties. Our approach extends previous work by two of the coauthors, which dealt with a deterministic control problem for which the Hilbert space could be generic (*ESAIM Control Optim. Calc. Var.* **25** (2019) 1–36).

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LOCALIZATION OF A ONE-DIMENSIONAL SIMPLE RANDOM WALK AMONG POWER-LAW RENEWAL OBSTACLES

BY JULIEN POISAT[a] AND FRANÇOIS SIMENHAUS[b]

Université Paris-Dauphine, CNRS, UMR 7534, CEREMADE, PSL Research University, [a]*poisat@ceremade.dauphine.fr,* [b]*simenhaus@ceremade.dauphine.fr*

We consider a one-dimensional simple random walk killed by quenched soft obstacles. The position of the obstacles is drawn according to a renewal process with a power-law increment distribution. In a previous work, we computed the large-time asymptotics of the quenched survival probability. In the present work we continue our study by describing the behaviour of the random walk conditioned to survive. We prove that with large probability, the walk quickly reaches a unique time-dependent optimal gap that is free from obstacles and gets localized there. We actually establish a dichotomy. If the renewal tail exponent is smaller than one then the walk hits the optimal gap and spends all of its remaining time inside, up to finitely many visits to the bottom of the gap. If the renewal tail exponent is larger than one then the random walk spends most of its time inside of the optimal gap but also performs short outward excursions, for which we provide matching upper and lower bounds on their length and cardinality. Our key tools include a Markov renewal interpretation of the survival probability as well as various comparison arguments for obstacle environments. Our results may also be rephrased in terms of localization properties for a directed polymer among multiple repulsive interfaces.

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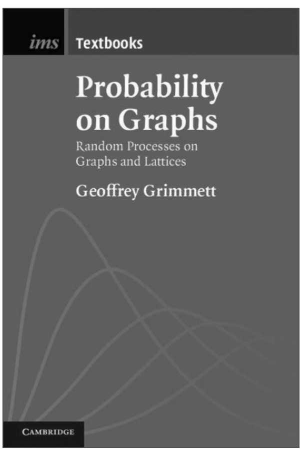

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