ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

continued

Vol. 34, No. 3—June 2024

ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles—*Continued from front cover*

Corrections

Vol. 34, No. 3—June 2024

THE ANNALS OF APPLIED PROBABILITY Vol. 34, No. 3, pp. 2539–3374 June 2024

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.

***The Annals of Applied Probability* [ISSN 1050-5164 (print); ISSN 2168-8737 (online)],** Volume 34, Number 3, June 2024. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, Ohio 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Applied Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2539–2565
https://doi.org/10.1214/22-AAP1910

QUANTITATIVE UNIVERSALITY FOR THE LARGEST EIGENVALUE OF SAMPLE COVARIANCE MATRICES

BY HAOYU WANG[a]

Department of Mathematics, Yale University, [a]*haoyu.wang@yale.edu*

We prove the first explicit rate of convergence to the Tracy–Widom distribution for the fluctuation of the largest eigenvalue of sample covariance matrices that are not integrable. Our primary focus is matrices of type X^*X and the proof follows the Erdös–Schlein–Yau dynamical method. We use a recent approach to the analysis of the Dyson Brownian motion from (*J. Eur. Math. Soc. (JEMS)* **24** (2022) 2823–2873) to obtain a quantitative error estimate for the local relaxation flow at the edge. Together with a quantitative version of the Green function comparison theorem, this gives the rate of convergence.

Combined with a result of Lee–Schnelli (*Ann. Appl. Probab.* **26** (2016) 3786–3839), some quantitative estimates also hold for more general separable sample covariance matrices $X^*\Sigma X$ with general diagonal population Σ.

REFERENCES

[1] ALT, J., ERDŐS, L. and KRÜGER, T. (2017). Local law for random Gram matrices. *Electron. J. Probab.* **22** Paper No. 25, 41. MR3622895 https://doi.org/10.1214/17-EJP42

[2] BAO, Z., PAN, G. and ZHOU, W. (2015). Universality for the largest eigenvalue of sample covariance matrices with general population. *Ann. Statist.* **43** 382–421. MR3311864 https://doi.org/10.1214/14-AOS1281

[3] BLOEMENDAL, A., ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2014). Isotropic local laws for sample covariance and generalized Wigner matrices. *Electron. J. Probab.* **19** no. 33, 53. MR3183577 https://doi.org/10.1214/ejp.v19-3054

[4] BLOEMENDAL, A., KNOWLES, A., YAU, H.-T. and YIN, J. (2016). On the principal components of sample covariance matrices. *Probab. Theory Related Fields* **164** 459–552. MR3449395 https://doi.org/10.1007/s00440-015-0616-x

[5] BOURGADE, P. (2022). Extreme gaps between eigenvalues of Wigner matrices. *J. Eur. Math. Soc.* (*JEMS*) **24** 2823–2873. MR4416591 https://doi.org/10.4171/jems/1141

[6] BOURGADE, P., ERDŐS, L., YAU, H.-T. and YIN, J. (2016). Fixed energy universality for generalized Wigner matrices. *Comm. Pure Appl. Math.* **69** 1815–1881. MR3541852 https://doi.org/10.1002/cpa.21624

[7] BOURGADE, P. and YAU, H.-T. (2017). The eigenvector moment flow and local quantum unique ergodicity. *Comm. Math. Phys.* **350** 231–278. MR3606475 https://doi.org/10.1007/s00220-016-2627-6

[8] BRU, M.-F. (1989). Diffusions of perturbed principal component analysis. *J. Multivariate Anal.* **29** 127–136. MR0991060 https://doi.org/10.1016/0047-259X(89)90080-8

[9] CACCIAPUOTI, C., MALTSEV, A. and SCHLEIN, B. (2013). Local Marchenko–Pastur law at the hard edge of sample covariance matrices. *J. Math. Phys.* **54** 043302, 13. MR3088802 https://doi.org/10.1063/1.4801856

[10] CHE, Z. and LOPATTO, P. (2019). Universality of the least singular value for sparse random matrices. *Electron. J. Probab.* **24** Paper No. 9, 53. MR3916329 https://doi.org/10.1214/19-EJP269

[11] DING, X. and YANG, F. (2018). A necessary and sufficient condition for edge universality at the largest singular values of covariance matrices. *Ann. Appl. Probab.* **28** 1679–1738. MR3809475 https://doi.org/10.1214/17-AAP1341

[12] EL KAROUI, N. (2006). A rate of convergence result for the largest eigenvalue of complex white Wishart matrices. *Ann. Probab.* **34** 2077–2117. MR2294977 https://doi.org/10.1214/009117906000000502

MSC2020 subject classifications. Primary 60B20; secondary 15B52.
Key words and phrases. Sample covariance matrix, Tracy–Widom distribution, universality, Dyson Brownian motion.

[13] EL KAROUI, N. (2007). Tracy–Widom limit for the largest eigenvalue of a large class of complex sample covariance matrices. *Ann*. *Probab*. **35** 663–714. MR2308592 https://doi.org/10.1214/009117906000000917

[14] ERDŐS, L., SCHLEIN, B. and YAU, H.-T. (2011). Universality of random matrices and local relaxation flow. *Invent*. *Math*. **185** 75–119. MR2810797 https://doi.org/10.1007/s00222-010-0302-7

[15] ERDŐS, L., SCHLEIN, B., YAU, H.-T. and YIN, J. (2012). The local relaxation flow approach to universality of the local statistics for random matrices. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **48** 1–46. MR2919197 https://doi.org/10.1214/10-AIHP388

[16] ERDŐS, L. and YAU, H.-T. (2015). Gap universality of generalized Wigner and β-ensembles. *J*. *Eur*. *Math*. *Soc*. (*JEMS*) **17** 1927–2036. MR3372074 https://doi.org/10.4171/JEMS/548

[17] ERDŐS, L. and YAU, H.-T. (2017). *A Dynamical Approach to Random Matrix Theory*. *Courant Lecture Notes in Mathematics* **28**. Courant Institute of Mathematical Sciences, New York, Amer. Math. Soc., Providence, RI. MR3699468

[18] ERDŐS, L., YAU, H.-T. and YIN, J. (2011). Universality for generalized Wigner matrices with Bernoulli distribution. *J*. *Comb*. **2** 15–81. MR2847916 https://doi.org/10.4310/JOC.2011.v2.n1.a2

[19] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Bulk universality for generalized Wigner matrices. *Probab*. *Theory Related Fields* **154** 341–407. MR2981427 https://doi.org/10.1007/s00440-011-0390-3

[20] HUANG, J. and LANDON, B. (2019). Rigidity and a mesoscopic central limit theorem for Dyson Brownian motion for general β and potentials. *Probab*. *Theory Related Fields* **175** 209–253. MR4009708 https://doi.org/10.1007/s00440-018-0889-y

[21] JOHANSSON, K. (2000). Shape fluctuations and random matrices. *Comm*. *Math*. *Phys*. **209** 437–476. MR1737991 https://doi.org/10.1007/s002200050027

[22] JOHNSTONE, I. M. (2001). On the distribution of the largest eigenvalue in principal components analysis. *Ann*. *Statist*. **29** 295–327. MR1863961 https://doi.org/10.1214/aos/1009210544

[23] KNOWLES, A. and YIN, J. (2017). Anisotropic local laws for random matrices. *Probab*. *Theory Related Fields* **169** 257–352. MR3704770 https://doi.org/10.1007/s00440-016-0730-4

[24] LANDON, B., SOSOE, P. and YAU, H.-T. (2019). Fixed energy universality of Dyson Brownian motion. *Adv*. *Math*. **346** 1137–1332. MR3914908 https://doi.org/10.1016/j.aim.2019.02.010

[25] LEE, J. O. and SCHNELLI, K. (2015). Edge universality for deformed Wigner matrices. *Rev*. *Math*. *Phys*. **27** 1550018, 94. MR3405746 https://doi.org/10.1142/S0129055X1550018X

[26] LEE, J. O. and SCHNELLI, K. (2016). Tracy–Widom distribution for the largest eigenvalue of real sample covariance matrices with general population. *Ann*. *Appl*. *Probab*. **26** 3786–3839. MR3582818 https://doi.org/10.1214/16-AAP1193

[27] MA, Z. (2012). Accuracy of the Tracy–Widom limits for the extreme eigenvalues in white Wishart matrices. *Bernoulli* **18** 322–359. MR2888709 https://doi.org/10.3150/10-BEJ334

[28] PÉCHÉ, S. (2009). Universality results for the largest eigenvalues of some sample covariance matrix ensembles. *Probab*. *Theory Related Fields* **143** 481–516. MR2475670 https://doi.org/10.1007/s00440-007-0133-7

[29] PILLAI, N. S. and YIN, J. (2012). Edge universality of correlation matrices. *Ann*. *Statist*. **40** 1737–1763. MR3015042 https://doi.org/10.1214/12-AOS1022

[30] PILLAI, N. S. and YIN, J. (2014). Universality of covariance matrices. *Ann*. *Appl*. *Probab*. **24** 935–1001. MR3199978 https://doi.org/10.1214/13-AAP939

[31] SHORACK, G. R. and WELLNER, J. A. (2009). *Empirical Processes with Applications to Statistics*. *Classics in Applied Mathematics* **59**. SIAM, Philadelphia, PA. Reprint of the 1986 original [MR0838963]. MR3396731 https://doi.org/10.1137/1.9780898719017.ch1

[32] SOSHNIKOV, A. (2002). A note on universality of the distribution of the largest eigenvalues in certain sample covariance matrices. *J*. *Stat*. *Phys*. **108** 1033–1056. MR1933444 https://doi.org/10.1023/A:1019739414239

[33] TAO, T. and VU, V. (2012). Random covariance matrices: Universality of local statistics of eigenvalues. *Ann*. *Probab*. **40** 1285–1315. MR2962092 https://doi.org/10.1214/11-AOP648

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2566–2599
https://doi.org/10.1214/23-AAP1942

STOCHASTIC INTEGRATION WITH RESPECT TO ARBITRARY COLLECTIONS OF CONTINUOUS SEMIMARTINGALES AND APPLICATIONS TO MATHEMATICAL FINANCE

BY CONSTANTINOS KARDARAS[a]

Department of Statistics, London School of Economics and Political Science, [a]*k.kardaras@lse.ac.uk*

Stochastic integrals are defined with respect to a collection $P = (P_i; i \in I)$ of continuous semimartingales, imposing no assumptions on the index set I and the subspace of $\mathbb{R}^I$ where P takes values. The integrals are constructed though finite-dimensional approximation, identifying the appropriate local geometry that allows extension to infinite dimensions. For local martingale integrators, the resulting space $\mathsf{S}(P)$ of stochastic integrals has an operational characterisation via a corresponding set of integrands $\mathsf{R}(C)$, constructed with only reference to the covariation structure C of P. This bijection between $\mathsf{R}(C)$ and the (closed in the semimartingale topology) set $\mathsf{S}(P)$ extends to families of continuous semimartingale integrators for which the drift process of P belongs to $\mathsf{R}(C)$. In the context of infinite-asset models in mathematical finance, the latter structural condition is equivalent to a certain natural form of market viability. The enriched class of wealth processes via extended stochastic integrals leads to exact analogues of optional decomposition and hedging duality as the finite-asset case. A corresponding characterisation of market completeness in this setting is provided.

REFERENCES

[1] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite Dimensional Analysis*: *A Hitchhiker's Guide*, 3rd ed. Springer, Berlin. MR2378491

[2] ARONSZAJN, N. (1950). Theory of reproducing kernels. *Trans*. *Amer*. *Math*. *Soc*. **68** 337–404. MR0051437 https://doi.org/10.2307/1990404

[3] BERLINET, A. and THOMAS-AGNAN, C. (2004). *Reproducing Kernel Hilbert Spaces in Probability and Statistics*. Kluwer Academic, Boston, MA. With a preface by Persi Diaconis. MR2239907 https://doi.org/10.1007/978-1-4419-9096-9

[4] BJÖRK, T., DI MASI, G., KABANOV, Y. and RUNGGALDIER, W. (1997). Towards a general theory of bond markets. *Finance Stoch*. **1** 141–174. MR2976683 https://doi.org/10.1007/s007800050020

[5] CARMONA, R. A. and TEHRANCHI, M. R. (2006). *Interest Rate Models*: *An Infinite Dimensional Stochastic Analysis Perspective*. *Springer Finance*. Springer, Berlin. MR2235463

[6] CHERNY, A. and SHIRYAEV, A. (2005). On stochastic integrals up to infinity and predictable criteria for integrability. In *Séminaire de Probabilités XXXVIII*. *Lecture Notes in Math*. **1857** 165–185. Springer, Berlin. MR2126973 https://doi.org/10.1007/978-3-540-31449-3_12

[7] CUCHIERO, C., KLEIN, I. and TEICHMANN, J. (2016). A new perspective on the fundamental theorem of asset pricing for large financial markets. *Theory Probab*. *Appl*. **60** 561–579. MR3583448 https://doi.org/10.1137/S0040585X97T987879

[8] DA PRATO, G. and ZABCZYK, J. (2014). *Stochastic Equations in Infinite Dimensions*, 2nd ed. *Encyclopedia of Mathematics and Its Applications* **152**. Cambridge Univ. Press, Cambridge. MR3236753 https://doi.org/10.1017/CBO9781107295513

[9] DE DONNO, M., GUASONI, P. and PRATELLI, M. (2005). Super-replication and utility maximization in large financial markets. *Stochastic Process*. *Appl*. **115** 2006–2022. MR2178505 https://doi.org/10.1016/j.spa.2005.06.010

[10] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math*. *Ann*. **300** 463–520. MR1304434 https://doi.org/10.1007/BF01450498

MSC2020 subject classifications. 60H05, 91G10.

Key words and phrases. Infinite-dimensional stochastic integration, continuous semimartingales, mathematical finance, fundamental theorem.

[11] El Karoui, N. and Quenez, M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim*. **33** 29–66. MR1311659 https://doi.org/10.1137/S0363012992232579

[12] Emery, M. (1979). Une topologie sur l'espace des semimartingales. In *Séminaire de Probabilités*, *XIII* (*Univ. Strasbourg*, *Strasbourg*, 1977/78). *Lecture Notes in Math*. **721** 260–280. Springer, Berlin. MR0544800

[13] Föllmer, H. and Kramkov, D. (1997). Optional decompositions under constraints. *Probab. Theory Related Fields* **109** 1–25. MR1469917 https://doi.org/10.1007/s004400050122

[14] Harrison, J. M. and Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process*. *Appl*. **11** 215–260. MR0622165 https://doi.org/10.1016/0304-4149(81)90026-0

[15] Heath, D., Jarrow, R. and Morton, A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica* **60** 77–105.

[16] Heath, D., Jarrow, R. and Morton, A. (2015). Optional decomposition for continuous semimartingales under arbitrary filtrations. *Electron*. *Commun*. *Probab*. **20**. https://doi.org/10

[17] Kabanov, Y., Kardaras, C. and Song, S. (2016). No arbitrage of the first kind and local martingale numéraires. *Finance Stoch*. **20** 1097–1108. MR3551861 https://doi.org/10.1007/s00780-016-0310-6

[18] Kabanov, Yu. M. (1997). On the FTAP of Kreps–Delbaen–Schachermayer. In *Statistics and Control of Stochastic Processes* (*Moscow*, 1995/1996) 191–203. World Sci. Publ., River Edge, NJ. MR1647282

[19] Kabanov, Yu. M. and Kramkov, D. O. (1994). Large financial markets: Asymptotic arbitrage and contiguity. *Teor. Veroyatn*. *Primen*. **39** 222–229. MR1348197 https://doi.org/10.1137/1139009

[20] Kabanov, Yu. M. and Kramkov, D. O. (1998). Asymptotic arbitrage in large financial markets. *Finance Stoch*. **2** 143–172. MR1806101 https://doi.org/10.1007/s007800050036

[21] Karatzas, I. and Kardaras, C. (2007). The numéraire portfolio in semimartingale financial models. *Finance Stoch*. **11** 447–493. MR2335830 https://doi.org/10.1007/s00780-007-0047-3

[22] Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[23] Kardaras, C. (2010). Finitely additive probabilities and the fundamental theorem of asset pricing. In *Contemporary Quantitative Finance* 19–34. Springer, Berlin. MR2732838 https://doi.org/10.1007/978-3-642-03479-4_2

[24] Kardaras, C. and Platen, E. (2011). On the semimartingale property of discounted asset-price processes. *Stochastic Process*. *Appl*. **121** 2678–2691. MR2832419 https://doi.org/10.1016/j.spa.2011.06.010

[25] Kramkov, D. and Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann*. *Appl*. *Probab*. **9** 904–950. MR1722287 https://doi.org/10.1214/aoap/1029962818

[26] Kramkov, D. and Schachermayer, W. (2003). Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann*. *Appl*. *Probab*. **13** 1504–1516. MR2023886 https://doi.org/10.1214/aoap/1069786508

[27] Kramkov, D. O. (1996). Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probab. Theory Related Fields* **105** 459–479. MR1402653 https://doi.org/10.1007/BF01191909

[28] Métivier, M. (1982). *Semimartingales*: *A Course on Stochastic Processes*. *De Gruyter Studies in Mathematics* **2**. de Gruyter, Berlin. MR0688144

[29] Mikulevicius, R. and Rozovskii, B. L. (1998). Normalized stochastic integrals in topological vector spaces. In *Séminaire de Probabilités*, *XXXII*. *Lecture Notes in Math*. **1686** 137–165. Springer, Berlin. MR1655149 https://doi.org/10.1007/BFb0101756

[30] Mostovyi, O. (2015). Necessary and sufficient conditions in the problem of optimal investment with intermediate consumption. *Finance Stoch*. **19** 135–159. MR3292127 https://doi.org/10.1007/s00780-014-0248-5

[31] Paulsen, V. I. and Raghupathi, M. (2016). *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*. *Cambridge Studies in Advanced Mathematics* **152**. Cambridge Univ. Press, Cambridge. MR3526117 https://doi.org/10.1017/CBO9781316219232

[32] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[33] Schweizer, M. (1995). On the minimal martingale measure and the Föllmer–Schweizer decomposition. *Stoch*. *Anal*. *Appl*. **13** 573–599. MR1353193 https://doi.org/10.1080/07362999508809418

[34] STRICKER, C. and YAN, J. A. (1998). Some remarks on the optional decomposition theorem. In *Séminaire de Probabilités, XXXII*. *Lecture Notes in Math*. **1686** 56–66. Springer, Berlin. MR1651229 https://doi.org/10.1007/BFb0101750

[35] TAKAOKA, K. and SCHWEIZER, M. (2014). A note on the condition of no unbounded profit with bounded risk. *Finance Stoch*. **18** 393–405. MR3177411 https://doi.org/10.1007/s00780-014-0229-8

[36] VERAAR, M. (2012). The stochastic Fubini theorem revisited. *Stochastics* **84** 543–551. MR2966093 https://doi.org/10.1080/17442508.2011.618883

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2600–2649
https://doi.org/10.1214/23-AAP2002

STATISTICAL INFERENCE FOR ROUGH VOLATILITY: CENTRAL LIMIT THEOREMS

BY CARSTEN H. CHONG[1,a], MARC HOFFMANN[2,b], YANGHUI LIU[3,c], MATHIEU ROSENBAUM[4,d] AND GRÉGOIRE SZYMANSKI[4,e]

[1]*Department of Information Systems, Business Statistics and Operations Management, The Hong Kong University of Science and Technology,* [a]*carstenchong@ust.hk*

[2]*CEREMADE, Université Paris Dauphine-PSL,* [b]*hoffmann@ceremade.dauphine.fr*

[3]*Department of Mathematics, Baruch College CUNY,* [c]*yanghui.liu@baruch.cuny.edu*

[4]*CMAP, École Polytechnique,* [d]*mathieu.rosenbaum@polytechnique.edu,* [e]*gregoire.szymanski@polytechnique.edu*

In recent years, there has been a substantive interest in rough volatility models. In this class of models, the local behavior of stochastic volatility is much more irregular than semimartingales and resembles that of a fractional Brownian motion with Hurst parameter $H < 0.5$. In this paper, we derive a consistent and asymptotically mixed normal estimator of H based on high-frequency price observations. In contrast to previous works, we work in a semiparametric setting and do not assume any a priori relationship between volatility estimators and true volatility. Furthermore, our estimator attains a rate of convergence that is known to be optimal in a minimax sense in parametric rough volatility models.

REFERENCES

[1] ABI JABER, E. and EL EUCH, O. (2019). Multifactor approximation of rough volatility models. *SIAM J. Financial Math*. **10** 309–349. MR3934104 https://doi.org/10.1137/18M1170236

[2] AÏT-SAHALIA, Y., FAN, J., LAEVEN, R. J. A., WANG, C. D. and YANG, X. (2017). Estimation of the continuous and discontinuous leverage effects. *J. Amer. Statist. Assoc*. **112** 1744–1758. MR3750896 https://doi.org/10.1080/01621459.2016.1240082

[3] AÏT-SAHALIA, Y., FAN, J. and LI, Y. (2013). The leverage effect puzzle: Disentangling sources of bias at high frequency. *J. Financ. Econ*. **109** 224–249.

[4] AÏT-SAHALIA, Y. and JACOD, J. (2009). Testing for jumps in a discretely observed process. *Ann. Statist*. **37** 184–222. MR2488349 https://doi.org/10.1214/07-AOS568

[5] AÏT-SAHALIA, Y. and JACOD, J. (2014). *High-Frequency Financial Econometrics*. Princeton Univ. Press, Princeton.

[6] ANDERSEN, T. G., BOLLERSLEV, T. and DIEBOLD, F. X. (2007). Roughing it up: Including jump components in the measurement, modeling, and forecasting of return volatility. *Rev. Econ. Stat*. **89** 701–720.

[7] ANDERSEN, T. G., BOLLERSLEV, T., DIEBOLD, F. X. and LABYS, P. (2003). Modeling and forecasting realized volatility. *Econometrica* **71** 579–625. MR1958138 https://doi.org/10.1111/1468-0262.00418

[8] BANDI, F. M. and RENÒ, R. (2012). Time-varying leverage effects. *J. Econometrics* **169** 94–113. MR2935266 https://doi.org/10.1016/j.jeconom.2012.01.010

[9] BARNDORFF-NIELSEN, O. E., CORCUERA, J. M. and PODOLSKIJ, M. (2011). Multipower variation for Brownian semistationary processes. *Bernoulli* **17** 1159–1194. MR2854768 https://doi.org/10.3150/10-BEJ316

[10] BARNDORFF-NIELSEN, O. E., CORCUERA, J. M. and PODOLSKIJ, M. (2013). Limit theorems for functionals of higher order differences of Brownian semi-stationary processes. In *Prokhorov and Contemporary Probability Theory*. *Springer Proc. Math. Stat*. **33** 69–96. Springer, Heidelberg. MR3070467 https://doi.org/10.1007/978-3-642-33549-5_4

[11] BAYER, C., FRIZ, P. and GATHERAL, J. (2016). Pricing under rough volatility. *Quant. Finance* **16** 887–904. MR3494612 https://doi.org/10.1080/14697688.2015.1099717

MSC2020 subject classifications. Primary 62G15, 62G20, 62M09; secondary 60F05, 62P20.

Key words and phrases. Central limit theorem, fractional Brownian motion, Hurst parameter, nonparametric estimation, rough volatility, spot volatility, volatility of volatility.

[12] BENNEDSEN, M., LUNDE, A. and PAKKANEN, M. S. (2017). Hybrid scheme for Brownian semistationary processes. *Finance Stoch*. **21** 931–965. MR3723378 https://doi.org/10.1007/s00780-017-0335-5

[13] BENNEDSEN, M., LUNDE, A. and PAKKANEN, M. S. (2022). Decoupling the short-and long-term behavior of stochastic volatility. *J. Financ. Econom*. **20** 961–1006.

[14] BOLKO, A. E., CHRISTENSEN, K., PAKKANEN, M. S. and VELIYEV, B. (2023). A GMM approach to estimate the roughness of stochastic volatility. *J. Econometrics* **235** 745–778. MR4602890 https://doi.org/10.1016/j.jeconom.2022.06.009

[15] BOLLERSLEV, T., LI, S. Z. and TODOROV, V. (2016). Roughing up beta: Continuous versus discontinuous betas and the cross section of expected stock returns. *J. Financ. Econ*. **120** 464–490.

[16] CHONG, C. H., DELERUE, T. and LI, G. (2022). When frictions are fractional: Rough noise in high-frequency data. Available at arXiv:2106.16149.

[17] CHONG, C. H., DELERUE, T. and MIES, F. (2022). Rate-optimal estimation of mixed semimartingales. Available at arXiv:2207.10464.

[18] CHONG, C. H., HOFFMANN, M., LIU, Y., ROSENBAUM, M. and SZYMANSKI, G. (2022). Statistical inference for rough volatility: Minimax theory. *Ann. Statist*. To appear. Available at arXiv:2210.01214.

[19] CHONG, C. H. and TODOROV, V. (2022). Short-time expansion of characteristic functions in a rough volatility setting with applications. Available at arXiv:2208.00830.

[20] COMTE, F. and RENAULT, E. (1998). Long memory in continuous-time stochastic volatility models. *Math. Finance* **8** 291–323. MR1645101 https://doi.org/10.1111/1467-9965.00057

[21] CORCUERA, J. M., HEDEVANG, E., PAKKANEN, M. S. and PODOLSKIJ, M. (2013). Asymptotic theory for Brownian semi-stationary processes with application to turbulence. *Stochastic Process. Appl*. **123** 2552–2574. MR3054536 https://doi.org/10.1016/j.spa.2013.03.011

[22] CORSI, F. (2009). A simple approximate long-memory model of realized volatility. *J. Financ. Econom*. **7** 174–196.

[23] DIEBOLD, F. X. and INOUE, A. (2001). Long memory and regime switching. *J. Econometrics* **105** 131–159. MR1864215 https://doi.org/10.1016/S0304-4076(01)00073-2

[24] EL EUCH, O., FUKASAWA, M. and ROSENBAUM, M. (2018). The microstructural foundations of leverage effect and rough volatility. *Finance Stoch*. **22** 241–280. MR3778355 https://doi.org/10.1007/s00780-018-0360-z

[25] EL EUCH, O. and ROSENBAUM, M. (2019). The characteristic function of rough Heston models. *Math. Finance* **29** 3–38. MR3905737 https://doi.org/10.1111/mafi.12173

[26] EL EUCH, O., GATHERAL, J. and ROSENBAUM, M. (2019). Roughening Heston. *Risk* 84–89.

[27] FUKASAWA, M. (2021). Volatility has to be rough. *Quant. Finance* **21** 1–8. MR4188876 https://doi.org/10.1080/14697688.2020.1825781

[28] FUKASAWA, M., TAKABATAKE, T. and WESTPHAL, R. (2022). Consistent estimation for fractional stochastic volatility model under high-frequency asymptotics. *Math. Finance* **32** 1086–1132. MR4485063

[29] GATHERAL, J., JAISSON, T. and ROSENBAUM, M. (2018). Volatility is rough. *Quant. Finance* **18** 933–949. MR3805308 https://doi.org/10.1080/14697688.2017.1393551

[30] GATHERAL, J., JUSSELIN, P. and ROSENBAUM, M. (2020). The quadratic rough Heston model and the joint S&P 500/VIX smile calibration problem. *Risk* 6 pp.

[31] GLOTER, A. (2000). Estimation du coefficient de diffusion de la volatilité d'un modèle à volatilité stochastique. *C. R. Acad. Sci. Paris Sér. I Math*. **330** 243–248. MR1748316 https://doi.org/10.1016/S0764-4442(00)00119-1

[32] GLOTER, A. and HOFFMANN, M. (2007). Estimation of the Hurst parameter from discrete noisy data. *Ann. Statist*. **35** 1947–1974. MR2363959 https://doi.org/10.1214/009053607000000316

[33] HÖRMANDER, L. (1990). *The Analysis of Linear Partial Differential Operators. I: Distribution Theory and Fourier Analysis*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **256**. Springer, Berlin. MR1065993 https://doi.org/10.1007/978-3-642-61497-2

[34] ISTAS, J. and LANG, G. (1997). Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Ann. Inst. Henri Poincaré Probab. Stat*. **33** 407–436. MR1465796 https://doi.org/10.1016/S0246-0203(97)80099-4

[35] JACOD, J. and PROTTER, P. (2012). *Discretization of Processes. Stochastic Modelling and Applied Probability* **67**. Springer, Heidelberg. MR2859096 https://doi.org/10.1007/978-3-642-24127-7

[36] JUSSELIN, P. and ROSENBAUM, M. (2020). No-arbitrage implies power-law market impact and rough volatility. *Math. Finance* **30** 1309–1336. MR4154771 https://doi.org/10.1111/mafi.12254

[37] KALNINA, I. and XIU, D. (2017). Nonparametric estimation of the leverage effect: A trade-off between robustness and efficiency. *J. Amer. Statist. Assoc*. **112** 384–396. MR3646579 https://doi.org/10.1080/01621459.2016.1141687

[38] Lampret, V. (2001). The Euler–Maclaurin and Taylor Formulas: Twin, Elementary Derivations. *Math. Mag.* **74** 109–122. MR1573513

[39] Li, Y., Liu, G. and Zhang, Z. (2022). Volatility of volatility: Estimation and tests based on noisy high frequency data with jumps. *J. Econometrics* **229** 422–451. MR4429572 https://doi.org/10.1016/j.jeconom.2021.02.007

[40] Liu, Y. and Tindel, S. (2020). Discrete rough paths and limit theorems. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 1730–1774. MR4116706 https://doi.org/10.1214/19-AIHP1015

[41] Livieri, G., Mouti, S., Pallavicini, A. and Rosenbaum, M. (2018). Rough volatility: Evidence from option prices. *IISE Trans.* **50** 767–776.

[42] Nourdin, I., Nualart, D. and Tudor, C. A. (2010). Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 1055–1079. MR2744886 https://doi.org/10.1214/09-AIHP342

[43] Račkauskas, A. and Suquet, C. (2004). Central limit theorems in Hölder topologies for Banach space valued random fields. *Teor. Veroyatn. Primen.* **49** 109–125. MR2141332 https://doi.org/10.1137/S0040585X97980889

[44] Shi, S., Liu, X. and Yu, J. (2021). Fractional stochastic volatility model. SSRN Preprint.

[45] Shi, S. and Yu, J. (2023). Volatility puzzle: Long memory or anti-persistency. *Manage. Sci.* **69** 3759–4361.

[46] Szymanski, G. (2024). Optimal estimation of the rough Hurst parameter in additive noise. *Stochastic Process. Appl.* **170** Paper no. 104302. MR4700236 https://doi.org/10.1016/j.spa.2024.104302

[47] Vetter, M. (2012). Estimation of correlation for continuous semimartingales. *Scand. J. Stat.* **39** 757–771. MR3000847 https://doi.org/10.1111/j.1467-9469.2012.00783.x

[48] Vetter, M. (2015). Estimation of integrated volatility of volatility with applications to goodness-of-fit testing. *Bernoulli* **21** 2393–2418. MR3378471 https://doi.org/10.3150/14-BEJ648

[49] Wang, C. D. and Mykland, P. A. (2014). The estimation of leverage effect with high-frequency data. *J. Amer. Statist. Assoc.* **109** 197–215. MR3180557 https://doi.org/10.1080/01621459.2013.864189

[50] Wang, X., Xiao, W. and Yu, J. (2023). Modeling and forecasting realized volatility with the fractional Ornstein–Uhlenbeck process. *J. Econometrics* **232** 389–415. MR4539492 https://doi.org/10.1016/j.jeconom.2021.08.001

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2650–2713
https://doi.org/10.1214/23-AAP2005

DIFFUSION APPROXIMATIONS FOR SELF-EXCITED SYSTEMS WITH APPLICATIONS TO GENERAL BRANCHING PROCESSES

BY WEI XU[a]

School of Mathematics and Statistics, Beijing Institute of Technology, [a]*xuwei.math@gmail.com*

In this work, several convergence results are established for nearly critical self-excited systems in which event arrivals are described by multivariate marked Hawkes point processes. Under some mild high-frequency assumptions, the rescaled density process behaves asymptotically like a multi-type continuous-state branching process with immigration, which is the unique solution to a multi-dimensional stochastic differential equation with dynamical mechanism similar to that of multivariate Hawkes processes. To illustrate the strength of these limit results, we further establish diffusion approximations for multi-type Crump–Mode–Jagers branching processes counted with various characteristics by linking them to marked Hawkes shot noise processes. In particular, an interesting phenomenon in queueing theory, well known as state space collapse, is observed in the behavior of the population structure at a large time scale. This phenomenon reveals that the rescaled complex biological system can be recovered from its population process by a lifting map.

REFERENCES

[1] AÏT-SAHALIA, Y., CACHO-DIAZ, J. and LAEVEN, R. J. (2015). Modeling financial contagion using mutually exciting jump processes. *J. Financ. Econ.* **117** 585–606.

[2] ALDOUS, D. (1978). Stopping times and tightness. *Ann. Probab.* **6** 335–340. MR0474446 https://doi.org/10.1214/aop/1176995579

[3] ATHREYA, K. B. and NEY, P. E. (1972). *Branching Processes*. *Die Grundlehren der Mathematischen Wissenschaften*, *Band* 196. Springer, New York. MR0373040

[4] BACRY, E., DELATTRE, S., HOFFMANN, M. and MUZY, J. F. (2013). Modelling microstructure noise with mutually exciting point processes. *Quant. Finance* **13** 65–77. MR3005350 https://doi.org/10.1080/14697688.2011.647054

[5] BACRY, E., DELATTRE, S., HOFFMANN, M. and MUZY, J. F. (2013). Some limit theorems for Hawkes processes and application to financial statistics. *Stochastic Process. Appl.* **123** 2475–2499. MR3054533 https://doi.org/10.1016/j.spa.2013.04.007

[6] BACRY, E., JAISSON, T. and MUZY, J. (2016). Estimation of slowly decreasing Hawkes kernels: Application to high-frequency order book dynamics. *Quant. Finance* **16** 1179–1201. MR3520321 https://doi.org/10.1080/14697688.2015.1123287

[7] BACRY, E., MASTROMATTEO, I. and MUZY, J. F. (2015). Hawkes processes in finance. *Mark. Microstruct. Liq.* **1** 1550005.

[8] BOWSHER, C. G. (2007). Modelling security market events in continuous time: Intensity based, multivariate point process models. *J. Econometrics* **141** 876–912. MR2413490 https://doi.org/10.1016/j.jeconom.2006.11.007

[9] BRAMSON, M. (1998). State space collapse with application to heavy traffic limits for multiclass queueing networks. *Queueing Syst. Theory Appl.* **30** 89–148. MR1663763 https://doi.org/10.1023/A:1019160803783

[10] BRÉMAUD, P. and MASSOULIÉ, L. (1996). Stability of nonlinear Hawkes processes. *Ann. Probab.* **24** 1563–1588. MR1411506 https://doi.org/10.1214/aop/1065725193

[11] BRÉMAUD, P. and MASSOULIÉ, L. (2002). Power spectra of general shot noises and Hawkes point processes with a random excitation. *Adv. in Appl. Probab.* **34** 205–222. MR1895338 https://doi.org/10.1239/aap/1019160957

MSC2020 subject classifications. Primary 60F17, 60G55; secondary 60J80, 62P10.

Key words and phrases. Scaling limit, marked Hawkes point measure, shot noise process, multi-type continuous-sate branching process, Crump–Mode–Jagers branching process, population structure, state space collapse.

[12] BRÉMAUD, P., NAPPO, G. and TORRISI, G. L. (2002). Rate of convergence to equilibrium of marked Hawkes processes. *J. Appl. Probab.* **39** 123–136. MR1895148 https://doi.org/10.1017/s0021900200021562

[13] CHAVEZ-DEMOULIN, V., DAVISON, A. C. and MCNEIL, A. J. (2005). Estimating value-at-risk: A point process approach. *Quant. Finance* **5** 227–234. MR2240248 https://doi.org/10.1080/14697680500039613

[14] CRUMP, K. S. and MODE, C. J. (1968). A general age-dependent branching process I. *J. Math. Anal. Appl.* **24** 494–508.

[15] CRUMP, K. S. and MODE, C. J. (1969). A general age-dependent branching process II. *J. Math. Anal. Appl.* **25** 8–17.

[16] DASSIOS, A. and ZHAO, H. (2011). A dynamic contagion process. *Adv. in Appl. Probab.* **43** 814–846. MR2858222 https://doi.org/10.1239/aap/1316792671

[17] DE ACOSTA, A. (1981). Inequalities for B-valued random vectors with applications to the strong law of large numbers. *Ann. Probab.* **9** 157–161. MR0606806

[18] DE ACOSTA, A. D. (1970). Existence and convergence of probability measures in Banach spaces. *Trans. Amer. Math. Soc.* **152** 273–298. MR0267614 https://doi.org/10.2307/1995651

[19] DUFFIE, D., FILIPOVIĆ, D. and SCHACHERMAYER, W. (2003). Affine processes and applications in finance. *Ann. Appl. Probab.* **13** 984–1053. MR1994043 https://doi.org/10.1214/aoap/1060202833

[20] EL EUCH, O., FUKASAWA, M. and ROSENBAUM, M. (2018). The microstructural foundations of leverage effect and rough volatility. *Finance Stoch.* **22** 241–280. MR3778355 https://doi.org/10.1007/s00780-018-0360-z

[21] EL KAROUI, N. and MÉLÉARD, S. (1990). Martingale measures and stochastic calculus. *Probab. Theory Related Fields* **84** 83–101. MR1027822 https://doi.org/10.1007/BF01288560

[22] ETHIER, S. N. and KURTZ, T. G. (2005). *Markov Processes: Characterization and Convergence*. Wiley, New York.

[23] FELLER, W. (1951). Two singular diffusion problems. *Ann. of Math.* (2) **54** 173–182. MR0054814 https://doi.org/10.2307/1969318

[24] FRIESEN, M., JIN, P. and RÜDIGER, B. (2020). On the boundary behavior of multi-type continuous-state branching processes with immigration. *Electron. Commun. Probab.* **25** 84. MR4195178 https://doi.org/10.3390/mca25010013

[25] GRIPENBERG, G. (1978). On positive, nonincreasing resolvents of Volterra equations. *J. Differ. Equ.* **30** 380–390. MR0521860 https://doi.org/10.1016/0022-0396(78)90007-4

[26] GRIPENBERG, G., LONDEN, S.-O. and STAFFANS, O. (1990). *Volterra Integral and Functional Equations. Encyclopedia of Mathematics and Its Applications* **34**. Cambridge Univ. Press, Cambridge. MR1050319 https://doi.org/10.1017/CBO9780511662805

[27] GROMOLL, H. C. (2004). Diffusion approximation for a processor sharing queue in heavy traffic. *Ann. Appl. Probab.* **14** 555–611. MR2052895 https://doi.org/10.1214/105051604000000035

[28] GUT, A. (1992). The weak law of large numbers for arrays. *Statist. Probab. Lett.* **14** 49–52. MR1172289 https://doi.org/10.1016/0167-7152(92)90209-N

[29] HAWKES, A. G. (1971). Point spectra of some mutually exciting point processes. *J. Roy. Statist. Soc. Ser. B* **33** 438–443. MR0358976

[30] HAWKES, A. G. (1971). Spectra of some self-exciting and mutually exciting point processes. *Biometrika* **58** 83–90. MR0278410 https://doi.org/10.1093/biomet/58.1.83

[31] HAWKES, A. G. and OAKES, D. (1974). A cluster process representation of a self-exciting process. *J. Appl. Probab.* **11** 493–503. MR0378093 https://doi.org/10.2307/3212693

[32] HEWLETT, P. (2006). Clustering of order arrivals, price impact and trade path optimisation. In *Workshop on Financial Modeling with Jump Processes* 6–8. Ecole Polytechnique, France.

[33] HOLBROOK, M. A. and MENNINGER, J. R. (2002). Erythromycin slows aging of Saccharomyces cerevisiae. *J. Gerontol., Ser. A, Biol. Sci. Med. Sci.* **57A** B29–B36.

[34] HORST, U. and XU, W. (2019). A scaling limit for limit order books driven by Hawkes processes. *SIAM J. Financial Math.* **10** 350–393. MR3939328 https://doi.org/10.1137/17M1148682

[35] HORST, U. and XU, W. (2021). Functional limit theorems for marked Hawkes point measures. *Stochastic Process. Appl.* **134** 94–131. MR4200854 https://doi.org/10.1016/j.spa.2020.12.002

[36] HORST, U. and XU, W. (2022). The microstructure of stochastic volatility models with self-exciting jump dynamics. *Ann. Appl. Probab.* **32** 4568–4610. MR4522360 https://doi.org/10.1214/22-aap1796

[37] HORST, U. and XU, W. (2023). Second-order regular variation and second-order approximation of Hawkes processes. arXiv preprint. Available at arXiv:2311.02655.

[38] HORST, U. and XU, W. (2024). Functional limit theorems for Hawkes processes. arXiv preprint. Available at arXiv:2401.11495.

[39] HORST, U., XU, W. and ZHANG, R. (2023). Convergence of heavy-tailed Hawkes processes and the microstructure of rough volatility. arXiv preprint. Available at arXiv:2312.08784.

[40] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. MR1011252

[41] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877 https://doi.org/10.1007/978-3-662-05265-5

[42] JAGERS, P. (1969). A general stochastic model for population development. *Skand. Aktuarietidskr.* 84–103. MR0431406 https://doi.org/10.1080/03461238.1969.10405220

[43] JAGERS, P. (1975). *Branching Processes with Biological Applications*. *Wiley Series in Probability and Mathematical Statistics—Applied Probability and Statistics*. Wiley-Interscience, London–New York–Sydney. MR0488341

[44] JAGERS, P. (2010). A plea for stochastic population dynamics. *J. Math. Biol.* **60** 761–764. MR2606524 https://doi.org/10.1007/s00285-010-0335-y

[45] JAISSON, T. and ROSENBAUM, M. (2015). Limit theorems for nearly unstable Hawkes processes. *Ann. Appl. Probab.* **25** 600–631. MR3313750 https://doi.org/10.1214/14-AAP1005

[46] JAISSON, T. and ROSENBAUM, M. (2016). Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes. *Ann. Appl. Probab.* **26** 2860–2882. MR3563196 https://doi.org/10.1214/15-AAP1164

[47] JIANG, J. C., JARUGA, E., REPNEVSKAYA, M. V. and JAZWINSKI, S. M. (2000). An intervention resembling caloric restriction prolongs life span and retards aging in yeast. *FASEB J.* **14** 2135–2137.

[48] JORION, P. and ZHANG, G. (2009). Credit contagion from counterparty risk. *J. Finance* **64** 2053–2087.

[49] KALASHNIKOV, V. (1997). *Geometric Sums*: *Bounds for Rare Events with Applications*: *Risk Analysis*, *Reliability*, *Queueing*. *Mathematics and Its Applications* **413**. Kluwer Academic, Dordrecht. MR1471479 https://doi.org/10.1007/978-94-017-1693-2

[50] KLÜPPELBERG, C. and MIKOSCH, T. (1995). Explosive Poisson shot noise processes with applications to risk reserves. *Bernoulli* **1** 125–147. MR1354458 https://doi.org/10.2307/3318683

[51] KURTZ, T. G. and PROTTER, P. (1991). Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann. Probab.* **19** 1035–1070. MR1112406

[52] KURTZ, T. G. and PROTTER, P. E. (1996). Weak convergence of stochastic integrals and differential equations. II. Infinite-dimensional case. In *Probabilistic Models for Nonlinear Partial Differential Equations* (*Montecatini Terme*, 1995). *Lecture Notes in Math.* **1627** 197–285. Springer, Berlin. MR1431303 https://doi.org/10.1007/BFb0093181

[53] KYPRIANOU, A. E. and PALAU, S. (2018). Extinction properties of multi-type continuous-state branching processes. *Stochastic Process. Appl.* **128** 3466–3489. MR3849816 https://doi.org/10.1016/j.spa.2017.11.006

[54] LAMBERT, A., SIMATOS, F. and ZWART, B. (2013). Scaling limits via excursion theory: Interplay between Crump–Mode–Jagers branching processes and processor-sharing queues. *Ann. Appl. Probab.* **23** 2357–2381. MR3127938 https://doi.org/10.1214/12-AAP904

[55] LARGE, J. (2007). Measuring the resiliency of an electronic limit order book. *J. Financ. Mark.* **10** 1–25.

[56] MAULIK, K. and RESNICK, S. (2003). Small and large time scale analysis of a network traffic model. *Queueing Syst.* **43** 221–250. MR1968784 https://doi.org/10.1023/A:1022894627652

[57] OGATA, Y. (1988). Statistical models for earthquake occurrences and residual analysis for point processes. *J. Amer. Statist. Assoc.* **83** 9–27.

[58] PHILLIPS, R. C. and GORSE, D. (2018). Mutual-excitation of cryptocurrency market returns and social media topics. In *Proceedings of the 4th International Conference on Frontiers of Educational Technologies* 80–86.

[59] ROSENBAUM, M. and TOMAS, M. (2021). From microscopic price dynamics to multidimensional rough volatility models. *Adv. in Appl. Probab.* **53** 425–462. MR4280453 https://doi.org/10.1017/apr.2020.60

[60] ROSS, S. M. (2010). *Introduction to Probability Models*, 10th ed. Academic Press, San Diego.

[61] SAGITOV, S. (1995). A key limit theorem for critical branching processes. *Stochastic Process. Appl.* **56** 87–100. MR1324323 https://doi.org/10.1016/0304-4149(94)00046-V

[62] SCHILLING, R. L., SONG, R. and VONDRAČEK, Z. (2012). *Bernstein Functions*: *Theory and Applications*, 2nd ed. *De Gruyter Studies in Mathematics* **37**. de Gruyter, Berlin. MR2978140 https://doi.org/10.1515/9783110269338

[63] STOLYAR, A. L. (2004). Maxweight scheduling in a generalized switch: State space collapse and workload minimization in heavy traffic. *Ann. Appl. Probab.* **14** 1–53. MR2023015 https://doi.org/10.1214/aoap/1075828046

[64] TORTORA, G. J., FUNKE, B. R., CASE, C. L. and JOHNSON, T. R. (2018). *Microbiology*: *An Introduction*. Pearson Education, Upper Saddle River.

[65] VERLOOP, I. M., AYESTA, U. and NÚÑEZ-QUEIJA, R. (2011). Heavy-traffic analysis of a multiple-phase network with discriminatory processor sharing. *Oper. Res.* **59** 648–660. MR2848544 https://doi.org/10.1287/opre.1110.0914

[66] WATANABE, S. (1969). On two dimensional Markov processes with branching property. *Trans. Amer. Math. Soc.* **136** 447–466. MR0234531 https://doi.org/10.2307/1994726

[67] WESTCOTT, M. (1971). On existence and mixing results for cluster point processes. *J. Roy. Statist. Soc. Ser. B* **33** 290–300. MR0317443

[68] WILLIAMS, R. J. (1998). Diffusion approximations for open multiclass queueing networks: Sufficient conditions involving state space collapse. *Queueing Syst. Theory Appl.* **30** 27–88. MR1663759 https://doi.org/10.1023/A:1019108819713

[69] WOOD, J. G., ROGINA, B., LAVU, S., HOWITZ, K., HELFAND, S. L., TATAR, M. and SINCLAIR, D. (2004). Sirtuin activators mimic caloric restriction and delay ageing in metazoans. *Nature* **430** 686–689.

[70] XU, W. (2014). Parameter estimation in two-type continuous-state branching processes with immigration. *Statist. Probab. Lett.* **91** 124–134. MR3208126 https://doi.org/10.1016/j.spl.2014.04.021

[71] XU, W. (2024). Stochastic Volterra equations for the local times of spectrally positive stable processes. To appear in *Ann. Appl. Probab.*

[72] YAMADA, T. and WATANABE, S. (1971). On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* **11** 155–167. MR0278420 https://doi.org/10.1215/kjm/1250523691

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2714–2732
https://doi.org/10.1214/23-AAP2009

PAIRWISE SEQUENCE ALIGNMENT AT ARBITRARILY LARGE EVOLUTIONARY DISTANCE

BY BRANDON LEGRIED[1,a] AND SEBASTIEN ROCH[2,b]

[1]*School of Mathematics, Georgia Institute of Technology,* [a]*blegried3@gatech.edu*
[2]*Department of Mathematics, University of Wisconsin–Madison,* [b]*roch@wisc.edu*

Ancestral sequence reconstruction is a key task in computational biology. It consists in inferring a molecular sequence at an ancestral species of a known phylogeny, given descendant sequences at the tip of the tree. In addition to its many biological applications, it has played a key role in elucidating the statistical performance of phylogeny estimation methods. Here we establish a formal connection to another important bioinformatics problem, multiple sequence alignment, where one attempts to best align a collection of molecular sequences under some mismatch penalty score by inserting gaps. Our result is counter-intuitive: we show that perfect pairwise sequence alignment with high probability is possible in principle *at arbitrary large evolutionary distances*—provided the phylogeny is known and dense enough. We use techniques from ancestral sequence reconstruction in the taxon-rich setting together with the probabilistic analysis of sequence evolution models involving insertions and deletions.

REFERENCES

[1] ALLMAN, E. S., RHODES, J. A. and SULLIVANT, S. (2017). Statistically consistent k-mer methods for phylogenetic tree reconstruction. *J. Comput. Biol.* **24** 153–171. MR3607847 https://doi.org/10.1089/cmb.2015.0216

[2] ANDONI, A., DASKALAKIS, C., HASSIDIM, A. and ROCH, S. (2012). Global alignment of molecular sequences via ancestral state reconstruction. *Stochastic Process. Appl.* **122** 3852–3874. MR2971717 https://doi.org/10.1016/j.spa.2012.08.004

[3] CHAKRABORTY, D., DAS, D. and KRAUTHGAMER, R. (2021). Approximate trace reconstruction via median string (in average-case). In 41*st IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science* (M. and C. , eds.) *LIPIcs. Leibniz Int. Proc. Inform.* **213** Art. No. 11, 23. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern. MR4459293

[4] CHANG, G. S., HONG, Y., KO, K. D., BHARDWAJ, G., HOLMES, E. C., PATTERSON, R. L. and VAN ROSSUM, D. B. (2008). Phylogenetic profiles reveal evolutionary relationships within the "twilight zone" of sequence similarity. *Proc. Natl. Acad. Sci. USA* **105** 13474–13479. https://doi.org/10.1073/pnas.0803860105

[5] CHASE, Z. and PERES, Y. Approximate trace reconstruction of random strings from a constant number of traces [math]. Available at arXiv:2107.06454.

[6] CHEN, X., DE, A., LEE, C. H., SERVEDIO, R. A. and SINHA, S. (2022). Near-optimal average-case approximate trace reconstruction from few traces. In *Proceedings of the* 2022 *Annual ACM-SIAM Symposium on Discrete Algorithms* (*SODA*) 779–821. SIAM, Philadelphia, PA. MR4415071 https://doi.org/10.1137/1.9781611977073.34

[7] CHEN, X., DE, A., LEE, C. H., SERVEDIO, R. A. and SINHA, S. (2023). Approximate trace reconstruction from a single trace. In *Proceedings of the* 2023 *Annual ACM-SIAM Symposium on Discrete Algorithms* (*SODA*) 605–637. SIAM, Philadelphia, PA. MR4538159 https://doi.org/10.1137/1.9781611977554.ch27

[8] DASKALAKIS, C. and ROCH, S. (2013). Alignment-free phylogenetic reconstruction: Sample complexity via a branching process analysis. *Ann. Appl. Probab.* **23** 693–721. MR3059273 https://doi.org/10.1214/12-AAP852

[9] DAVIES, S., RÁCZ, M. Z. and RASHTCHIAN, C. (2021). Reconstructing trees from traces. *Ann. Appl. Probab.* **31** 2772–2810. MR4350973 https://doi.org/10.1214/21-aap1662

MSC2020 subject classifications. Primary 60K35; secondary 92D15.
Key words and phrases. Reconstruction, branching processes.

[10] DAVIES, S., RÁCZ, M. Z., SCHIFFER, B. G. and RASHTCHIAN, C. (2021). Approximate trace reconstruction: Algorithms. *Abstr. Pap.—IEEE Int. Symp. Inf. Theory* 2525–2530. https://doi.org/10.1109/ISIT45174.2021.9517926

[11] EVANS, W., KENYON, C., PERES, Y. and SCHULMAN, L. J. (2000). Broadcasting on trees and the Ising model. *Ann. Appl. Probab.* **10** 410–433. MR1768240 https://doi.org/10.1214/aoap/1019487349

[12] FAN, W.-T. and ROCH, S. (2020). Statistically consistent and computationally efficient inference of ancestral DNA sequences in the TKF91 model under dense taxon sampling. *Bull. Math. Biol.* **82** Paper No. 21, 32. MR4056716 https://doi.org/10.1007/s11538-020-00693-3

[13] FAN, W.-T. L., LEGRIED, B. and ROCH, S. (2020). Impossibility of consistent distance estimation from sequence lengths under the TKF91 model. *Bull. Math. Biol.* **82** Paper No. 123, 12. MR4150869 https://doi.org/10.1007/s11538-020-00801-3

[14] FISCHER, M. and THATTE, B. D. (2009). Maximum parsimony on subsets of taxa. *J. Theoret. Biol.* **260** 290–293. MR2973084 https://doi.org/10.1016/j.jtbi.2009.06.010

[15] FITCH, W. M. (1971). Toward defining the course of evolution: Minimum change for a specific tree topology. *Syst. Zool.* **20** 406–416.

[16] GANESH, A. and ZHANG, Q. (2019). Optimal sequence length requirements for phylogenetic tree reconstruction with indels. In *STOC'19—Proceedings of the* 51*st Annual ACM SIGACT Symposium on Theory of Computing* 721–732. ACM, New York. MR4003378 https://doi.org/10.1145/3313276.3316345

[17] GASCUEL, O. and STEEL, M. (2010). Inferring ancestral sequences in taxon-rich phylogenies. *Math. Biosci.* **227** 125–135. MR2732449 https://doi.org/10.1016/j.mbs.2010.07.002

[18] HERBST, L. and FISCHER, M. (2017). Ancestral sequence reconstruction with maximum parsimony. *Bull. Math. Biol.* **79** 2865–2886. MR3736430 https://doi.org/10.1007/s11538-017-0354-6

[19] HERBST, L. and FISCHER, M. (2018). On the accuracy of ancestral sequence reconstruction for ultrametric trees with parsimony. *Bull. Math. Biol.* **80** 864–879. MR3778384 https://doi.org/10.1007/s11538-018-0407-5

[20] HOLDEN, N. and LYONS, R. (2020). Lower bounds for trace reconstruction. *Ann. Appl. Probab.* **30** 503–525. MR4108114 https://doi.org/10.1214/19-AAP1506

[21] HOLENSTEIN, T., MITZENMACHER, M., PANIGRAHY, R. and WIEDER, U. (2008). Trace reconstruction with constant deletion probability and related results. In *Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms* 389–398. ACM, New York. MR2487606

[22] IOFFE, D. (1996). On the extremality of the disordered state for the Ising model on the Bethe lattice. *Lett. Math. Phys.* **37** 137–143. MR1391195 https://doi.org/10.1007/BF00416016

[23] LI, G., STEEL, M. and ZHANG, L. (2008). More taxa are not necessarily better for the reconstruction of ancestral character states. *Syst. Biol.* **57** 647–653. https://doi.org/10.1080/10635150802203898

[24] LIBERLES, D. A., ed. (2007). *Ancestral Sequence Reconstruction* Oxford Univ. Press, Oxford. https://doi.org/10.1093/acprof:oso/9780199299188.001.0001

[25] LUNTER, G., DRUMMOND, A. J., MIKLÓS, I. and HEIN, J. (2005). Statistical alignment: Recent progress, new applications, and challenges. In *Statistical Methods in Molecular Evolution*. *Stat. Biol. Health* 375–405. Springer, New York. MR2161838 https://doi.org/10.1007/0-387-27733-1_14

[26] MITROPHANOV, A. Y. and BORODOVSKY, M. (2007). Convergence rate estimation for the TKF91 model of biological sequence length evolution. *Math. Biosci.* **209** 470–485. MR2359560 https://doi.org/10.1016/j.mbs.2007.02.011

[27] MOSSEL, E. (1998). Recursive reconstruction on periodic trees. *Random Structures Algorithms* **13** 81–97. MR1634344 https://doi.org/10.1002/(SICI)1098-2418(199808)13:1<81::AID-RSA5>3.3.CO;2-K

[28] MOSSEL, E. (2001). Reconstruction on trees: Beating the second eigenvalue. *Ann. Appl. Probab.* **11** 285–300. MR1825467 https://doi.org/10.1214/aoap/998926994

[29] MOSSEL, E. (2003). On the impossibility of reconstructing ancestral data and phylogenies. *J. Comput. Biol.* **10** 669–676. Publisher: Mary Ann Liebert, Inc., publishers. https://doi.org/10.1089/106652703322539015

[30] MOSSEL, E. (2004). Phase transitions in phylogeny. *Trans. Amer. Math. Soc.* **356** 2379–2404. MR2048522 https://doi.org/10.1090/S0002-9947-03-03382-8

[31] NAZAROV, F. and PERES, Y. (2017). Trace reconstruction with $\exp(O(n^{1/3}))$ samples. In *STOC'17—Proceedings of the* 49*th Annual ACM SIGACT Symposium on Theory of Computing* 1042–1046. ACM, New York. MR3678249

[32] RANWEZ, V. and CHANTRET, N. N. (2020). Strengths and limits of multiple sequence alignment and filtering methods. In *Phylogenetics in the Genomic Era* (C. Scornavacca, F. Delsuc and N. Galtier, eds.) 2.2:1–2.2:36. No commercial publisher. Authors open access book.

[33] ROCH, S. (2010). Toward extracting all phylogenetic information from matrices of evolutionary distances. *Science* **327** 1376–1379. MR2650508 https://doi.org/10.1126/science.1182300

[34] ROCH, S. and SLY, A. (2017). Phase transition in the sample complexity of likelihood-based phylogeny inference. *Probab. Theory Related Fields* **169** 3–62. MR3704765 https://doi.org/10.1007/s00440-017-0793-x

[35] ROCH, S. and WANG, K.-C. (2021). Sufficient condition for root reconstruction by parsimony on binary trees with general weights. *Electron. Commun. Probab.* **26** Paper No. 55, 13. MR4319989 https://doi.org/10.1214/21-ecp423

[36] ROST, B. (1999). Twilight zone of protein sequence alignments. *Protein Eng. Des. Sel.* **12** 85–94. https://doi.org/10.1093/protein/12.2.85

[37] SEMPLE, C. and STEEL, M. (2003). *Phylogenetics. Oxford Lecture Series in Mathematics and Its Applications* **24**. Oxford Univ. Press, Oxford. MR2060009

[38] STEEL, M. and CHARLESTON, M. (1995). Five surprising properties of parsimoniously colored trees. *Bull. Math. Biol.* **57** 367–375. https://doi.org/10.1007/BF02460622

[39] THATTE, B. D. (2006). Invertibility of the TKF model of sequence evolution. *Math. Biosci.* **200** 58–75. MR2211928 https://doi.org/10.1016/j.mbs.2005.12.025

[40] THORNE, J. L., KISHINO, H. and FELSENSTEIN, J. (1991). An evolutionary model for maximum likelihood alignment of DNA sequences. *J. Mol. Evol.* **33** 114–124. https://doi.org/10.1007/BF02193625

[41] YANG, Z. (2014). *Molecular Evolution: A Statistical Approach*. Oxford Univ. Press, Oxford. https://doi.org/10.1093/acprof:oso/9780199602605.001.0001

[42] ZHANG, L., SHEN, J., YANG, J. and LI, G. (2010). Analyzing the Fitch method for reconstructing ancestral states of ultrametric phylogenetic trees. *Bull. Math. Biol.* **72** 1760–1782. MR2728004 https://doi.org/10.1007/s11538-010-9505-8

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2733–2798
https://doi.org/10.1214/23-AAP2017

STOCHASTIC VOLTERRA EQUATIONS FOR THE LOCAL TIMES OF SPECTRALLY POSITIVE STABLE PROCESSES

BY WEI XU[a]

School of Mathematics and Statistics, Beijing Institute of Technology, [a]*xuwei.math@gmail.com*

This paper is concerned with the evolution dynamics of local times of a spectrally positive stable process in the spatial direction. The main results state that conditioned on the finiteness of the first time at which the local time at zero exceeds a given value, the local times at positive half line are equal in distribution to the unique solution of a stochastic Volterra equation driven by a Poisson random measure whose intensity coincides with the Lévy measure. This helps us to provide not only a simple proof for the Hölder regularity, but also a uniform upper bound for all moments of the Hölder coefficient as well as a maximal inequality for the local times. Moreover, based on this stochastic Volterra equation, we extend the method of duality to establish an exponential-affine representation of the Laplace functional in terms of the unique solution of a nonlinear Volterra integral equation associated with the Laplace exponent of the stable process.

REFERENCES

[1] ABI JABER, E. (2021). Weak existence and uniqueness for affine stochastic Volterra equations with L^1-kernels. *Bernoulli* **27** 1583–1615. MR4260503 https://doi.org/10.3150/20-bej1284
[2] ABI JABER, E., CUCHIERO, C., LARSSON, M. and PULIDO, S. (2021). A weak solution theory for stochastic Volterra equations of convolution type. *Ann. Appl. Probab.* **31** 2924–2952. MR4350978 https://doi.org/10.1214/21-aap1667
[3] ABI JABER, E., LARSSON, M. and PULIDO, S. (2019). Affine Volterra processes. *Ann. Appl. Probab.* **29** 3155–3200. MR4019885 https://doi.org/10.1214/19-AAP1477
[4] AÏDÉKON, E., HU, Y. and SHI, Z. (2024). An infinite-dimensional representation of the Ray–Knight theorems. *Sci. China Math.* **67** 149–162. MR4685742 https://doi.org/10.1007/s11425-022-2068-0
[5] BARLOW, M. T. (1988). Necessary and sufficient conditions for the continuity of local time of Lévy processes. *Ann. Probab.* **16** 1389–1427. MR0958195
[6] BARLOW, M. T., PERKINS, E. A. and TAYLOR, S. J. (1986). Two uniform intrinsic constructions for the local time of a class of Lévy processes. *Illinois J. Math.* **30** 19–65. MR0822383
[7] BARNDORFF-NIELSEN, O. E., MIKOSCH, T. and RESNICK, S. I. (2001). *Lévy Processes: Theory and Applications* Birkhäuser, Boston, MA. MR1833689 https://doi.org/10.1007/978-1-4612-0197-7
[8] BASS, R. F. (2003). Stochastic differential equations driven by symmetric stable processes. In *Séminaire de Probabilités, XXXVI. Lecture Notes in Math.* **1801** 302–313. Springer, Berlin. MR1971592 https://doi.org/10.1007/978-3-540-36107-7_11
[9] BERTOIN, J. (1995). On the Hilbert transform of the local times of a Lévy process. *Bull. Sci. Math.* **119** 147–156. MR1324841
[10] BERTOIN, J. (1996). *Lévy Processes. Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564
[11] BERTOIN, J. and LE GALL, J.-F. (2006). Stochastic flows associated to coalescent processes. III. Limit theorems. *Illinois J. Math.* **50** 147–181. MR2247827
[12] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics: Probability and Statistics*. Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962
[13] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR0898871 https://doi.org/10.1017/CBO9780511721434

MSC2020 subject classifications. Primary 60G52, 60J55, 60H20; secondary 60G22, 60F17, 60G55.
Key words and phrases. Local time, stable process, stochastic Volterra equation, heavy tail, Poisson random measure, marked Hawkes point measure, Laplace functional, Ray–Knight theorem.

[14] BOYLAN, E. S. (1964). Local times for a class of Markoff processes. *Illinois J. Math.* **8** 19–39. MR0158434

[15] CALLEGARO, G., GRASSELLI, M. and PAGÈS, G. (2021). Fast hybrid schemes for fractional Riccati equations (rough is not so tough). *Math. Oper. Res.* **46** 221–254. MR4224426 https://doi.org/10.1287/moor.2020.1054

[16] DAWSON, D. A. and LI, Z. (2006). Skew convolution semigroups and affine Markov processes. *Ann. Probab.* **34** 1103–1142. MR2243880 https://doi.org/10.1214/009117905000000747

[17] DAWSON, D. A. and LI, Z. (2012). Stochastic equations, flows and measure-valued processes. *Ann. Probab.* **40** 813–857. MR2952093 https://doi.org/10.1214/10-AOP629

[18] DELLACHERIE, C. and MEYER, P. A. (1980). *Séminaire de Probabilités*, *Vol. II. Théprie des Martingales*. Hermann, Paris.

[19] DENTON, Z. and VATSALA, A. S. (2010). Fractional integral inequalities and applications. *Comput. Math. Appl.* **59** 1087–1094. MR2579473 https://doi.org/10.1016/j.camwa.2009.05.012

[20] DUFFIE, D., FILIPOVIĆ, D. and SCHACHERMAYER, W. (2003). Affine processes and applications in finance. *Ann. Appl. Probab.* **13** 984–1053. MR1994043 https://doi.org/10.1214/aoap/1060202833

[21] DUQUESNE, T. and LE GALL, J.-F. (2002). Random trees, Lévy processes and spatial branching processes. *Astérisque* **281** vi+147. MR1954248

[22] EISENBAUM, N. and KASPI, H. (1993). A necessary and sufficient condition for the Markov property of the local time process. *Ann. Probab.* **21** 1591–1598. MR1235430

[23] EISENBAUM, N., KASPI, H., MARCUS, M. B., ROSEN, J. and SHI, Z. (2000). A Ray–Knight theorem for symmetric Markov processes. *Ann. Probab.* **28** 1781–1796. MR1813843 https://doi.org/10.1214/aop/1019160507

[24] EL EUCH, O., FUKASAWA, M. and ROSENBAUM, M. (2018). The microstructural foundations of leverage effect and rough volatility. *Finance Stoch.* **22** 241–280. MR3778355 https://doi.org/10.1007/s00780-018-0360-z

[25] EL EUCH, O. and ROSENBAUM, M. (2019). The characteristic function of rough Heston models. *Math. Finance* **29** 3–38. MR3905737 https://doi.org/10.1111/mafi.12173

[26] FITZSIMMONS, P. J. and GETOOR, R. K. (1992). Limit theorems and variation properties for fractional derivatives of the local time of a stable process. *Ann. Inst. Henri Poincaré Probab. Stat.* **28** 311–333. MR1162577

[27] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2018). Uniform control of local times of spectrally positive stable processes. *Ann. Appl. Probab.* **28** 2592–2634. MR3843837 https://doi.org/10.1214/17-AAP1370

[28] FU, Z. and LI, Z. (2010). Stochastic equations of non-negative processes with jumps. *Stochastic Process. Appl.* **120** 306–330. MR2584896 https://doi.org/10.1016/j.spa.2009.11.005

[29] GARSIA, A. M., RODEMICH, E. and RUMSEY, H. JR. (1970/71). A real variable lemma and the continuity of paths of some Gaussian processes. *Indiana Univ. Math. J.* **20** 565–578. MR0267632 https://doi.org/10.1512/iumj.1970.20.20046

[30] GEMAN, D. and HOROWITZ, J. (1980). Occupation densities. *Ann. Probab.* **8** 1–67. MR0556414

[31] GRIPENBERG, G., LONDEN, S.-O. and STAFFANS, O. (1990). *Volterra Integral and Functional Equations*. *Encyclopedia of Mathematics and Its Applications* **34**. Cambridge Univ. Press, Cambridge. MR1050319 https://doi.org/10.1017/CBO9780511662805

[32] HAUBOLD, H. J., MATHAI, A. M. and SAXENA, R. K. (2011). Mittag-Leffler functions and their applications. *J. Appl. Math.* **2011** Art. ID 298628, 51 pp. MR2800586 https://doi.org/10.1155/2011/298628

[33] HORST, U. and XU, W. (2021). Functional limit theorems for marked Hawkes point measures. *Stochastic Process. Appl.* **134** 94–131. MR4200854 https://doi.org/10.1016/j.spa.2020.12.002

[34] HORST, U. and XU, W. (2022). The microstructure of stochastic volatility models with self-exciting jump dynamics. *Ann. Appl. Probab.* **32** 4568–4610. MR4522360 https://doi.org/10.1214/22-aap1796

[35] HORST, U., XU, W. and ZHANG, R. (2023). The microstructure of rough volatility models with self-exciting sharp-rises. In preparation.

[36] HORST, U., XU, W. and ZHANG, R. (2023). Convergence of heavy-tailed Hawkes processes and the microstructure of rough volatility. Preprint. Available at arXiv:2312.08784.

[37] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland/Kodansha, Amsterdam/Tokyo. MR1011252

[38] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877 https://doi.org/10.1007/978-3-662-05265-5

[39] JAISSON, T. and ROSENBAUM, M. (2015). Limit theorems for nearly unstable Hawkes processes. *Ann. Appl. Probab.* **25** 600–631. MR3313750 https://doi.org/10.1214/14-AAP1005

[40] JAISSON, T. and ROSENBAUM, M. (2016). Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes. *Ann. Appl. Probab.* **26** 2860–2882. MR3563196 https://doi.org/10.1214/15-AAP1164

[41] KAWAZU, K. and WATANABE, S. (1971). Branching processes with immigration and related limit theorems. *Theory Probab. Appl.* **16** 36–54.

[42] KNIGHT, F. B. (1963). Random walks and a sojourn density process of Brownian motion. *Trans. Amer. Math. Soc.* **109** 56–86. MR0154337 https://doi.org/10.2307/1993647

[43] KURTZ, T. G. and PROTTER, P. (1991). Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann. Probab.* **19** 1035–1070. MR1112406

[44] KURTZ, T. G. and PROTTER, P. E. (1996). Weak convergence of stochastic integrals and differential equations. II. Infinite-dimensional case. In *Probabilistic Models for Nonlinear Partial Differential Equations* (*Montecatini Terme*, 1995). *Lecture Notes in Math.* **1627** 197–285. Springer, Berlin. MR1431303 https://doi.org/10.1007/BFb0093181

[45] KYPRIANOU, A. E. (2014). *Fluctuations of Lévy Processes with Applications*: *Introductory Lectures*, 2nd ed. *Universitext*. Springer, Heidelberg. MR3155252 https://doi.org/10.1007/978-3-642-37632-0

[46] LAMBERT, A. (2010). The contour of splitting trees is a Lévy process. *Ann. Probab.* **38** 348–395. MR2599603 https://doi.org/10.1214/09-AOP485

[47] LAMBERT, A. and SIMATOS, F. (2015). Asymptotic behavior of local times of compound Poisson processes with drift in the infinite variance case. *J. Theoret. Probab.* **28** 41–91. MR3320960 https://doi.org/10.1007/s10959-013-0492-1

[48] LAMBERT, A., SIMATOS, F. and ZWART, B. (2013). Scaling limits via excursion theory: Interplay between Crump–Mode–Jagers branching processes and processor-sharing queues. *Ann. Appl. Probab.* **23** 2357–2381. MR3127938 https://doi.org/10.1214/12-AAP904

[49] LAMBERT, A. and URIBE BRAVO, G. (2018). Totally ordered measured trees and splitting trees with infinite variation. *Electron. J. Probab.* **23** Paper No. 120, 41 pp. MR3896857 https://doi.org/10.1214/18-EJP251

[50] LE GALL, J.-F. (1999). *Spatial Branching Processes*, *Random Snakes and Partial Differential Equations*. *Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR1714707 https://doi.org/10.1007/978-3-0348-8683-3

[51] LE GALL, J.-F. and LE JAN, Y. (1998). Branching processes in Lévy processes: The exploration process. *Ann. Probab.* **26** 213–252. MR1617047 https://doi.org/10.1214/aop/1022855417

[52] LE GALL, J.-F. and LE JAN, Y. (1998). Branching processes in Lévy processes: Laplace functionals of snakes and superprocesses. *Ann. Probab.* **26** 1407–1432. MR1675019 https://doi.org/10.1214/aop/1022855868

[53] LÉPINGLE, D. and MÉMIN, J. (1978). Sur l'intégrabilité uniforme des martingales exponentielles. *Z. Wahrsch. Verw. Gebiete* **42** 175–203. MR0489492 https://doi.org/10.1007/BF00641409

[54] LI, Z. (2006). A limit theorem for discrete Galton–Watson branching processes with immigration. *J. Appl. Probab.* **43** 289–295. MR2225068 https://doi.org/10.1239/jap/1143936261

[55] LI, Z. (2020). Continuous-state branching processes with immigration. In *From Probability to Finance—Lecture Notes of BICMR Summer School on Financial Mathematics*. *Math. Lect. Peking Univ.* 1–69. Springer, Singapore. MR4339413 https://doi.org/10.1007/978-981-15-1576-7_1

[56] MAINARDI, F. (2014). On some properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$, completely monotone for $t > 0$ with $0 < \alpha < 1$. *Discrete Contin. Dyn. Syst. Ser. B* **19** 2267–2278. MR3253257 https://doi.org/10.3934/dcdsb.2014.19.2267

[57] MARINELLI, C. and RÖCKNER, M. (2014). On maximal inequalities for purely discontinuous martingales in infinite dimensions. In *Séminaire de Probabilités XLVI*. *Lecture Notes in Math.* **2123** 293–315. Springer, Cham. MR3330821 https://doi.org/10.1007/978-3-319-11970-0_10

[58] MATHAI, A. M. and HAUBOLD, H. J. (2008). *Special Functions for Applied Scientists*. Springer, Berlin.

[59] PARDOUX, É. and PROTTER, P. (1990). Stochastic Volterra equations with anticipating coefficients. *Ann. Probab.* **18** 1635–1655. MR1071815

[60] PROTTER, P. (1985). Volterra equations driven by semimartingales. *Ann. Probab.* **13** 519–530. MR0781420

[61] PROTTER, P. E. (2005). *Stochastic Integration and Differential Equations*. *Stochastic Modelling and Applied Probability* **21**. Springer, Berlin. MR2273672 https://doi.org/10.1007/978-3-662-10061-5

[62] RAY, D. (1963). Sojourn times of diffusion processes. *Illinois J. Math.* **7** 615–630. MR0156383

[63] RESNICK, S. I. (2007). *Heavy-Tail Phenomena*: *Probabilistic and Statistical Modeling*. *Springer Series in Operations Research and Financial Engineering*. Springer, New York. MR2271424

[64] REVUZ, D. and YOR, M. (2005). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[65] ROSENBAUM, M. and TOMAS, M. (2021). From microscopic price dynamics to multidimensional rough volatility models. *Adv. in Appl. Probab.* **53** 425–462. MR4280453 https://doi.org/10.1017/apr.2020.60

[66] SABOT, C. and TARRES, P. (2016). Inverting Ray–Knight identity. *Probab. Theory Related Fields* **165** 559–580. MR3520013 https://doi.org/10.1007/s00440-015-0640-x

[67] XU, W. (2021). Asymptotic results for rough continuous-state branching processes. Preprint. Available at arXiv:2107.05888.

[68] XU, W. (2024). Diffusion approximations for self-excited systems with applications to general branching processes. *Ann. Appl. Probab.* To appear.

[69] ZWART, B. (2022). Conjectures on symmetric queues in heavy traffic. *Queueing Syst.* **100** 369–371. MR4434087 https://doi.org/10.1007/s11134-022-09746-x

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2799–2843
https://doi.org/10.1214/23-AAP2020

CORRELATION DETECTION IN TREES FOR PLANTED GRAPH ALIGNMENT

BY LUCA GANASSALI[1,a], MARC LELARGE[1,b] AND LAURENT MASSOULIÉ[1,2,c]

[1]*Inria, DI/ENS, PSL Research University,* [a]*luca.ganassali@inria.fr,* [b]*marc.lelarge@inria.fr*
[2]*MSR-Inria Joint Centre,* [c]*laurent.massoulie@inria.fr*

Motivated by alignment of correlated sparse random graphs, we introduce a hypothesis testing problem of deciding whether or not two random trees are correlated. We study the likelihood ratio test and obtain sufficient conditions under which this task is impossible or feasible. We propose `MPAlign`, a message-passing algorithm for graph alignment inspired by the tree correlation detection problem. We prove `MPAlign` to succeed in polynomial time at partial alignment whenever tree detection is feasible. As a result our analysis of tree detection reveals new ranges of parameters for which partial alignment of sparse random graphs is feasible in polynomial time.[1]

REFERENCES

[1] ARVIND, V. and KURUR, P. P. (2002). Graph isomorphism is in SPP. In *The* 43*rd Annual IEEE Symposium on Foundations of Computer Science*, 2002. *Proceedings*. 743–750.

[2] BANKS, J. E., MOORE, C., NEEMAN, J. and NETRAPALLI, P. (2016). Information-theoretic thresholds for community detection in sparse networks. Available at arXiv:1607.01760.

[3] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966 https://doi.org/10.1017/CBO9780511814068

[4] BORDENAVE, C., LELARGE, M. and MASSOULIÉ, L. (2018). Nonbacktracking spectrum of random graphs: Community detection and nonregular Ramanujan graphs. *Ann. Probab.* **46** 1–71. MR3758726 https://doi.org/10.1214/16-AOP1142

[5] CONTE, D., FOGGIA, P., VENTO, M. and SANSONE, C. (2004). Thirty Years Of Graph Matching In Pattern Recognition. *Int. J. Pattern Recognit. Artif. Intell.* **18** 265–298.

[6] CULLINA, D. and KIYAVASH, N. (2017). Exact alignment recovery for correlated Erdős–Rényi graphs. Available at arXiv:1711.06783.

[7] CULLINA, D., KIYAVASH, N., MITTAL, P. and POOR, H. V. (2018). Partial Recovery of Erdős–Rényi Graph Alignment via k-Core Alignment. Available at arXiv:1809.03553.

[8] DING, J. and DU, H. (2023). Matching recovery threshold for correlated random graphs. *Ann. Statist.* **51** 1718–1743. MR4658574 https://doi.org/10.1214/23-aos2305

[9] DING, J., MA, Z., WU, Y. and XU, J. (2021). Efficient random graph matching via degree profiles. *Probab. Theory Related Fields* **179** 29–115. MR4221654 https://doi.org/10.1007/s00440-020-00997-4

[10] EMRE DAI, O., CULLINA, D. and KIYAVASH, N. (2019). Database Alignment with Gaussian Features. ArXiv E-prints. Available at arXiv:1903.01422.

[11] ERDŐS, P. and RÉNYI, A. (1959). On random graphs. I. *Publ. Math. Debrecen* **6** 290–297. MR0120167 https://doi.org/10.5486/pmd.1959.6.3-4.12

[12] FAN, Z., MAO, C., WU, Y. and XU, J. (2019). Spectral Graph Matching and Regularized Quadratic Relaxations II: Erdős–Rényi Graphs and Universality. Available at arXiv:1907.08883.

[13] GANASSALI, L. and MASSOULIÉ, L. (2020). From tree matching to sparse graph alignment. In *Proceedings of Thirty Third Conference on Learning Theory* (J. Abernethy and S. Agarwal, eds.). *Proceedings of Machine Learning Research* **125** 1633–1665. PMLR.

[14] GANASSALI, L., MASSOULIE, L. and LELARGE, M. (2021). Impossibility of Partial Recovery in the Graph Alignment Problem. In *Proceedings of Thirty Fourth Conference on Learning Theory* (M. Belkin and S. Kpotufe, eds.). *Proceedings of Machine Learning Research* **134** 2080–2102. PMLR.

MSC2020 subject classifications. Primary 05C80, 05C85; secondary 62F03.
Key words and phrases. Inference in random graphs, graph matching, graph alignment, hypothesis testing.

[1]An early, short version of this work has been presented at the ITCS'22 conference [15].

[15] GANASSALI, L., MASSOULIÉ, L. and LELARGE, M. (2022). Correlation Detection in Trees for Planted Graph Alignment. In 13*th Innovations in Theoretical Computer Science Conference*, *ITCS* 2022, *January* 31–*February* 3, 2022, *Berkeley*, *CA*, *USA* (M. Braverman, ed.). *LIPIcs* **215** 74:1–74:8. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

[16] HALL, G. and MASSOULIÉ, L. (2023). Partial recovery in the graph alignment problem. *Oper. Res.* **71** 259–272. MR4585566

[17] KRZAKALA, F., MOORE, C., MOSSEL, E., NEEMAN, J., SLY, A., ZDEBOROVÁ, L. and ZHANG, P. (2013). Spectral redemption in clustering sparse networks. *Proc. Natl. Acad. Sci. USA* **110** 20935–20940. MR3174850 https://doi.org/10.1073/pnas.1312486110

[18] ŁUCZAK, T. (1988). The automorphism group of random graphs with a given number of edges. *Math. Proc. Cambridge Philos. Soc.* **104** 441–449. MR0957250 https://doi.org/10.1017/S0305004100065646

[19] MAKARYCHEV, K., MANOKARAN, R. and SVIRIDENKO, M. (2014). Maximum quadratic assignment problem: Reduction from maximum label cover and LP-based approximation algorithm. *ACM Trans. Algorithms* **10** Art. 18. MR3254506 https://doi.org/10.1145/2629672

[20] MOSSEL, E., NEEMAN, J. and SLY, A. (2016). Belief propagation, robust reconstruction and optimal recovery of block models. *Ann. Appl. Probab.* **26** 2211–2256. MR3543895 https://doi.org/10.1214/15-AAP1145

[21] NARAYANAN, A. and SHMATIKOV, V. (2008). Robust De-anonymization of Large Sparse Datasets. In 2008 *IEEE Symposium on Security and Privacy* (*sp* 2008) 111–125.

[22] PARDALOS, P. M., RENDL, F. and WOLKOWICZ, H. (1994). The quadratic assignment problem: A survey and recent developments. In *Quadratic Assignment and Related Problems* (*New Brunswick*, *NJ*, 1993). *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* **16** 1–42. Amer. Math. Soc., Providence, RI. MR1290345 https://doi.org/10.1090/dimacs/016/01

[23] PICCIOLI, G., SEMERJIAN, G., SICURO, G. and ZDEBOROVÁ, L. (2022). Aligning random graphs with a sub-tree similarity message-passing algorithm. *J. Stat. Mech. Theory Exp.* **6** Paper No. 063401. MR4483763 https://doi.org/10.1088/1742-5468/ac70d2

[24] SINGH, R., XU, J. and BERGER, B. (2008). Global alignment of multiple protein interaction networks with application to functional orthology detection. *Proc. Natl. Acad. Sci. USA* **105** 12763–12768.

[25] WU, Y., XU, J. and YU, S. H. (2022). Settling the sharp reconstruction thresholds of random graph matching. *IEEE Trans. Inf. Theory* **68** 5391–5417. MR4476403

[26] WU, Y., XU, J. and YU, S. H. (2023). Testing correlation of unlabeled random graphs. *Ann. Appl. Probab.* **33** 2519–2558. MR4612649 https://doi.org/10.1214/22-aap1786

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2844–2884
https://doi.org/10.1214/23-AAP2022

GENERATING GALTON–WATSON TREES USING RANDOM WALKS AND PERCOLATION FOR THE GAUSSIAN FREE FIELD

BY ALEXANDER DREWITZ[1,a], GIOELE GALLO[1,b] AND ALEXIS PRÉVOST[2,c]

[1]*Department of Mathematics and Computer Science, Universität zu Köln,* [a]*adrewitz@uni-koeln.de,* [b]*gioelegj@gmail.com*
[2]*Section of Mathematics, University of Geneva,* [c]*alexis.prevost@unige.ch*

The study of Gaussian free field level sets on supercritical Galton–Watson trees has been initiated by Abächerli and Sznitman in 2018. By means of entirely different tools, we continue this investigation and generalize their main result on the positivity of the associated percolation critical parameter h_* to the setting of arbitrary supercritical offspring distribution and random conductances. In our setting, this establishes a rigorous proof of the physics literature mantra that positive correlations facilitate percolation when compared to the independent case. Our proof proceeds by constructing the Galton–Watson tree through an exploration via finite random walk trajectories. This exploration of the tree progressively unveils an infinite connected component in the random interlacements set on the tree, which is stable under small quenched noise. Using a Dynkin-type isomorphism theorem, we then infer the strict positivity of the critical parameter h_*. As a byproduct, we obtain transience results for the above-mentioned sets.

REFERENCES

[1] ABÄCHERLI, A. and ČERNÝ, J. (2020). Level-set percolation of the Gaussian free field on regular graphs I: Regular trees. *Electron. J. Probab.* **25** Paper No. 65, 24. MR4115734 https://doi.org/10.1214/20-ejp468

[2] ABÄCHERLI, A. and ČERNÝ, J. (2020). Level-set percolation of the Gaussian free field on regular graphs II: Finite expanders. *Electron. J. Probab.* **25** Paper No. 130, 39. MR4169171 https://doi.org/10.1214/20-ejp532

[3] ABÄCHERLI, A. and SZNITMAN, A.-S. (2018). Level-set percolation for the Gaussian free field on a transient tree. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 173–201. MR3765885 https://doi.org/10.1214/16-AIHP799

[4] BRICMONT, J., LEBOWITZ, J. L. and MAES, C. (1987). Percolation in strongly correlated systems: The massless Gaussian field. *J. Stat. Phys.* **48** 1249–1268. MR0914444 https://doi.org/10.1007/BF01009544

[5] BRYDGES, D., FRÖHLICH, J. and SPENCER, T. (1982). The random walk representation of classical spin systems and correlation inequalities. *Comm. Math. Phys.* **83** 123–150. MR0648362

[6] ČERNÝ, J. (2023). Giant component for the supercritical level-set percolation of the Gaussian free field on regular expander graphs. *Comm. Pure Appl. Math.* **76** 3346–3373. MR4642820

[7] ČERNÝ, J. and LOCHER, R. (2023). Critical and near-critical level-set percolation of the Gaussian free field on regular trees. Preprint, available at arXiv:2302.02753.

[8] ČERNÝ, J. and TEIXEIRA, A. Q. (2012). *From Random Walk Trajectories to Random Interlacements*. *Ensaios Matemáticos* [*Mathematical Surveys*] **23**. Sociedade Brasileira de Matemática, Rio de Janeiro. MR3014964

[9] CHIARINI, A. and NITZSCHNER, M. (2021). Disconnection and entropic repulsion for the harmonic crystal with random conductances. *Comm. Math. Phys.* **386** 1685–1745. MR4299133 https://doi.org/10.1007/s00220-021-04153-4

[10] COLLEVECCHIO, A. (2006). On the transience of processes defined on Galton–Watson trees. *Ann. Probab.* **34** 870–878. MR2243872 https://doi.org/10.1214/009117905000000837

[11] CONCHON-KERJAN, G. (2023). Anatomy of a Gaussian giant: Supercritical level-sets of the free field on regular graphs. *Electron. J. Probab.* **28** Paper No. 35, 60. MR4552959 https://doi.org/10.1214/23-ejp920

MSC2020 subject classifications. 60K35, 60G15, 60G60, 82B43.

[12] DREWITZ, A., PRÉVOST, A. and RODRIGUEZ, P.-F. (2018). The sign clusters of the massless Gaussian free field percolate on $\mathbb{Z}^d$, $d \geq 3$ (and more). *Comm. Math. Phys.* **362** 513–546. MR3843421 https://doi.org/10.1007/s00220-018-3209-6

[13] DREWITZ, A., PRÉVOST, A. and RODRIGUEZ, P.-F. (2018). Geometry of Gaussian free field sign clusters and random interlacements. Preprint, available at arXiv:1811.05970.

[14] DREWITZ, A., PRÉVOST, A. and RODRIGUEZ, P.-F. (2022). Cluster capacity functionals and isomorphism theorems for Gaussian free fields. *Probab. Theory Related Fields* **183** 255–313. MR4421175 https://doi.org/10.1007/s00440-021-01090-0

[15] DREWITZ, A., RÁTH, B. and SAPOZHNIKOV, A. (2014). *An Introduction to Random Interlacements*. *SpringerBriefs in Mathematics*. Springer, Cham. MR3308116 https://doi.org/10.1007/978-3-319-05852-8

[16] DUMINIL-COPIN, H., GOSWAMI, S., RODRIGUEZ, P.-F. and SEVERO, F. (2023). Equality of critical parameters for percolation of Gaussian free field level sets. *Duke Math. J.* **172** 839–913. MR4568695 https://doi.org/10.1215/00127094-2022-0017

[17] EISENBAUM, N., KASPI, H., MARCUS, M. B., ROSEN, J. and SHI, Z. (2000). A Ray–Knight theorem for symmetric Markov processes. *Ann. Probab.* **28** 1781–1796. MR1813843 https://doi.org/10.1214/aop/1019160507

[18] GANTERT, N., MÜLLER, S., POPOV, S. and VACHKOVSKAIA, M. (2012). Random walks on Galton–Watson trees with random conductances. *Stochastic Process. Appl.* **122** 1652–1671. MR2914767 https://doi.org/10.1016/j.spa.2012.01.004

[19] GOSWAMI, S., RODRIGUEZ, P.-F. and SEVERO, F. (2022). On the radius of Gaussian free field excursion clusters. *Ann. Probab.* **50** 1675–1724. MR4474499 https://doi.org/10.1214/22-aop1569

[20] LOÈVE, M. (1977). *Probability Theory. I*, 4th ed. *Graduate Texts in Mathematics* **45**. Springer, New York. MR0651017

[21] LUPU, T. (2016). From loop clusters and random interlacements to the free field. *Ann. Probab.* **44** 2117–2146. MR3502602 https://doi.org/10.1214/15-AOP1019

[22] LYONS, R. (1990). Random walks and percolation on trees. *Ann. Probab.* **18** 931–958. MR1062053

[23] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks*. *Cambridge Series in Statistical and Probabilistic Mathematics* **42**. Cambridge Univ. Press, New York. MR3616205 https://doi.org/10.1017/9781316672815

[24] MARINOV, V. I. and LEBOWITZ, J. L. (2006). Percolation in the harmonic crystal and voter model in three dimensions. *Phys. Rev. E* (3) **74** 031120, 7. MR2282124 https://doi.org/10.1103/PhysRevE.74.031120

[25] PITT, L. D. (1982). Positively correlated normal variables are associated. *Ann. Probab.* **10** 496–499. MR0665603

[26] PRAKASH, S., HAVLIN, S., SCHWARTZ, M. and STANLEY, H. E. (1992). Structural and dynamical properties of long-range correlated percolation. *Phys. Rev. A* **46** R1724–R1727. https://doi.org/10.1103/PhysRevA.46.R1724

[27] RÁTH, B. and SAPOZHNIKOV, A. (2011). On the transience of random interlacements. *Electron. Commun. Probab.* **16** 379–391. MR2819660 https://doi.org/10.1214/ECP.v16-1637

[28] RÁTH, B. and SAPOZHNIKOV, A. (2013). The effect of small quenched noise on connectivity properties of random interlacements. *Electron. J. Probab.* **18** no. 4, 20. MR3024098 https://doi.org/10.1214/EJP.v18-2122

[29] RODRIGUEZ, P.-F. and SZNITMAN, A.-S. (2013). Phase transition and level-set percolation for the Gaussian free field. *Comm. Math. Phys.* **320** 571–601. MR3053773 https://doi.org/10.1007/s00220-012-1649-y

[30] SYMANZIK, K. (1966). Euclidean quantum field theory. I. Equations for a scalar model. *J. Math. Phys.* **7** 510–525. MR0187686 https://doi.org/10.1063/1.1704960

[31] SZNITMAN, A.-S. (2010). Vacant set of random interlacements and percolation. *Ann. of Math.* (2) **171** 2039–2087. MR2680403 https://doi.org/10.4007/annals.2010.171.2039

[32] SZNITMAN, A.-S. (2012). An isomorphism theorem for random interlacements. *Electron. Commun. Probab.* **17** no. 9, 9. MR2892408 https://doi.org/10.1214/ECP.v17-1792

[33] SZNITMAN, A.-S. (2012). *Topics in Occupation Times and Gaussian Free Fields*. *Zurich Lectures in Advanced Mathematics*. Eur. Math. Soc., Zürich. MR2932978 https://doi.org/10.4171/109

[34] SZNITMAN, A.-S. (2015). Disconnection and level-set percolation for the Gaussian free field. *J. Math. Soc. Japan* **67** 1801–1843. MR3417515 https://doi.org/10.2969/jmsj/06741801

[35] SZNITMAN, A.-S. (2016). Coupling and an application to level-set percolation of the Gaussian free field. *Electron. J. Probab.* **21** Paper No. 35, 26. MR3492939 https://doi.org/10.1214/16-EJP4563

[36] TASSY, M. (2010). Random interlacements on Galton–Watson trees. *Electron. Commun. Probab.* **15** 562–571. MR2737713 https://doi.org/10.1214/ECP.v15-1586

[37] TEIXEIRA, A. (2009). Interlacement percolation on transient weighted graphs. *Electron. J. Probab.* **14** 1604–1628. MR2525105 https://doi.org/10.1214/EJP.v14-670

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2885–2939
https://doi.org/10.1214/23-AAP2025

TYPICAL STRUCTURE OF SPARSE EXPONENTIAL RANDOM GRAPH MODELS

BY NICHOLAS A. COOK[1,a] AND AMIR DEMBO[2,b]

[1]*Department of Mathematics, Duke University,* [a]*nickcook@math.duke.edu*
[2]*Department of Mathematics, Stanford University,* [b]*adembo@stanford.edu*

We consider general exponential random graph models (ERGMs) where the sufficient statistics are functions of homomorphism counts for a fixed collection of simple graphs F_k. Whereas previous work has shown a degeneracy phenomenon in dense ERGMs, we show this can be cured by raising the sufficient statistics to a fractional power. We rigorously establish the naïve mean-field approximation for the partition function of the corresponding Gibbs measures, and in case of "ferromagnetic" models with vanishing edge density show that typical samples resemble a typical Erdős–Rényi graph with a planted clique and/or a planted complete bipartite graph of appropriate sizes. We establish such behavior also for the conditional structure of the Erdős–Rényi graph in the large deviations regime for excess F_k-homomorphism counts. These structural results are obtained by combining quantitative large deviation principles, established in previous works, with a novel stability form of a result of (*Adv. Math.* **319** (2017) 313–347) on the asymptotic solution for the associated entropic variational problem. A technical ingredient of independent interest is a stability form of Finner's generalized Hölder inequality.

REFERENCES

[1] AUGERI, F. (2020). Nonlinear large deviation bounds with applications to Wigner matrices and sparse Erdős-Rényi graphs. *Ann. Probab.* **48** 2404–2448. MR4152647 https://doi.org/10.1214/20-AOP1427
[2] BASAK, A. and BASU, R. (2023). Upper tail large deviations of regular subgraph counts in Erdős–Rényi graphs in the full localized regime. *Comm. Pure Appl. Math.* **76** 3–72. MR4544794 https://doi.org/10.1002/cpa.22036
[3] BATTISTON, F., CENCETTI, G., IACOPINI, I., LATORA, V., LUCAS, M., PATANIA, A., YOUNG, J.-G. and PETRI, G. (2020). Networks beyond pairwise interactions: Structure and dynamics. *Phys. Rep.* **874** 1–92. MR4147650 https://doi.org/10.1016/j.physrep.2020.05.004
[4] BHAMIDI, S., BRESLER, G. and SLY, A. (2011). Mixing time of exponential random graphs. *Ann. Appl. Probab.* **21** 2146–2170. MR2895412 https://doi.org/10.1214/10-AAP740
[5] BHATTACHARYA, B. B., GANGULY, S., LUBETZKY, E. and ZHAO, Y. (2017). Upper tails and independence polynomials in random graphs. *Adv. Math.* **319** 313–347. MR3695877 https://doi.org/10.1016/j.aim.2017.08.003
[6] BHATTACHARYA, S. and DEMBO, A. (2021). Upper tail for homomorphism counts in constrained sparse random graphs. *Random Structures Algorithms* **59** 315–338. MR4295566 https://doi.org/10.1002/rsa.21011
[7] CALDERÓN, A.-P. (1976). An inequality for integrals. *Studia Math.* **57** 275–277. MR0422544 https://doi.org/10.4064/sm-57-3-275-277
[8] CHATTERJEE, S. (2017). *Large Deviations for Random Graphs*. *Lecture Notes in Math.* **2197**. Springer, Cham. MR3700183 https://doi.org/10.1007/978-3-319-65816-2
[9] CHATTERJEE, S. and DEMBO, A. (2016). Nonlinear large deviations. *Adv. Math.* **299** 396–450. MR3519474 https://doi.org/10.1016/j.aim.2016.05.017
[10] CHATTERJEE, S. and DIACONIS, P. (2013). Estimating and understanding exponential random graph models. *Ann. Statist.* **41** 2428–2461. MR3127871 https://doi.org/10.1214/13-AOS1155

MSC2020 subject classifications. 60F10, 05C80, 60C05, 82B26.
Key words and phrases. Large deviations, Erdős, Rényi graphs, homomorphism counts, Gibbs measures, variational problems, upper tails, Brascamp, Lieb inequality.

[11] CHATTERJEE, S. and VARADHAN, S. R. S. (2011). The large deviation principle for the Erdős–Rényi random graph. *European J. Combin.* **32** 1000–1017. MR2825532 https://doi.org/10.1016/j.ejc.2011.03.014

[12] CHRIST, M. (2019). Near equality in the Riesz–Sobolev inequality. *Acta Math. Sin.* (*Engl. Ser.*) **35** 783–814. MR3952692 https://doi.org/10.1007/s10114-019-8412-7

[13] CHRIST, M. and O'NEILL, K. Maximizers of Rogers–Brascamp–Lieb–Luttinger functionals in higher dimensions. Available at arXiv:1712.00109.

[14] COHEN ANTONIR, A. (2024). The upper tail problem for induced 4-cycles in sparse random graphs. *Random Structures Algorithms* **64** 401–459. https://doi.org/10.1002/rsa.21187

[15] COOK, N. and DEMBO, A. (2020). Large deviations of subgraph counts for sparse Erdős–Rényi graphs. *Adv. Math.* **373** 107289. MR4130460 https://doi.org/10.1016/j.aim.2020.107289

[16] COOK, N. A., DEMBO, A. and PHAM, H. T. Regularity method and large deviation principles for the Erdős–Rényi hypergraph. *Duke Math. J.* To appear. Available at arXiv:2102.09100.

[17] DEMBO, A. and ZEITOUNI, O. (2002). Large deviations and applications. In *Handbook of Stochastic Analysis and Applications*. *Statist. Textbooks Monogr.* **163** 361–416. Dekker, New York. MR1882715

[18] DEMUSE, R. Extremal behavior in exponential random graphs. Available at arXiv:1906.00525.

[19] ELDAN, R. (2018). Gaussian-width gradient complexity, reverse log-Sobolev inequalities and nonlinear large deviations. *Geom. Funct. Anal.* **28** 1548–1596. MR3881829 https://doi.org/10.1007/s00039-018-0461-z

[20] ELDAN, R. and GROSS, R. (2018). Exponential random graphs behave like mixtures of stochastic block models. *Ann. Appl. Probab.* **28** 3698–3735. MR3861824 https://doi.org/10.1214/18-AAP1402

[21] ELDAN, R. and GROSS, R. (2018). Decomposition of mean-field Gibbs distributions into product measures. *Electron. J. Probab.* **23** Paper No. 35. MR3798245 https://doi.org/10.1214/18-EJP159

[22] ELLIS, D., FRIEDGUT, E., KINDLER, G. and YEHUDAYOFF, A. (2016). Geometric stability via information theory. *Discrete Anal.* Paper No. 10. MR3555193 https://doi.org/10.19086/da.784

[23] FIENBERG, S. E. (2010). Introduction to papers on the modeling and analysis of network data. *Ann. Appl. Stat.* **4** 1–4. MR2758081 https://doi.org/10.1214/10-AOAS346

[24] FIENBERG, S. E. (2010). Introduction to papers on the modeling and analysis of network data—II. *Ann. Appl. Stat.* **4** 533–534. MR2744531 https://doi.org/10.1214/10-AOAS365

[25] FIGALLI, A. (2014). Quantitative stability results for the Brunn–Minkowski inequality. In *Proceedings of the International Congress of Mathematicians—Seoul* 2014. *Vol. III* 237–256. Kyung Moon Sa, Seoul. MR3729026

[26] FIGALLI, A. and JERISON, D. (2017). Quantitative stability for the Brunn–Minkowski inequality. *Adv. Math.* **314** 1–47. MR3658711 https://doi.org/10.1016/j.aim.2016.12.018

[27] FINNER, H. (1992). A generalization of Hölder's inequality and some probability inequalities. *Ann. Probab.* **20** 1893–1901. MR1188047

[28] FRANK, O. and STRAUSS, D. (1986). Markov graphs. *J. Amer. Statist. Assoc.* **81** 832–842. MR0860518

[29] GANGULY, S. and NAM, K. (2024). Sub-critical exponential random graphs: Concentration of measure and some applications. *Trans. Amer. Math. Soc.* To appear. https://doi.org/10.1090/tran/8690

[30] GUNBY, B. Upper tails of subgraph counts in sparse regular graphs. Available at arXiv:2010.00658.

[31] HANDCOCK, M. S. (2003). Assessing degeneracy in statistical models of social networks. Working Paper 39. Tech. Rep., Center for Statistics and Social Sciences, University of Washington.

[32] HAREL, M., MOUSSET, F. and SAMOTIJ, W. (2022). Upper tails via high moments and entropic stability. *Duke Math. J.* **171** 2089–2192. MR4484206 https://doi.org/10.1215/00127094-2021-0067

[33] HOLLAND, P. W. and LEINHARDT, S. (1981). An exponential family of probability distributions for directed graphs. *J. Amer. Statist. Assoc.* **76** 33–65. MR0608176

[34] HORVÁT, S., CZABARKA, É. and TOROCZKAI, Z. (2015). Reducing degeneracy in maximum entropy models of networks. *Phys. Rev. Lett.* **114** 158701. https://doi.org/10.1103/PhysRevLett.114.158701

[35] KENYON, R., RADIN, C., REN, K. and SADUN, L. (2017). Multipodal structure and phase transitions in large constrained graphs. *J. Stat. Phys.* **168** 233–258. MR3667360 https://doi.org/10.1007/s10955-017-1804-0

[36] KENYON, R. and YIN, M. (2017). On the asymptotics of constrained exponential random graphs. *J. Appl. Probab.* **54** 165–180. MR3632612 https://doi.org/10.1017/jpr.2016.93

[37] KOZMA, G. and SAMOTIJ, W. (2023). Lower tails via relative entropy. *Ann. Probab.* **51** 665–698. MR4546629 https://doi.org/10.1214/22-aop1610

[38] LIU, Y. P. and ZHAO, Y. (2021). On the upper tail problem for random hypergraphs. *Random Structures Algorithms* **58** 179–220. MR4201796 https://doi.org/10.1002/rsa.20975

[39] LUBETZKY, E. and ZHAO, Y. (2015). On replica symmetry of large deviations in random graphs. *Random Structures Algorithms* **47** 109–146. MR3366814 https://doi.org/10.1002/rsa.20536

[40] LUBETZKY, E. and ZHAO, Y. (2017). On the variational problem for upper tails in sparse random graphs. *Random Structures Algorithms* **50** 420–436. MR3632418 https://doi.org/10.1002/rsa.20658

[41] MUKHERJEE, S. and BHATTACHARYA, B. B. (2020). Replica symmetry in upper tails of mean-field hypergraphs. *Adv. in Appl. Math.* **119** 102047. MR4092987 https://doi.org/10.1016/j.aam.2020.102047

[42] NEEMAN, J., RADIN, C. and SADUN, L. (2020). Phase transitions in finite random networks. *J. Stat. Phys.* **181** 305–328. MR4142953 https://doi.org/10.1007/s10955-020-02582-4

[43] RADIN, C., REN, K. and SADUN, L. (2014). The asymptotics of large constrained graphs. *J. Phys. A* **47** 175001. MR3197554 https://doi.org/10.1088/1751-8113/47/17/175001

[44] RADIN, C. and SADUN, L. (2013). Phase transitions in a complex network. *J. Phys. A* **46** 305002. MR3083277 https://doi.org/10.1088/1751-8113/46/30/305002

[45] SNIJDERS, T. A. (2002). Markov chain Monte Carlo estimation of exponential random graph models. *J. Soc. Struct.* **3**.

[46] SNIJDERS, T. A., PATTISON, P., ROBBINS, G. and HANDCOCK, M. (2006). New specifications for exponential random graph models. *Sociol. Method.* **36** 99–153.

[47] STASI, D., SADEGHI, K., RINALDO, A., PETROVIC, S. and FIENBERG, S. (2014). β models for random hypergraphs with a given degree sequence. In *Proceedings of COMPSTAT* 2014—21*st International Conference on Computational Statistics* 593–600. Internat. Statist. Inst., The Hague. MR3372442

[48] STRAUSS, D. (1986). On a general class of models for interaction. *SIAM Rev.* **28** 513–527. MR0867682 https://doi.org/10.1137/1028156

[49] WASSERMAN, S. and PATTISON, P. (1996). Logit models and logistic regressions for social networks. I. An introduction to Markov graphs and p. *Psychometrika* **61** 401–425. MR1424909 https://doi.org/10.1007/BF02294547

[50] YIN, M., RINALDO, A. and FADNAVIS, S. (2016). Asymptotic quantization of exponential random graphs. *Ann. Appl. Probab.* **26** 3251–3285. MR3582803 https://doi.org/10.1214/16-AAP1175

[51] YIN, M. and ZHU, L. (2017). Asymptotics for sparse exponential random graph models. *Braz. J. Probab. Stat.* **31** 394–412. MR3635912 https://doi.org/10.1214/16-BJPS319

[52] ZUEV, K., BOGUÑÁ, M., BIANCONI, G. and KRIOUKOV, D. (2015). Emergence of soft communities from geometric preferential attachment. *Sci. Rep.* **5** 9421. https://doi.org/10.1038/srep09421

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2940–2985
https://doi.org/10.1214/23-AAP2030

OPEN MARKETS AND HYBRID JACOBI PROCESSES

BY DAVID ITKIN[1,a] AND MARTIN LARSSON[2,b]

[1]*Department of Mathematics, Imperial College London,* [a]*d.itkin@imperial.ac.uk*
[2]*Department of Mathematical Sciences, Carnegie Mellon University,* [b]*martinl@andrew.cmu.edu*

We propose a unified approach to several problems in stochastic portfolio theory (SPT), which is a framework for equity markets with a large number d of stocks. Our approach combines *open markets*, where trading is confined to the top N capitalized stocks as well as the market portfolio consisting of all d assets, with a parametric family of models which we call *hybrid Jacobi processes*. We provide a detailed analysis of ergodicity, particle collisions, and boundary attainment, and use these results to study the associated financial markets. Their properties include (1) stability of the capital distribution curve and (2) explicit and not artificially leveraged growth optimal strategies. The sub-class of *rank Jacobi models* are additionally shown to (3) serve as the worst-case model for a robust asymptotic growth problem under model ambiguity and (4) exhibit stability in the large-d limit. Our definition of an open market is a relaxation of existing definitions which is essential to make the analysis tractable.

REFERENCES

[1] BANNER, A. D., FERNHOLZ, R. and KARATZAS, I. (2005). Atlas models of equity markets. *Ann. Appl. Probab.* **15** 2296–2330. MR2187296 https://doi.org/10.1214/105051605000000449
[2] BANNER, A. D. and GHOMRASNI, R. (2008). Local times of ranked continuous semimartingales. *Stochastic Process. Appl.* **118** 1244–1253. MR2428716 https://doi.org/10.1016/j.spa.2007.08.001
[3] CAMPBELL, S. and WONG, T.-K. L. (2022). Functional portfolio optimization in stochastic portfolio theory. *SIAM J. Financial Math.* **13** 576–618. MR4416997 https://doi.org/10.1137/21M1417715
[4] CUCHIERO, C. (2019). Polynomial processes in stochastic portfolio theory. *Stochastic Process. Appl.* **129** 1829–1872. MR3944786 https://doi.org/10.1016/j.spa.2018.06.007
[5] EBERLE, A. (2009). Markov processes. Lecture Notes at University of Bonn.
[6] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658
[7] FERNHOLZ, E. R. (2002). *Stochastic Portfolio Theory: Stochastic Modelling and Applied Probability. Applications of Mathematics (New York)* **48**. Springer, New York. MR1894767 https://doi.org/10.1007/978-1-4757-3699-1
[8] FERNHOLZ, R. (1999). On the diversity of equity markets. *J. Math. Econom.* **31** 393–417. MR1684221 https://doi.org/10.1016/S0304-4068(97)00018-9
[9] FERNHOLZ, R. (2018). Numeraire markets. ArXiv preprint. Available at arXiv:1801.07309.
[10] FERNHOLZ, R., ICHIBA, T. and KARATZAS, I. (2013). A second-order stock market model. *Ann. Finance* **9** 439–454. MR3082660 https://doi.org/10.1007/s10436-012-0193-2
[11] FERNHOLZ, R. and KARATZAS, I. (2005). Relative arbitrage in volatility-stabilized markets. *Ann. Finance* **1** 149–177.
[12] FILIPOVIĆ, D. and LARSSON, M. (2016). Polynomial diffusions and applications in finance. *Finance Stoch.* **20** 931–972. MR3551857 https://doi.org/10.1007/s00780-016-0304-4
[13] FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (1994). *Dirichlet Forms and Symmetric Markov Processes. De Gruyter Studies in Mathematics* **19**. de Gruyter, Berlin. MR1303354 https://doi.org/10.1515/9783110889741
[14] GOURIEROUX, C. and JASIAK, J. (2006). Multivariate Jacobi process with application to smooth transitions. *J. Econometrics* **131** 475–505. MR2276008 https://doi.org/10.1016/j.jeconom.2005.01.014

MSC2020 subject classifications. Primary 60G44, 60J60, 91G15; Secondary 60J46.
Key words and phrases. Stochastic portfolio theory, open markets, growth optimality, Dirichlet forms, boundary attainment, robust finance, Poisson–Dirichlet distribution.

[15] ICHIBA, T., PAPATHANAKOS, V., BANNER, A., KARATZAS, I. and FERNHOLZ, R. (2011). Hybrid atlas models. *Ann. Appl. Probab.* **21** 609–644. MR2807968 https://doi.org/10.1214/10-AAP706

[16] ITKIN, D., KOCH, B., LARSSON, M. and TEICHMANN, J. (2022). Ergodic robust maximization of asymptotic growth under stochastic volatility. ArXiv preprint. Available at arXiv:2211.15628.

[17] ITKIN, D. and LARSSON, M. (2021). On A class of rank-based continuous semimartingales. ArXiv preprint. Available at arXiv:2104.04396.

[18] ITKIN, D. and LARSSON, M. (2022). Robust asymptotic growth in stochastic portfolio theory under long-only constraints. *Math. Finance* **32** 114–171. MR4370804 https://doi.org/10.1111/mafi.12331

[19] KARATZAS, I. and KARDARAS, C. (2021). *Portfolio Theory and Arbitrage—a Course in Mathematical Finance*. *Graduate Studies in Mathematics* **214**. Amer. Math. Soc., Providence, RI. MR4422028 https://doi.org/10.1090/gsm/214

[20] KARATZAS, I. and KIM, D. (2021). Open markets. *Math. Finance* **31** 1111–1161. MR4332842 https://doi.org/10.1111/mafi.12294

[21] KARATZAS, I. and RUF, J. (2017). Trading strategies generated by Lyapunov functions. *Finance Stoch.* **21** 753–787. MR3663643 https://doi.org/10.1007/s00780-017-0332-8

[22] KARDARAS, C. and ROBERTSON, S. (2012). Robust maximization of asymptotic growth. *Ann. Appl. Probab.* **22** 1576–1610. MR2985170 https://doi.org/10.1214/11-AAP802

[23] KARDARAS, C. and ROBERTSON, S. (2021). Ergodic robust maximization of asymptotic growth. *Ann. Appl. Probab.* **31** 1787–1819. MR4312847 https://doi.org/10.1214/20-aap1634

[24] LEONI, G. (2017). *A First Course in Sobolev Spaces*, 2nd ed. *Graduate Studies in Mathematics* **181**. Amer. Math. Soc., Providence, RI. MR3726909 https://doi.org/10.1090/gsm/181

[25] MA, Z. M. and RÖCKNER, M. (1992). *Introduction to the Theory of* (*Nonsymmetric*) *Dirichlet Forms*. *Universitext*. Springer, Berlin. MR1214375 https://doi.org/10.1007/978-3-642-77739-4

[26] PICKOVÁ, R. (2014). Generalized volatility-stabilized processes. *Ann. Finance* **10** 101–125. MR3159207 https://doi.org/10.1007/s10436-013-0230-9

[27] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[28] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. *Grundlehren der Mathematischen Wissenschaften* **233**. Springer, Berlin. MR0532498

The Annals of Applied Probability
2024, Vol. 34, No. 3, 2986–3017
https://doi.org/10.1214/23-AAP2031

ARITHMETIC OSCILLATIONS OF THE CHEMICAL DISTANCE IN LONG-RANGE PERCOLATION ON $\mathbb{Z}^d$

BY MAREK BISKUP[a] AND ANDREW KRIEGER[b]

Department of Mathematics, University of California, Los Angeles, [a]*biskup@math.ucla.edu,* [b]*akrieger@math.ucla.edu*

We consider a long-range percolation graph on $\mathbb{Z}^d$ where, in addition to the nearest-neighbor edges of $\mathbb{Z}^d$, distinct $x, y \in \mathbb{Z}^d$ are connected by an edge independently with probability asymptotic to $\beta|x-y|^{-s}$, for $s \in (d, 2d)$, $\beta > 0$ and $|\cdot|$ a norm on $\mathbb{R}^d$. We first show that, for all but perhaps a countably many $\beta > 0$, the graph-theoretical (a.k.a. chemical) distance between typical vertices at $|\cdot|$-distance r is, with high probability as $r \to \infty$, asymptotic to $\phi_\beta(r)(\log r)^\Delta$, where $\Delta^{-1} := \log_2(2d/s)$ and ϕ_β is a deterministic, positive, bounded and continuous function subject to log-log-periodicity constraint $\phi_\beta(r^\gamma) = \phi_\beta(r)$ for $\gamma := s/(2d)$. The proof parallels the arguments developed in a continuum version of the model where a similar scaling was shown earlier by the first author and J. Lin. That work also conjectured that ϕ_β is constant which we show to be false by proving that $(\log \beta)^\Delta \phi_\beta$ tends, as $\beta \to \infty$, to a nonconstant limit that is independent of the specifics of the model. The proof reveals arithmetic rigidity of the shortest paths that maintain a hierarchical (dyadic) structure all the way to unit scales.

REFERENCES

[1] ANTAL, P. and PISZTORA, A. (1996). On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.* **24** 1036–1048. MR1404543 https://doi.org/10.1214/aop/1039639377

[2] AUFFINGER, A., DAMRON, M. and HANSON, J. (2017). 50 *Years of First-Passage Percolation*. *University Lecture Series* **68**. Amer. Math. Soc., Providence, RI. MR3729447 https://doi.org/10.1090/ulect/068

[3] BÄUMLER, J. (2023). Distances in $\frac{1}{\|x-y\|^{2d}}$ Percolation Models for all Dimensions. *Comm. Math. Phys.* **404** 1495–1570. MR4677584 https://doi.org/10.1007/s00220-023-04861-z

[4] BÄUMLER, J. and BERGER, N. (2022). Isoperimetric lower bounds for critical exponents for long-range percolation. Available at arXiv:2204.12410.

[5] BENJAMINI, I. and BERGER, N. (2001). The diameter of long-range percolation clusters on finite cycles. *Random Structures Algorithms* **19** 102–111. MR1848786 https://doi.org/10.1002/rsa.1022

[6] BENJAMINI, I., BERGER, N. and YADIN, A. (2008). Long-range percolation mixing time. *Combin. Probab. Comput.* **17** 487–494. MR2433935 https://doi.org/10.1017/S0963548308008948

[7] BENJAMINI, I., KESTEN, H., PERES, Y. and SCHRAMM, O. (2004). Geometry of the uniform spanning forest: Transitions in dimensions $4, 8, 12, \ldots$. *Ann. of Math.* (2) **160** 465–491. MR2123930 https://doi.org/10.4007/annals.2004.160.465

[8] BERGER, N. (2002). Transience, recurrence and critical behavior for long-range percolation. *Comm. Math. Phys.* **226** 531–558. MR1896880 https://doi.org/10.1007/s002200200617

[9] BERGER, N. (2004). A lower bound for the chemical distance in sparse long-range percolation models. Available at arXiv:math.PR/0409021.

[10] BISKUP, M. (2004). On the scaling of the chemical distance in long-range percolation models. *Ann. Probab.* **32** 2938–2977. MR2094435 https://doi.org/10.1214/009117904000000577

[11] BISKUP, M. (2011). Graph diameter in long-range percolation. *Random Structures Algorithms* **39** 210–227. MR2850269 https://doi.org/10.1002/rsa.20349

[12] BISKUP, M., CHEN, X., KUMAGAI, T. and WANG, J. (2021). Quenched invariance principle for a class of random conductance models with long-range jumps. *Probab. Theory Related Fields* **180** 847–889. MR4288333 https://doi.org/10.1007/s00440-021-01059-z

[13] BISKUP, M. and LIN, J. (2019). Sharp asymptotic for the chemical distance in long-range percolation. *Random Structures Algorithms* **55** 560–583. MR3997479 https://doi.org/10.1002/rsa.20849

MSC2020 subject classifications. Primary 60K35, 82B43; secondary 05C12, 05C82.
Key words and phrases. Long range percolation, graph-theoretical distance, asymptotic.

[14] CAN, V. H., CROYDON, D. A. and KUMAGAI, T. (2022). Spectral dimension of simple random walk on a long-range percolation cluster. *Electron. J. Probab.* **27** 1–37. MR4417202 https://doi.org/10.1214/22-ejp783

[15] CHATTERJEE, S. and DEY, P. S. (2016). Multiple phase transitions in long-range first-passage percolation on square lattices. *Comm. Pure Appl. Math.* **69** 203–256. MR3434612 https://doi.org/10.1002/cpa.21571

[16] COPPERSMITH, D., GAMARNIK, D. and SVIRIDENKO, M. (2002). The diameter of a long-range percolation graph. *Random Structures Algorithms* **21** 1–13. MR1913075 https://doi.org/10.1002/rsa.10042

[17] CRAWFORD, N. and SLY, A. (2012). Simple random walk on long range percolation clusters I: Heat kernel bounds. *Probab. Theory Related Fields* **154** 753–786. MR3000561 https://doi.org/10.1007/s00440-011-0383-2

[18] CRAWFORD, N. and SLY, A. (2013). Simple random walk on long-range percolation clusters II: Scaling limits. *Ann. Probab.* **41** 445–502. MR3077517 https://doi.org/10.1214/12-AOP774

[19] DEIJFEN, M., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. (2013). Scale-free percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 817–838. MR3112435 https://doi.org/10.1214/12-AIHP480

[20] DEPREZ, P., HAZRA, R. S. and WÜTTRICH, M. (2015). Inhomogeneous long-range percolation for real-life network modeling. *Risks* **3** 1–23.

[21] DING, J. and SLY, A. (2013). Distances in critical long range percolation. Available at arXiv:1303.3995.

[22] GARET, O. and MARCHAND, R. (2004). Asymptotic shape for the chemical distance and first-passage percolation on the infinite Bernoulli cluster. *ESAIM Probab. Stat.* **8** 169–199. MR2085613 https://doi.org/10.1051/ps:2004009

[23] HAMMERSLEY, J. M. and WELSH, D. J. A. (1965). First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, Calif.*, 1963 61–110. Springer, New York. MR0198576

[24] HAO, N. and HEYDENREICH, M. (2023). Graph distances in scale-free percolation: The logarithmic case. *J. Appl. Probab.* **60** 295–313. MR4546124 https://doi.org/10.1017/jpr.2022.44

[25] HEYDENREICH, M., HULSHOF, T. and JORRITSMA, J. (2017). Structures in supercritical scale-free percolation. *Ann. Appl. Probab.* **27** 2569–2604. MR3693534 https://doi.org/10.1214/16-AAP1270

[26] HUTCHCROFT, T. (2021). Power-law bounds for critical long-range percolation below the upper-critical dimension. *Probab. Theory Related Fields* **181** 533–570. MR4341081 https://doi.org/10.1007/s00440-021-01043-7

[27] KINGMAN, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B* **30** 499–510. MR0254907

[28] KINGMAN, J. F. C. (1973). Subadditive ergodic theory. *Ann. Probab.* **1** 883–899. MR0356192 https://doi.org/10.1214/aop/1176996798

[29] KUMAGAI, T. and MISUMI, J. (2008). Heat kernel estimates for strongly recurrent random walk on random media. *J. Theoret. Probab.* **21** 910–935. MR2443641 https://doi.org/10.1007/s10959-008-0183-5

[30] MISUMI, J. (2008). Estimates on the effective resistance in a long-range percolation on $\mathbb{Z}^d$. *J. Math. Kyoto Univ.* **48** 389–400. MR2436744 https://doi.org/10.1215/kjm/1250271419

[31] TRAPMAN, P. (2010). The growth of the infinite long-range percolation cluster. *Ann. Probab.* **38** 1583–1608. MR2663638 https://doi.org/10.1214/09-AOP517

[32] VAN DEN BERG, J. and KESTEN, H. (1985). Inequalities with applications to percolation and reliability. *J. Appl. Probab.* **22** 556–569. MR0799280 https://doi.org/10.1017/s0021900200029326

[33] WU, T. (2022). A sharp leading order asymptotic of the diameter of a long range percolation graph. Available at arXiv:2211.16500.

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3018–3092
https://doi.org/10.1214/23-AAP2032

INVASION PERCOLATION ON POWER-LAW BRANCHING PROCESSES

BY ROWEL GÜNDLACH[a] AND REMCO VAN DER HOFSTAD[b]

Mathematics and Computer Science, Eindhoven University of Technology, [a]*r.c.gundlach@tue.nl,* [b]*r.w.v.d.hofstad@tue.nl*

We analyse the cluster discovered by invasion percolation on a branching process with a power-law offspring distribution. Invasion percolation is a paradigm model of *self-organised criticality*, where criticality is approach without tuning any parameter. By performing invasion percolation for n steps, and letting $n \to \infty$, we find an infinite subtree, called *the invasion percolation cluster* (IPC). A notable feature of the IPC is its geometry that consists of a unique path to infinity (also called the *backbone*) onto which finite forests are attached. The main theorem shows the volume scaling limit of the k*-cut IPC*, which is the cluster containing the root when the edge between the kth and $(k+1)$st backbone vertices is cut.

We assume a power-law offspring distribution with exponent α and analyse the IPC for different power-law regimes. In a finite-variance setting ($\alpha > 2$) the results, are a natural extension of previous works on the branching process tree (*Electron. J. Probab.* **24** (2019) 1–35) and the regular tree (*Ann. Probab.* **35** (2008) 420–466). However, for an infinite-variance setting ($\alpha \in (1, 2)$) or even an infinite-mean setting ($\alpha \in (0, 1)$), results significantly change. This is illustrated by the volume scaling of the k-cut IPC, which scales as k^2 for $\alpha > 2$, but as $k^{\alpha/(\alpha-1)}$ for $\alpha \in (1, 2)$ and exponentially for $\alpha \in (0, 1)$.

REFERENCES

[1] ADDARIO-BERRY, L. (2013). The local weak limit of the minimum spanning tree of the complete graph. Preprint. Available at arXiv:1301.1667v2 [math.PR].

[2] ADDARIO-BERRY, L., BROUTIN, N., GOLDSCHMIDT, C. and MIERMONT, G. (2017). The scaling limit of the minimum spanning tree of the complete graph. *Ann. Probab.* **45** 3075–3144. MR3706739 https://doi.org/10.1214/16-AOP1132

[3] ADDARIO-BERRY, L., GRIFFITHS, S. and KANG, R. J. (2012). Invasion percolation on the Poisson-weighted infinite tree. *Ann. Appl. Probab.* **22** 931–970. MR2977982 https://doi.org/10.1214/11-AAP761

[4] ADDARIO-BERRY, L. and SEN, S. (2021). Geometry of the minimal spanning tree of a random 3-regular graph. *Probab. Theory Related Fields* **180** 553–620. MR4288328 https://doi.org/10.1007/s00440-021-01071-3

[5] ALDOUS, D. and STEELE, J. M. (2004). The objective method: Probabilistic combinatorial optimization and local weak convergence. In *Probability on Discrete Structures. Encyclopaedia Math. Sci.* **110** 1–72. Springer, Berlin. MR2023650 https://doi.org/10.1007/978-3-662-09444-0_1

[6] ANGEL, O., GOODMAN, J., DEN HOLLANDER, F. and SLADE, G. (2008). Invasion percolation on regular trees. *Ann. Probab.* **36** 420–466. MR2393988 https://doi.org/10.1214/07-AOP346

[7] BAK, P., TANG, C. and WIESENFELD, K. (1988). Self-organized criticality. *Phys. Rev. A* (3) **38** 364–374. MR0949160 https://doi.org/10.1103/PhysRevA.38.364

[8] BARABÁSI, (1996). Invasion percolation and global optimization. *Phys. Rev. Lett.* **76** 3750–3753. https://doi.org/10.1103/PhysRevLett.76.3750

[9] BEN AROUS, G., MOLCHANOV, S. and RAMÍREZ, A. F. (2005). Transition from the annealed to the quenched asymptotics for a random walk on random obstacles. *Ann. Probab.* **33** 2149–2187. MR2184094 https://doi.org/10.1214/009117905000000404

[10] BENGS, V. and HOLZMANN, H. (2019). Uniform approximation in classical weak convergence theory. Preprint. Available at arXiv:1903.09864 [math.PR].

MSC2020 subject classifications. Primary 60J80, 60K35; secondary 05C80.
Key words and phrases. Invasion percolation, branching process tree, volume growth, power-law offspring distribution.

[11] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962

[12] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation*. *Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR0898871 https://doi.org/10.1017/CBO9780511721434

[13] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966 https://doi.org/10.1017/CBO9780511814068

[14] BOLLOBÁS, B. and RIORDAN, O. (2012). *Percolation*. Cambridge Univ. Press, New York.

[15] BONNAR, J. (2017). The gamma function. Independently published.

[16] BOYCE, W. E. and DIPRIMA, R. C. (2003). *Elementary Differential Equations and Boundary Value Problems*. 5th ed. Wiley, New York.

[17] BROADBENT, S. R. and HAMMERSLEY, J. M. (1957). Percolation processes. I. Crystals and mazes. *Proc. Camb. Philos. Soc.* **53** 629–641. MR0091567 https://doi.org/10.1017/s0305004100032680

[18] CAMES VAN BATENBURG, W. P. S. (2015). The dimension of the incipient infinite cluster. *Electron. Commun. Probab.* **20** 1–10. MR3342167 https://doi.org/10.1214/ECP.v20-3570

[19] CHANDLER, R., KOPLIK, J., LERMAN, K. and WILLEMSEN, J. F. (1982). Capillary displacement and percolation in porous media. *J. Fluid Mech.* **119** 249–267. https://doi.org/10.1017/S0022112082001335

[20] CHAYES, J. T., CHAYES, L. and NEWMAN, C. M. (1985). The stochastic geometry of invasion percolation. *Comm. Math. Phys.* **101** 383–407. MR0815191

[21] CHAYES, J. T., CHAYES, L. and NEWMAN, C. M. (1987). Bernoulli percolation above threshold: An invasion percolation analysis. *Ann. Probab.* **15** 1272–1287. MR0905331

[22] DAMRON, M., SAPOZHNIKOV, A. and VÁGVÖLGYI, B. (2009). Relations between invasion percolation and critical percolation in two dimensions. *Ann. Probab.* **37** 2297–2331. MR2573559 https://doi.org/10.1214/09-AOP462

[23] EBRAHIMI, F. (2010). Invasion percolation: A computational algorithm for complex phenomena. *Comput. Sci. Eng.* **12** 84–93. https://doi.org/10.1109/MCSE.2010.42

[24] FISHER, M. E. (1961). Critical probabilities for cluster size and percolation problems. *J. Math. Phys.* **2** 620–627. MR0126306 https://doi.org/10.1063/1.1703746

[25] FRISTEDT, B. and GRAY, L. (1997). *A Modern Approach to Probability Theory*. *Probability and Its Applications*. Birkhäuser, Boston, MA. MR1422917 https://doi.org/10.1007/978-1-4899-2837-5

[26] GABRIELLI, A. and CALDARELLI, G. (2007). Invasion percolation and critical transient in the Barabási model of human dynamics. *Phys. Rev. Lett.* **98** 1–4. https://doi.org/10.1103/PhysRevLett.98.208701

[27] GABRIELLI, A. and CALDARELLI, G. (2009). Invasion percolation on a tree and queueing models. *Phys. Rev. E, Stat. Nonlin. Soft Matter Phys.* **79** 1–7. https://doi.org/10.1103/PhysRevE.79.041133

[28] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1968). *Limit Distributions for Sums of Independent Random Variables*, Revised ed. Addison-Wesley, Reading Translated from the Russian, annotated, and revised by K. L. Chung. MR0233400

[29] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339 https://doi.org/10.1007/978-3-662-03981-6

[30] GÜNDLACH, R. (2019). On the local behaviour of the minimal spanning tree on the configuration model Master's thesis, Eindhoven university of technology, Eindhoven.

[31] GÜNDLACH, R. and VAN DER HOFSTAD, R. (2024). Supplement to "Invasion Percolation on Power-Law Branching Processes." https://doi.org/10.1214/23-AAP2032SUPP

[32] HALIN, R. (1964). Über unendliche Wege in Graphen. *Math. Ann.* **157** 125–137. MR0170340 https://doi.org/10.1007/BF01362670

[33] HAMMERSLEY, J. M. (1957). Percolation processes: Lower bounds for the critical probability. *Ann. Math. Stat.* **28** 790–795. MR0101564 https://doi.org/10.1214/aoms/1177706894

[34] HARRIS, T. E. (1960). A lower bound for the critical probability in a certain percolation process. *Proc. Camb. Philos. Soc.* **56** 13–20. MR0115221

[35] HEYDENREICH, M., VAN DER HOFSTAD, R. and HULSHOF, T. (2014). High-dimensional incipient infinite clusters revisited. *J. Stat. Phys.* **155** 966–1025. MR3198669 https://doi.org/10.1007/s10955-014-0979-x

[36] JANSON, S. (2006). Random cutting and records in deterministic and random trees. *Random Structures Algorithms* **29** 139–179. MR2245498 https://doi.org/10.1002/rsa.20086

[37] JANSON, S., ŁUCZAK, T. and RUCINSKI, A. (2000). *Random Graphs*. *Wiley-Interscience Series in Discrete Mathematics and Optimization*. Wiley-Interscience, New York. MR1782847 https://doi.org/10.1002/9781118032718

[38] JÁRAI, A. A. (2003). Invasion percolation and the incipient infinite cluster in 2D. *Comm. Math. Phys.* **236** 311–334. MR1981994 https://doi.org/10.1007/s00220-003-0796-6

[39] KARLIN, S. and TAYLOR, H. M. (1975). *A First Course in Stochastic Processes*, 2nd ed. Academic Press, San Diego. MR0356197

[40] KESTEN, H. (1980). The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.* **74** 41–59. MR0575895

[41] KESTEN, H. (1982). *Percolation Theory for Mathematicians*. *Progress in Probability and Statistics* **2**. Birkhäuser, Boston, MA. MR0692943

[42] KESTEN, H. (1986). The incipient infinite cluster in two-dimensional percolation. *Probab. Theory Related Fields* **73** 369–394. MR0859839 https://doi.org/10.1007/BF00776239

[43] KORTCHEMSKI, I. (2017). Sub-exponential tail bounds for conditioned stable Bienaymé–Galton–Watson trees. *Probab. Theory Related Fields* **168** 1–40. MR3651047 https://doi.org/10.1007/s00440-016-0704-6

[44] KRANTZ, S. G. and PARKS, H. R. (2003). *The Implicit Function Theorem: History, Theory, and Applications*. Birkhäuser, Boston, MA. MR1894435 https://doi.org/10.1007/978-1-4612-0059-8

[45] KULIK, R. and SOULIER, P. (2020). *Heavy-Tailed Time Series*. *Springer Series in Operations Research and Financial Engineering*. Springer, New York. MR4174389 https://doi.org/10.1007/978-1-0716-0737-4

[46] LAIDLAW, W. G., WILSON, W. G. and COOMBE, D. A. (1993). A lattice model of foam flow in porous media: A percolation approach. *Transp. Porous Media* **11** 139–159. https://doi.org/10.1007/BF01059631

[47] LAST, G. and PENROSE, M. (2017). *Lectures on the Poisson Process*. *Institute of Mathematical Statistics Textbooks* **7**. Cambridge Univ. Press, Cambridge. MR3791470

[48] MICHELEN, M. (2019). Critical percolation and the incipient infinite cluster on Galton–Watson trees. *Electron. Commun. Probab.* **24** 1–13. MR3916342 https://doi.org/10.1214/19-ECP216

[49] MICHELEN, M., PEMANTLE, R. and ROSENBERG, J. (2019). Invasion percolation on Galton–Watson trees. *Electron. J. Probab.* **24** 1–35. MR3940761 https://doi.org/10.1214/19-EJP281

[50] MICHELEN, M., PEMANTLE, R. and ROSENBERG, J. (2020). Quenched survival of Bernoulli percolation on Galton–Watson trees. *J. Stat. Phys.* **181** 1323–1364. MR4163504 https://doi.org/10.1007/s10955-020-02629-6

[51] MIKOSCH, T. (1999). Regular variation, subexponentiality and their applications in probability theory. Report Eurandom. Eurandom.

[52] NICKEL, B. and WILKINSON, D. (1983). Invasion percolation on the Cayley tree: Exact solution of a modified percolation model. *Phys. Rev. Lett.* **51** 71–74. MR0708864 https://doi.org/10.1103/PhysRevLett.51.71

[53] NORRIS, J. Q., TURCOTTE, D. L. and RUNDLE, J. B. (2014). Loopless nontrapping invasion-percolation model for fracking. *Phys. Rev. E, Stat. Nonlin. Soft Matter Phys.* **89** 1–11. https://doi.org/10.1103/PhysRevE.89.022119

[54] PRIM, R. C. (1957). Shortest connection networks and some generalizations. *Bell Syst. Tech. J.* **36** 1389–1401. https://doi.org/10.1002/j.1538-7305.1957.tb01515.x

[55] SAPOZHNIKOV, A. (2011). The incipient infinite cluster does not stochastically dominate the invasion percolation cluster in two dimensions. *Electron. Commun. Probab.* **16** 775–780. MR2861441 https://doi.org/10.1214/ECP.v16-1684

[56] SHAO, J., HAVLIN, S. and STANLEY, E. H. (2009). Dynamic opinion model and invasion percolation. *Phys. Rev. Lett.* **103** 1–4. https://doi.org/10.1103/PhysRevLett.103.018701

[57] VAN DER HOFSTAD, R. (2017). *Random Graphs and Complex Networks. Vol. 1*. *Cambridge Series in Statistical and Probabilistic Mathematics*, [43]. Cambridge Univ. Press, Cambridge. MR3617364 https://doi.org/10.1017/9781316779422

[58] VAN DER HOFSTAD, R. (2024). *Random Graphs and Complex Networks. Vol. 2*. *Cambridge Series in Statistical and Probabilistic Mathematics*, [54]. Cambridge Univ. Press, Cambridge. https://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf. https://doi.org/10.1017/9781316795552

[59] VAN DER HOFSTAD, R. and JÁRAI, A. A. (2004). The incipient infinite cluster for high-dimensional unoriented percolation. *J. Stat. Phys.* **114** 625–663. MR2035627 https://doi.org/10.1023/B:JOSS.0000012505.39213.6a

[60] WHITT, W. (2002). *Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Application to Queues*. *Springer Series in Operations Research*. Springer, New York. MR1876437

[61] WILKINSON, D. and WILLEMSEN, J. F. (1983). Invasion percolation: A new form of percolation theory. *J. Phys. A* **16** 3365–3376. MR0725616

[62] WILLIAMS, D. (1991). *Probability with Martingales*. *Cambridge Mathematical Textbooks*. Cambridge Univ. Press, Cambridge. MR1155402 https://doi.org/10.1017/CBO9780511813658

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3093–3124
https://doi.org/10.1214/23-AAP2033

SURFACE AREA AND VOLUME OF EXCURSION SETS OBSERVED ON POINT CLOUD BASED POLYTOPIC TESSELLATIONS

BY RYAN COTSAKIS[1,a], ELENA DI BERNARDINO[1,b] AND CÉLINE DUVAL[2,c]

[1]*Université Côte d'Azur, Laboratoire J.A. Dieudonné, UMR CNRS 7351,* [a]*Ryan.Cotsakis@univ-cotedazur.fr,* [b]*Elena.DI_BERNARDINO@univ-cotedazur.fr*

[2]*Université de Lille, Laboratoire Paul Painlevé, UMR 8524 CNRS,* [c]*celine.duval@univ-lille.fr*

The excursion set of a C^2 smooth random field carries relevant information in its various geometric measures. From a computational viewpoint, one never has access to the continuous observation of the excursion set, but rather to observations at discrete points in space. It has been reported that for specific regular lattices of points in dimensions 2 and 3, the usual approximation of the surface area of the excursions does not converge when the lattice becomes dense in the domain of observation to the desired limit. In the present work, under the key assumptions of stationarity and isotropy, we demonstrate that this limiting factor is invariant to the locations of the observation points. Indeed, we identify an explicit formula for the correction factor, showing that it only depends on the spatial dimension d. This enables us to define an approximation for the surface area of excursion sets for general tessellations of polytopes in $\mathbb{R}^d$, including Poisson–Voronoi tessellations. We also establish a joint central limit theorem for the surface area and volume of excursion sets observed over hypercubic lattices.

REFERENCES

[1] AARON, C., CHOLAQUIDIS, A. and FRAIMAN, R. (2022). Estimation of surface area. *Electron. J. Stat.* **16** 3751–3788. MR4448877 https://doi.org/10.1214/22-ejs2031

[2] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. *Springer Monographs in Mathematics*. Springer, New York. MR2319516

[3] ADLER, R. J. and TAYLOR, J. E. (2011). *Topological Complexity of Smooth Random Functions*: *Lectures from the* 39*th Probability Summer School held in Saint-Flour*, 2009, *École d'Été de Probabilités de Saint-Flour.* [*Saint-Flour Probability Summer School*]. *Lecture Notes in Math.* **2019**. Springer, Heidelberg. MR2768175 https://doi.org/10.1007/978-3-642-19580-8

[4] AZAÏS, J.-M. and WSCHEBOR, M. (2009). *Level Sets and Extrema of Random Processes and Fields*. Wiley, Hoboken, NJ. MR2478201 https://doi.org/10.1002/9780470434642

[5] BACCELLI, F., KLEIN, M., LEBOURGES, M. and ZUYEV, S. (1997). Stochastic geometry and architecture of communication networks. *Telecommun. Syst.* **7** 209–227.

[6] BACCELLI, F., TCHOUMATCHENKO, K. and ZUYEV, S. (2000). Markov paths on the Poisson–Delaunay graph with applications to routing in mobile networks. *Adv. in Appl. Probab.* **32** 1–18. MR1765174 https://doi.org/10.1239/aap/1013540019

[7] BERZIN, C. (2021). Estimation of local anisotropy based on level sets. *Electron. J. Probab.* **26** 1–72. MR4347380 https://doi.org/10.1214/21-ejp721

[8] BERZIN, C., LATOUR, A. and LEÓN, J. R. (2017). *Kac-Rice Formulas for Random Fields and Theirs Applications in*: *Random Geometry*, *Roots of Random Polynomials and Some Engineering Problems*. Ediciones IVIC.

[9] BEUTLER, F. J. and LENEMAN, O. A. Z. (1966). The theory of stationary point processes. *Acta Math.* **116** 159–190. MR0199895 https://doi.org/10.1007/BF02392816

[10] BIERMÉ, H. and DESOLNEUX, A. (2020). Mean geometry for 2D random fields: Level perimeter and level total curvature integrals. *Ann. Appl. Probab.* **30** 561–607. MR4108116 https://doi.org/10.1214/19-AAP1508

MSC2020 subject classifications. 62R30, 62M40, 60G60, 62H11.

Key words and phrases. Bias correction, Crofton formula, crossing probabilities, excursion sets, Lipschitz–Killing curvatures, surface area, Voronoi tessellations.

[11] BIERMÉ, H. and DESOLNEUX, A. (2021). The effect of discretization on the mean geometry of a 2D random field. *Ann. Henri Lebesgue* **4** 1295–1345. MR4353964 https://doi.org/10.5802/ahl.103

[12] BIERMÉ, H., DI BERNARDINO, E., DUVAL, C. and ESTRADE, A. (2019). Lipschitz–Killing curvatures of excursion sets for two-dimensional random fields. *Electron. J. Stat.* **13** 536–581. MR3911693 https://doi.org/10.1214/19-EJS1530

[13] BOLTHAUSEN, E. (1982). On the central limit theorem for stationary mixing random fields. *Ann. Probab.* **10** 1047–1050. MR0672305

[14] BOOS, D. D. (1985). A converse to Scheffé's theorem. *Ann. Statist.* **13** 423–427. MR0773179 https://doi.org/10.1214/aos/1176346604

[15] BRADLEY, R. C. (2005). Basic properties of strong mixing conditions. A survey and some open questions. *Probab. Surv.* **2** 107–144. MR2178042 https://doi.org/10.1214/154957805100000104

[16] BULINSKI, A. (2010). Central limit theorem for random fields and applications. In *Advances in Data Analysis*. *Stat. Ind. Technol.* 141–150. Birkhäuser, Boston, MA. MR2641266 https://doi.org/10.1007/978-0-8176-4799-5_13

[17] BULINSKI, A. and SHASHKIN, A. (2007). *Limit Theorems for Associated Random Fields and Related Systems*. *Advanced Series on Statistical Science & Applied Probability* **10**. World Scientific, Hackensack, NJ. MR2375106 https://doi.org/10.1142/9789812709417

[18] BULINSKI, A., SPODAREV, E. and TIMMERMANN, F. (2012). Central limit theorems for the excursion set volumes of weakly dependent random fields. *Bernoulli* **18** 100–118. MR2888700 https://doi.org/10.3150/10-BEJ339

[19] CABAÑA, E. M. (1987). Affine processes: A test of isotropy based on level sets. *SIAM J. Appl. Math.* **47** 886–891. MR0898839 https://doi.org/10.1137/0147059

[20] CASAPONSA, B., CRILL, B., COLOMBO, L., DANESE, L., BOCK, J., CATALANO, A., BONALDI, A., BASAK, S., BONAVERA, L. et al. (2016). Planck 2015 results: Xvi. Isotropy and statistics of the CMB.

[21] CHENG, D. and SCHWARTZMAN, A. (2018). Expected number and height distribution of critical points of smooth isotropic Gaussian random fields. *Bernoulli* **24** 3422–3446. MR3788177 https://doi.org/10.3150/17-BEJ964

[22] COTSAKIS, R., DI BERNARDINO, E. and DUVAL, C. (2022). Surface area and volume of excursion sets observed on point cloud based polytopic tessellations. Preprint hal-03782649.

[23] COTSAKIS, R., DI BERNARDINO, E. and OPITZ, T. (2023). On the perimeter estimation of pixelated excursion sets of 2D anisotropic random fields. *Scand. J. Stat.* **51** 268–301.

[24] DEDECKER, J. (1998). A central limit theorem for stationary random fields. *Probab. Theory Related Fields* **110** 397–426. MR1616496 https://doi.org/10.1007/s004400050153

[25] DI BERNARDINO, E., ESTRADE, A. and LEÓN, J. R. (2017). A test of Gaussianity based on the Euler characteristic of excursion sets. *Electron. J. Stat.* **11** 843–890. MR3629017 https://doi.org/10.1214/17-EJS1248

[26] DI BERNARDINO, E., ESTRADE, A. and OPITZ, T. (2022). Spatial extremes and stochastic geometry for Gaussian-based peaks-over-threshold processes. Preprint hal-03825701.

[27] DOMBRY, C. and EYI-MINKO, F. (2012). Strong mixing properties of max-infinitely divisible random fields. *Stochastic Process. Appl.* **122** 3790–3811. MR2965926 https://doi.org/10.1016/j.spa.2012.06.013

[28] DOUKHAN, P. (1994). *Mixing*: *Properties and Examples*. *Lecture Notes in Statistics* **85**. Springer, New York. MR1312160 https://doi.org/10.1007/978-1-4612-2642-0

[29] EDELSBRUNNER, H. and NIKITENKO, A. (2021). Average and expected distortion of Voronoi paths and scapes. arXiv:2012.03350v2.

[30] EDELSBRUNNER, H. and PAUSINGER, F. (2016). Approximation and convergence of the intrinsic volume. *Adv. Math.* **287** 674–703. MR3422689 https://doi.org/10.1016/j.aim.2015.10.004

[31] EYMARD, R., GALLOUËT, T. and HERBIN, R. (2000). Finite volume methods. In *Handbook of Numerical Analysis*, *Vol. VII*. *Handb. Numer. Anal.*, *VII* 713–1020. North-Holland, Amsterdam. MR1804748 https://doi.org/10.1016/S1570-8659(00)07005-8

[32] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications*, *Vol. II*. Wiley, New York. MR0210154

[33] GOTT, J. R., COLLEY, W. N., PARK, C.-G., PARK, C. and MUGNOLO, C. (2007). Genus topology of the cosmic microwave background from the WMAP 3-year data. *Mon. Not. R. Astron. Soc.* **377** 1668–1678.

[34] GOTT, J. R., HAMBRICK, D. C., VOGELEY, M. S., KIM, J., PARK, C., CHOI, Y.-Y., CEN, R., OSTRIKER, J. P. and NAGAMINE, K. (2008). Genus topology of structure in the sloan digital sky survey: Model testing. *Astrophys. J.* **675** 16.

[35] GUDERLEI, R., KLENK, S., MAYER, J., SCHMIDT, V. and SPODAREV, E. (2007). Algorithms for the computation of the Minkowski functionals of deterministic and random polyconvex sets. *Image Vis. Comput.* **25** 464–474.

[36] IRIBARREN, I. (1989). Asymptotic behaviour of the integral of a function on the level set of a mixing random field. *Probab. Math. Statist.* **10** 45–56. MR0990398

[37] KRATZ, M. and VADLAMANI, S. (2018). Central limit theorem for Lipschitz–Killing curvatures of excursion sets of Gaussian random fields. *J. Theoret. Probab.* **31** 1729–1758. MR3842168 https://doi.org/10.1007/s10959-017-0760-6

[38] KRATZ, M. F. (2006). Level crossings and other level functionals of stationary Gaussian processes. *Probab. Surv.* **3** 230–288. MR2264709 https://doi.org/10.1214/154957806000000087

[39] LACHAUD, J.-O., COEURJOLLY, D., LABART, C., ROMON, P. and THIBERT, B. (2023). Lightweight curvature estimation on point clouds with randomized corrected curvature measures. *Computer Graphics Forum* **42**.

[40] LACHAUD, J.-O., ROMON, P., THIBERT, B. and COEURJOLLY, D. (2020). Interpolated corrected curvature measures for polygonal surfaces. *Computer Graphics Forum* **39**.

[41] LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. *Springer Series in Statistics*. Springer, New York-Berlin. MR0691492

[42] LEHMANN, G. and LEGLAND, D. (2012). Efficient N-dimensional surface estimation using Crofton formula and run-length encoding. *The Insight Journal*, *INRA*.

[43] LINDGREN, G. (2000). Wave analysis by Slepian models. *Probab. Eng. Mech.* **15** 49–57.

[44] LONGUET-HIGGINS, M. S. (1957). The statistical analysis of a random, moving surface. *Philos. Trans. R. Soc. Lond. Ser. A* **249** 321–387. MR0087257 https://doi.org/10.1098/rsta.1957.0002

[45] MESCHENMOSER, D. and SHASHKIN, A. (2011). Functional central limit theorem for the volume of excursion sets generated by associated random fields. *Statist. Probab. Lett.* **81** 642–646. MR2783860 https://doi.org/10.1016/j.spl.2011.02.012

[46] MESCHENMOSER, D. and SHASHKIN, A. (2013). Functional central limit theorem for the measures of level surfaces of the Gaussian random field. *Theory Probab. Appl.* **57** 162–172. MR3201646 https://doi.org/10.1137/S0040585X97985844

[47] MESCHENMOSER, D. and SPODAREV, E. (2010). On the computation of intrinsic volumes. Preprint.

[48] MILLER, E. (1999). Alternative tilings for improved surface area estimates by local counting algorithms. *Comput. Vis. Image Underst.* **74** 193–211.

[49] MÜLLER, D. (2017). A central limit theorem for Lipschitz–Killing curvatures of Gaussian excursions. *J. Math. Anal. Appl.* **452** 1040–1081. MR3632692 https://doi.org/10.1016/j.jmaa.2017.03.036

[50] REDDY, T. R., VADLAMANI, S. and YOGESHWARAN, D. (2018). Central limit theorem for exponentially quasi-local statistics of spin models on Cayley graphs. *J. Stat. Phys.* **173** 941–984. MR3876914 https://doi.org/10.1007/s10955-018-2026-9

[51] ROGERS, C. A. (1998). *Hausdorff Measures*. *Cambridge Mathematical Library*. Cambridge Univ. Press, Cambridge. Reprint of the 1970 original, With a foreword by K. J. Falconer. MR1692618

[52] SCHMALZING, J. and GÓRSKI, K. M. (1998). Minkowski functionals used in the morphological analysis of cosmic microwave background anisotropy maps. *Monthly Notices of the Royal Astronomical Society* **297** 355–365.

[53] SCHNEIDER, R. and WEIL, W. (2008). *Stochastic and Integral Geometry*. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2455326 https://doi.org/10.1007/978-3-540-78859-1

[54] SPODAREV, E. (2014). Limit theorems for excursion sets of stationary random fields. In *Modern Stochastics and Applications*. *Springer Optim. Appl.* **90** 221–241. Springer, Cham. MR3236077 https://doi.org/10.1007/978-3-319-03512-3_13

[55] SVANE, A. M. (2014). On multigrid convergence of local algorithms for intrinsic volumes. *J. Math. Imaging Vision* **49** 148–172. MR3180961 https://doi.org/10.1007/s10851-013-0450-7

[56] SWEETING, T. J. (1986). On a converse to Scheffé's theorem. *Ann. Statist.* **14** 1252–1256. MR0856821 https://doi.org/10.1214/aos/1176350065

[57] THÄLE, C. and YUKICH, J. E. (2016). Asymptotic theory for statistics of the Poisson–Voronoi approximation. *Bernoulli* **22** 2372–2400. MR3498032 https://doi.org/10.3150/15-BEJ732

[58] WSCHEBOR, M. (1985). *Surfaces Aléatoires*: *Mesure géométrique des ensembles de niveau*. *Lecture Notes in Math.* **1147**. Springer, Berlin. MR0871689 https://doi.org/10.1007/BFb0075073

[59] YUKICH, J. E. (2015). Surface order scaling in stochastic geometry. *Ann. Appl. Probab.* **25** 177–210. MR3297770 https://doi.org/10.1214/13-AAP992

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3125–3180
https://doi.org/10.1214/23-AAP2035

LARGE POPULATION ASYMPTOTICS FOR A MULTITYPE STOCHASTIC SIS EPIDEMIC MODEL IN RANDOMLY SWITCHING ENVIRONMENT

BY ADRIEN PRODHOMME[1,a] AND EDOUARD STRICKLER[2,b]

[1]*Institut Denis Poisson, Université de Tours,* [a]*adrien-prodhomme@orange.fr*
[2]*Université de Lorraine, CNRS, Inria, IECL, Nancy,* [b]*edouard.strickler@univ-lorraine.fr*

We consider an epidemic SIS model described by a multitype birth-and-death process in a randomly switching environment. That is, the infection and cure rates of the process depend on the state of a finite Markov jump process (the environment), whose transitions also depend on the number of infectives. The total size of the population is constant and equal to some $K \in \mathbf{N}^*$, and the number of infectives vanishes almost surely in finite time. We prove that, as $K \to \infty$, the process composed of the proportions of infectives of each type X^K and the state of the environment Ξ^K, converges to a piecewise deterministic Markov process (PDMP) given by a system of randomly switching ODEs. The long term behaviour of this PDMP has been previously investigated by Benaïm and Strickler, and depends only on the sign of the top Lyapunov exponent Λ of the linearised PDMP at 0: if $\Lambda < 0$, the proportion of infectives in each group converges to zero, while if $\Lambda > 0$, the disease becomes endemic. In this paper, we show that the large population asymptotics of X^K also strongly depend on the sign of Λ: if negative, then from fixed initial proportions of infectives the disease disappears in a time of order at most $\log(K)$, while if positive, the typical extinction time grows at least as a power of K. We prove that in the situation where the origin is accessible for the linearised PDMP, the mean extinction time of X^K is logarithmically equivalent to K^{p^*}, where $p^* > 0$ is fully characterised. We also investigate the quasi-stationary distribution μ^K of (X^K, Ξ^K) and show that, when $\Lambda < 0$, weak limit points of $(\mu^K)_{K>0}$ are supported by the extinction set, while when $\Lambda > 0$, limit points belong to the (nonempty) set of stationary distributions of the limiting PDMP which do not give mass to the extinction set.

REFERENCES

[1] ANDERSSON, H. and DJEHICHE, B. (1998). A threshold limit theorem for the stochastic logistic epidemic. *J. Appl. Probab.* **35** 662–670. MR1659540 https://doi.org/10.1239/jap/1032265214

[2] ARNOLD, L. (1984). A formula connecting sample and moment stability of linear stochastic systems. *SIAM J. Appl. Math.* **44** 793–802. MR0750951 https://doi.org/10.1137/0144057

[3] ARNOLD, L. (1998). *Random Dynamical Systems. Springer Monographs in Mathematics*. Springer, Berlin. MR1723992 https://doi.org/10.1007/978-3-662-12878-7

[4] ARNOLD, L., KLIEMANN, W. and OELJEKLAUS, E. (1986). Lyapunov exponents of linear stochastic systems. In *Lyapunov Exponents* (*Bremen*, 1984). *Lecture Notes in Math.* **1186** 85–125. Springer, Berlin. MR0850072 https://doi.org/10.1007/BFb0076835

[5] ARNOLD, L., OELJEKLAUS, E. and PARDOUX, É. (1986). Almost sure and moment stability for linear Itô equations. In *Lyapunov Exponents* (*Bremen*, 1984). *Lecture Notes in Math.* **1186** 129–159. Springer, Berlin. MR0850074 https://doi.org/10.1007/BFb0076837

[6] ARTALEJO, J. R., ECONOMOU, A. and LOPEZ-HERRERO, M. J. (2013). Stochastic epidemic models with random environment: Quasi-stationarity, extinction and final size. *J. Math. Biol.* **67** 799–831. MR3096540 https://doi.org/10.1007/s00285-012-0570-5

[7] BACAËR, N. (2016). Le modèle stochastique SIS pour une épidémie dans un environnement aléatoire. *J. Math. Biol.* **73** 847–866. MR3545521 https://doi.org/10.1007/s00285-016-0974-8

MSC2020 subject classifications. Primary 60J25, 60J80; secondary 37H15, 92D30.
Key words and phrases. Epidemiological model, SIS, birth-and-death process, piecewise deterministic Markov process, quasi-stationary distribution, large population, Lyapunov exponent, stochastic persistence, extinction.

[8] BARDET, J.-B., GUÉRIN, H. and MALRIEU, F. (2010). Long time behavior of diffusions with Markov switching. *ALEA Lat. Am. J. Probab. Math. Stat.* **7** 151–170. MR2653702

[9] BENAÏM, M. (2018). Stochastic persistence. Preprint. Available at arXiv:1806.08450.

[10] BENAÏM, M. and HIRSCH, M. W. (1999). Differential and stochastic epidemic models. In *Differential Equations with Applications to Biology* (*Halifax*, *NS*, 1997). *Fields Inst. Commun.* **21** 31–44. Amer. Math. Soc., Providence, RI. MR1662601 https://doi.org/10.1007/bf02218617

[11] BENAÏM, M., LE BORGNE, S., MALRIEU, F. and ZITT, P.-A. (2015). Qualitative properties of certain piecewise deterministic Markov processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 1040–1075. MR3365972 https://doi.org/10.1214/14-AIHP619

[12] BENAÏM, M. and LOBRY, C. (2016). Lotka–Volterra with randomly fluctuating environments or "How switching between beneficial environments can make survival harder". *Ann. Appl. Probab.* **26** 3754–3785. MR3582817 https://doi.org/10.1214/16-AAP1192

[13] BENAÏM, M. and STRICKLER, E. (2019). Random switching between vector fields having a common zero. *Ann. Appl. Probab.* **29** 326–375. MR3910006 https://doi.org/10.1214/18-AAP1418

[14] CHAMPAGNAT, N. and VILLEMONAIS, D. (2023). General criteria for the study of quasi-stationarity. *Electron. J. Probab.* **28** Paper No. 22. MR4546021 https://doi.org/10.1214/22-ejp880

[15] CHAZOTTES, J.-R., COLLET, P., MARTÍNEZ, S. and MÉLÉARD, S. (2020). Quasi-stationary distributions and resilience: What to get from a sample? *J. Éc. Polytech. Math.* **7** 943–980. MR4115739 https://doi.org/10.5802/jep.132

[16] CHAZOTTES, J.-R., COLLET, P. and MÉLÉARD, S. (2016). Sharp asymptotics for the quasi-stationary distribution of birth-and-death processes. *Probab. Theory Related Fields* **164** 285–332. MR3449391 https://doi.org/10.1007/s00440-014-0612-6

[17] CHAZOTTES, J.-R., COLLET, P. and MÉLÉARD, S. (2019). On time scales and quasi-stationary distributions for multitype birth-and-death processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 2249–2294. MR4029154 https://doi.org/10.1214/18-AIHP948

[18] COGBURN, R. and TORREZ, W. C. (1981). Birth and death processes with random environments in continuous time. *J. Appl. Probab.* **18** 19–30. MR0598919 https://doi.org/10.2307/3213163

[19] COSTA, M. (2016). A piecewise deterministic model for a prey-predator community. *Ann. Appl. Probab.* **26** 3491–3530. MR3582809 https://doi.org/10.1214/16-AAP1182

[20] CRUDU, A., DEBUSSCHE, A., MULLER, A. and RADULESCU, O. (2012). Convergence of stochastic gene networks to hybrid piecewise deterministic processes. *Ann. Appl. Probab.* **22** 1822–1859. MR3025682 https://doi.org/10.1214/11-AAP814

[21] DAVIS, M. H. A. (1984). Piecewise-deterministic Markov processes: A general class of nondiffusion stochastic models. *J. Roy. Statist. Soc. Ser. B* **46** 353–388. MR0790622

[22] DOERING, C. R., SARGSYAN, K. V. and SANDER, L. M. (2005). Extinction times for birth-death processes: Exact results, continuum asymptotics, and the failure of the Fokker–Planck approximation. *Multiscale Model. Simul.* **3** 283–299. MR2122989 https://doi.org/10.1137/030602800

[23] DU, N. H. and DANG, N. H. (2014). Asymptotic behavior of Kolmogorov systems with predator-prey type in random environment. *Commun. Pure Appl. Anal.* **13** 2693–2712. MR3248409 https://doi.org/10.3934/cpaa.2014.13.2693

[24] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

[25] FANG, Y. (1994). Stability analysis of linear control systems with uncertain parameters. PhD thesis, Case Western Reserve University. Available at http://www.fang.ece.ufl.edu/mypaper/cwru.pdf.

[26] FAURE, M. and SCHREIBER, S. J. (2014). Quasi-stationary distributions for randomly perturbed dynamical systems. *Ann. Appl. Probab.* **24** 553–598. MR3178491 https://doi.org/10.1214/13-AAP923

[27] GRAY, A., GREENHALGH, D., MAO, X. and PAN, J. (2012). The SIS epidemic model with Markovian switching. *J. Math. Anal. Appl.* **394** 496–516. MR2927473 https://doi.org/10.1016/j.jmaa.2012.05.029

[28] HAIRER, M. (2006). Ergodic properties of Markov processes. Lecture Notes. Available at https://www.hairer.org/notes/Markov.pdf.

[29] HENING, A., QI, W., SHEN, Z. and YI, Y. (2022). Population dynamics under demographic and environmental stochasticity. Preprint. Available at arXiv:2207.08883.

[30] HENING, A. and STRICKLER, E. (2019). On a predator-prey system with random switching that never converges to its equilibrium. *SIAM J. Math. Anal.* **51** 3625–3640. MR4000212 https://doi.org/10.1137/18M1196042

[31] HILLE, E. and PHILLIPS, R. S. (1957). *Functional Analysis and Semi-Groups*. *American Mathematical Society Colloquium Publications*, *Vol.* 31. Amer. Math. Soc., Providence, RI. MR0089373

[32] HIRSCH, M. W. (1994). Positive equilibria and convergence in subhomogeneous monotone dynamics. In *Comparison Methods and Stability Theory* (*Waterloo*, *ON*, 1993). *Lecture Notes in Pure and Applied Mathematics* **162** 169–188. Dekker, New York. MR1291618

[33] KRYSCIO, R. J. and LEFÈVRE, C. (1989). On the extinction of the S-I-S stochastic logistic epidemic. *J. Appl. Probab.* **26** 685–694. MR1025386 https://doi.org/10.1017/s002190020002756x

[34] KURTZ, T. G. (1981). *Approximation of Population Processes*. *CBMS-NSF Regional Conference Series in Applied Mathematics* **36**. SIAM, Philadelphia, PA. MR0610982

[35] LAJMANOVICH, A. and YORKE, J. A. (1976). A deterministic model for gonorrhea in a nonhomogeneous population. *Math. Biosci.* **28** 221–236. MR0403726 https://doi.org/10.1016/0025-5564(76)90125-5

[36] LEIZAROWITZ, A. (1991). Eigenvalue representation for the Lyapunov exponents of certain Markov processes. In *Lyapunov Exponents* (*Oberwolfach*, 1990). *Lecture Notes in Math.* **1486** 51–63. Springer, Berlin. MR1178946 https://doi.org/10.1007/BFb0086657

[37] LI, D., LIU, S. and CUI, J. (2017). Threshold dynamics and ergodicity of an SIRS epidemic model with Markovian switching. *J. Differ. Equ.* **263** 8873–8915. MR3710707 https://doi.org/10.1016/j.jde.2017.08.066

[38] MÉLÉARD, S. and VILLEMONAIS, D. (2012). Quasi-stationary distributions and population processes. *Probab. Surv.* **9** 340–410. MR2994898 https://doi.org/10.1214/11-PS191

[39] NÅSELL, I. (1996). The quasi-stationary distribution of the closed endemic SIS model. *Adv. in Appl. Probab.* **28** 895–932. MR1404315 https://doi.org/10.2307/1428186

[40] NÅSELL, I. (1999). On the quasi-stationary distribution of the stochastic logistic epidemic. *Math. Biosci.* **156** 21–40.

[41] NGUYEN, D. H. and STRICKLER, E. (2020). A method to deal with the critical case in stochastic population dynamics. *SIAM J. Appl. Math.* **80** 1567–1589. MR4116754 https://doi.org/10.1137/20M131134X

[42] RAMI, M. A., BOKHARAIE, V. S., MASON, O. and WIRTH, F. R. (2014). Stability criteria for SIS epidemiological models under switching policies. *Discrete Contin. Dyn. Syst. Ser. B* **19** 2865–2887. MR3261402 https://doi.org/10.3934/dcdsb.2014.19.2865

[43] SCHREIBER, S. J. (2017). Coexistence in the face of uncertainty. In *Recent Progress and Modern Challenges in Applied Mathematics*, *Modeling and Computational Science*. *Fields Inst. Commun.* **79** 349–384. Springer, New York. MR3700056

[44] SCHREIBER, S. J., HUANG, S., JIANG, J. and WANG, H. (2021). Extinction and quasi-stationarity for discrete-time, endemic SIS and SIR models. *SIAM J. Appl. Math.* **81** 2195–2217. MR4320897 https://doi.org/10.1137/20M1339015

[45] SENETA, E. (2006). *Non-negative Matrices and Markov Chains*. *Springer Series in Statistics*. Springer, New York. MR2209438

[46] STRICKLER, E. (2019). Persistance de processus de Markov déterministes par morceaux. Ph.D. thesis, Université de Neuchâtel. Available at https://doc.rero.ch/record/326883.

[47] TAKÁČ, P. (1990). Asymptotic behavior of discrete-time semigroups of sublinear, strongly increasing mappings with applications to biology. *Nonlinear Anal.* **14** 35–42. MR1028245 https://doi.org/10.1016/0362-546X(90)90133-2

[48] TAKEUCHI, Y., DU, N. H., HIEU, N. T. and SATO, K. (2006). Evolution of predator-prey systems described by a Lotka–Volterra equation under random environment. *J. Math. Anal. Appl.* **323** 938–957. MR2260154 https://doi.org/10.1016/j.jmaa.2005.11.009

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3181–3226
https://doi.org/10.1214/23-AAP2036

ERGODICITY OF THE UNDERDAMPED MEAN-FIELD LANGEVIN DYNAMICS

BY ANNA KAZEYKINA[1,a], ZHENJIE REN[2,b], XIAOLU TAN[3,c] AND JUNJIAN YANG[4,d]

[1]*Département de Mathématiques, Faculté des Sciences d'Orsay, Université Paris-Saclay,* [a]*anna.kazeykina@math.u-psud.fr*
[2]*CEREMADE, Université Paris-Dauphine, PSL,* [b]*ren@ceremade.dauphine.fr*
[3]*Department of Mathematics, The Chinese University of Hong Kong,* [c]*xiaolu.tan@cuhk.edu.hk*
[4]*FAM, Fakultät für Mathematik und Geoinformation, Vienna University of Technology,* [d]*junjian.yang@tuwien.ac.at*

We study the long time behavior of an underdamped mean-field Langevin (MFL) equation, and provide a general convergence as well as an exponential convergence rate result under different conditions. The results on the MFL equation can be applied to study the convergence of the Hamiltonian gradient descent algorithm for the overparametrized optimization. We then provide some numerical examples of the algorithm to train a generative adversarial network (GAN).

REFERENCES

[1] ARJOVSKY, M., CHINTALA, S. and BOTTOU, L. (2017). Wasserstein generative adversarial networks. In *Proceedings of the* 34*th International Conference on Machine Learning* (D. Precup and Y. W. Teh, eds.). *Proceedings of Machine Learning Research* **70** 214–223. PMLR.

[2] ARMSTRONG, S. and MOURRAT, J. C. (2019). Variational methods for the kinetic Fokker–Planck equation. Preprint. Available at arXiv:1902.04037.

[3] BAKRY, D., CATTIAUX, P. and GUILLIN, A. (2008). Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. *J. Funct. Anal.* **254** 727–759. MR2381160 https://doi.org/10.1016/j.jfa.2007.11.002

[4] BALLY, V. (1991). On the connection between the Malliavin covariance matrix and Hörmander's condition. *J. Funct. Anal.* **96** 219–255. MR1101258 https://doi.org/10.1016/0022-1236(91)90062-A

[5] BEN AROUS, G., CRANSTON, M. and KENDALL, W. S. (1995). Coupling constructions for hypoelliptic diffusions: Two examples. In *Stochastic Analysis* (*Ithaca*, *NY*, 1993). *Proc. Sympos. Pure Math.* **57** 193–212. Amer. Math. Soc., Providence, RI. MR1335472 https://doi.org/10.1090/pspum/057/1335472

[6] BOLLEY, F., GUILLIN, A., LE BRIS, P. and MONMARCHÉ, P. Wasserstein contraction for kinetic mean field particles system. Ongoing.

[7] BOLLEY, F., GUILLIN, A. and MALRIEU, F. (2010). Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov–Fokker–Planck equation. *ESAIM Math. Model. Numer. Anal.* **44** 867–884. MR2731396 https://doi.org/10.1051/m2an/2010045

[8] BONILLA, L. L., CARRILLO, J. A. and SOLER, J. (1997). Asymptotic behavior of an initial-boundary value problem for the Vlasov–Poisson–Fokker–Planck system. *SIAM J. Appl. Math.* **57** 1343–1372. MR1470927 https://doi.org/10.1137/S0036139995291544

[9] BOU-RABEE, N., EBERLE, A. and ZIMMER, R. (2020). Coupling and convergence for Hamiltonian Monte Carlo. *Ann. Appl. Probab.* **30** 1209–1250. MR4133372 https://doi.org/10.1214/19-AAP1528

[10] BOU-RABEE, N. and SCHUH, K. (2023). Convergence of unadjusted Hamiltonian Monte Carlo for mean-field models. *Electron. J. Probab.* **28** 1–40. MR4610714 https://doi.org/10.1214/23-ejp970

[11] BRÜNGER, A., BROOKS, C. III and KARPLUS, M. (1984). Stochastic boundary conditions for molecular dynamics simulations of ST2 water. *Chem. Phys. Lett.* **105** 495–500.

[12] CAO, Y., LU, J. and WANG, L. (2023). On explicit L^2-convergence rate estimate for underdamped Langevin dynamics. *Arch. Ration. Mech. Anal.* **247** Paper No. 90, 34. MR4632836 https://doi.org/10.1007/s00205-023-01922-4

[13] CARDALIAGUET, P. (2017). A short course on mean field games. Preprint.

[14] CARDALIAGUET, P., DELARUE, F., LASRY, J.-M. and LIONS, P.-L. (2019). *The Master Equation and the Convergence Problem in Mean Field Games*. *Annals of Mathematics Studies* **201**. Princeton Univ. Press, Princeton, NJ. MR3967062 https://doi.org/10.2307/j.ctvckq7qf

MSC2020 subject classifications. Primary 37M25, 60H10; secondary 60H30, 60J60.
Key words and phrases. Underdamped mean-field Langevin dynamics, ergodicity, coupling, GAN.

[15] CARMONA, R. (2016). *Lectures on BSDES, Stochastic Control, and Stochastic Differential Games with Financial Applications*. *Financial Mathematics* **1**. SIAM, Philadelphia, PA. MR3629171 https://doi.org/10.1137/1.9781611974249

[16] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications*. *I*: *Mean Field FBSDEs*, *Control*, *and Games*. *Probability Theory and Stochastic Modelling* **83**. Springer, Cham. MR3752669

[17] CATTIAUX, P. and MESNAGER, L. (2002). Hypoelliptic non-homogeneous diffusions. *Probab. Theory Related Fields* **123** 453–483. MR1921010 https://doi.org/10.1007/s004400100194

[18] CHENG, X., CHATTERJI, N., ABBASI-YADKORI, Y., BARTLETT, P. L. and JORDAN, M. I. (2020). Sharp convergence rates for Langevin dynamics in the nonconvex setting. Preprint.

[19] CHENG, X., CHATTERJI, N. S., BARTLETT, P. L. and JORDAN, M. I. (2018). Underdamped Langevin MCMC: A non-asymptotic analysis. *Proc*. *Mach*. *Learn*. *Res*. **75** 1–24.

[20] CHIZAT, L. and BACH, F. (2018). On the global convergence of gradient descent for over-parameterized models using optimal transport. In *Advances in Neural Information Processing Systems* 3040–3050.

[21] CONFORTI, G., KAZEYKINA, A. and REN, Z. (2023). Game on random environment, mean-field Langevin system, and neural networks. *Math*. *Oper. Res*. **48** 78–99. MR4567278 https://doi.org/10.1287/moor.2022.1252

[22] DALALYAN, A. S. (2017). Theoretical guarantees for approximate sampling from smooth and log-concave densities. *J*. *R*. *Stat*. *Soc*. *Ser. B*. *Stat*. *Methodol*. **79** 651–676. MR3641401 https://doi.org/10.1111/rssb.12183

[23] DOLBEAULT, J., KLAR, A., MOUHOT, C. and SCHMEISER, C. (2013). Exponential rate of convergence to equilibrium for a model describing fiber lay-down processes. *Appl*. *Math*. *Res*. *Express*. *AMRX* 2 165–175. MR3106879 https://doi.org/10.1093/amrx/abs015

[24] DOLBEAULT, J., MOUHOT, C. and SCHMEISER, C. (2009). Hypocoercivity for kinetic equations with linear relaxation terms. *C. R. Math*. *Acad*. *Sci*. *Paris* **347** 511–516. MR2576899 https://doi.org/10.1016/j.crma.2009.02.025

[25] DOLBEAULT, J., MOUHOT, C. and SCHMEISER, C. (2015). Hypocoercivity for linear kinetic equations conserving mass. *Trans*. *Amer. Math*. *Soc*. **367** 3807–3828. MR3324910 https://doi.org/10.1090/S0002-9947-2015-06012-7

[26] DOMINGO-ENRICH, C., JELASSI, S., MENSCH, A., ROTSKOFF, G. and BRUNA, J. (2020). A mean-field analysis of two-player zero-sum games. *Adv. Neural Inf. Process*. *Syst*. **33** 20215–20226.

[27] DUONG, M. H. and TUGAUT, J. (2018). The Vlasov–Fokker–Planck equation in non-convex landscapes: Convergence to equilibrium. *Electron*. *Commun*. *Probab.* **23** 1–10. MR3779816 https://doi.org/10.1214/18-ECP116

[28] DUPUIS, P. and ELLIS, R. S. (1997). *A Weak Convergence Approach to the Theory of Large Deviations*. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1431744 https://doi.org/10.1002/9781118165904

[29] DURMUS, A. and MOULINES, E. (2016). Sampling from strongly log-concave distributions with the Unadjusted Langevin Algorithm. Preprint. Available at arXiv:1605.01559.

[30] EBERLE, A. (2016). Reflection couplings and contraction rates for diffusions. *Probab. Theory Related Fields* **166** 851–886. MR3568041 https://doi.org/10.1007/s00440-015-0673-1

[31] EBERLE, A., GUILLIN, A. and ZIMMER, R. (2019). Couplings and quantitative contraction rates for Langevin dynamics. *Ann*. *Probab.* **47** 1982–2010. MR3980913 https://doi.org/10.1214/18-AOP1299

[32] EBERLE, A., GUILLIN, A. and ZIMMER, R. (2019). Quantitative Harris-type theorems for diffusions and McKean–Vlasov processes. *Trans*. *Amer. Math*. *Soc*. **371** 7135–7173. MR3939573 https://doi.org/10.1090/tran/7576

[33] EINSTEIN, A. (1905). Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Ann*. *Phys*. **322** 549–560.

[34] FÖLLMER, H. (1986). Time reversal on Wiener space. In *Stochastic Processes—Mathematics and Physics* (*Bielefeld*, 1984) (S. A. Albeverio, P. Blanchard and L. Streit, eds.). *Lecture Notes in Math*. **1158** 119–129. Springer, Berlin. MR0838561 https://doi.org/10.1007/BFb0080212

[35] FONTBONA, J. and JOURDAIN, B. (2016). A trajectorial interpretation of the dissipations of entropy and Fisher information for stochastic differential equations. *Ann. Probab.* **44** 131–170. MR3456334 https://doi.org/10.1214/14-AOP969

[36] GELMAN, A., ROBERTS, G. O. and GILKS, W. R. (1996). Efficient Metropolis jumping rules. In *Bayesian Statistics*, 5 (*Alicante*, 1994). *Oxford Sci*. *Publ*. 599–607. Oxford Univ. Press, New York. MR1425429

[37] GOODFELLOW, I. J., POUGET-ABADIE, J., MIRZA, M., XU, B., WARDE-FARLEY, D., OZAIR, S., COURVILLE, A. and BENGIO, Y. (2014). Generative adversarial nets. In *NIPS*'14 : *Proceedings of the* 27*th International Conference on Neural Information Processing Systems* **2** 2672–2680. MIT Press, Cambridge, MA, USA.

[38] GROTHAUS, M. and STILGENBAUER, P. (2016). Hilbert space hypocoercivity for the Langevin dynamics revisited. *Methods Funct. Anal. Topology* **22** 152–168. MR3522857

[39] GUILLIN, A., LE BRIS, P. and MONMARCHÉ, P. (2022). Convergence rates for the Vlasov–Fokker–Planck equation and uniform in time propagation of chaos in non convex cases. *Electron. J. Probab.* **27** 1–44. MR4489825 https://doi.org/10.1214/22-ejp853

[40] GUILLIN, A., LIU, W., WU, L. and ZHANG, C. (2021). The kinetic Fokker–Planck equation with mean field interaction. *J. Math. Pures Appl.* (9) **150** 1–23. MR4248461 https://doi.org/10.1016/j.matpur.2021.04.001

[41] GUILLIN, A. and MONMARCHÉ, P. (2021). Uniform long-time and propagation of chaos estimates for mean field kinetic particles in non-convex landscapes. *J. Stat. Phys.* **185** Paper No. 15, 20. MR4333408 https://doi.org/10.1007/s10955-021-02839-6

[42] HAUSSMANN, U. G. and PARDOUX, É. (1986). Time reversal of diffusions. *Ann. Probab.* **14** 1188–1205. MR0866342

[43] HENRY, D. (1981). *Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Math.* **840**. Springer, Berlin. MR0610244

[44] HÉRAU, F. (2006). Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation. *Asymptot. Anal.* **46** 349–359. MR2215889

[45] HU, K., KAZEYKINA, A. and REN, Z. (2019). Mean-field Langevin system, optimal control and deep neural networks. Preprint. Available at arXiv:1909.07278.

[46] HU, K., REN, Z., ŠIŠKA, D. and SZPRUCH, Ł. (2021). Mean-field Langevin dynamics and energy landscape of neural networks. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 2043–2065. MR4328560 https://doi.org/10.1214/20-aihp1140

[47] IACOBUCCI, A., OLLA, S. and STOLTZ, G. (2019). Convergence rates for nonequilibrium Langevin dynamics. *Ann. Math. Qué.* **43** 73–98. MR3925138 https://doi.org/10.1007/s40316-017-0091-0

[48] JABIR, J. F., ŠIŠKA, D. and SZPRUCH, L. (2019). Mean-field neural ODEs via relaxed optimal control. Preprint. Available at arXiv:1912.05475.

[49] JACOD, J. and MÉMIN, J. (1981). Weak and strong solutions of stochastic differential equations: Existence and stability. In *Stochastic Integrals* (*Proc. Sympos., Univ. Durham, Durham*, 1980) (D. Williams, ed.). *Lecture Notes in Math.* **851** 169–212. Springer, Berlin. MR0620991

[50] KOZLOV, S. M. (1989). Effective diffusion for the Fokker–Planck equation. *Mat. Zametki* **45** 360–368. MR1005458 https://doi.org/10.1007/BF01157929

[51] LANGEVIN, P. (1908). Sur la théorie du mouvement brownien. *C. R. Acad. Sci. Paris* **146** 530–533.

[52] LEIMKUHLER, B. and MATTHEWS, C. (2015). *Molecular Dynamics: With Deterministic and Stochastic Numerical Methods. Interdisciplinary Applied Mathematics* **39**. Springer, Cham. MR3362507

[53] LELIÈVRE, T., ROUSSET, M. and STOLTZ, G. (2010). *Free Energy Computations: A Mathematical Perspective*. Imperial College Press, London. MR2681239 https://doi.org/10.1142/9781848162488

[54] LIONS, P. L. In Cours Au Collège de France. www.college-de-france.fr.

[55] LU, Y., MA, C., LU, Y., LU, J. and YING, L. (2020). A mean field analysis of deep ResNet and beyond: Towards provably optimization via overparameterization from depth. In *Proceedings of the* 37*th International Conference on Machine Learning* (A. Daumé III and A. Singh, eds.). *Proceedings of Machine Learning Research* **119** 6426–6436. PMLR.

[56] LUO, D. and WANG, J. (2016). Exponential convergence in L^p-Wasserstein distance for diffusion processes without uniformly dissipative drift. *Math. Nachr.* **289** 1909–1926. MR3563910 https://doi.org/10.1002/mana.201500351

[57] MATTINGLY, J. C., STUART, A. M. and HIGHAM, D. J. (2002). Ergodicity for SDEs and approximations: Locally Lipschitz vector fields and degenerate noise. *Stochastic Process. Appl.* **101** 185–232. MR1931266 https://doi.org/10.1016/S0304-4149(02)00150-3

[58] MEI, S., MONTANARI, A. and NGUYEN, P.-M. (2018). A mean field view of the landscape of two-layer neural networks. *Proc. Natl. Acad. Sci. USA* **115** E7665–E7671. MR3845070 https://doi.org/10.1073/pnas.1806579115

[59] METAFUNE, G., PALLARA, D. and PRIOLA, E. (2002). Spectrum of Ornstein–Uhlenbeck operators in L^p spaces with respect to invariant measures. *J. Funct. Anal.* **196** 40–60. MR1941990 https://doi.org/10.1006/jfan.2002.3978

[60] MILLET, A., NUALART, D. and SANZ, M. (1989). Integration by parts and time reversal for diffusion processes. *Ann. Probab.* **17** 208–238. MR0972782

[61] MONMARCHÉ, P. (2017). Long-time behaviour and propagation of chaos for mean field kinetic particles. *Stochastic Process. Appl.* **127** 1721–1737. MR3646428 https://doi.org/10.1016/j.spa.2016.10.003

[62] NEAL, R. M. (2011). MCMC using Hamiltonian dynamics. In *Handbook of Markov Chain Monte Carlo. Chapman & Hall/CRC Handb. Mod. Stat. Methods* 113–162. CRC Press, Boca Raton, FL. MR2858447

[63] NELSON, E. (1967). *Dynamical Theories of Brownian Motion*. Princeton Univ. Press, Princeton, NJ. MR0214150

[64] PAVLIOTIS, G. A. (2014). *Stochastic Processes and Applications*: *Diffusion Processes*, *the Fokker–Planck and Langevin Equations*. *Texts in Applied Mathematics* **60**. Springer, New York. MR3288096 https://doi.org/10.1007/978-1-4939-1323-7

[65] REY-BELLET, L. and THOMAS, L. E. (2002). Exponential convergence to non-equilibrium stationary states in classical statistical mechanics. *Comm*. *Math*. *Phys*. **225** 305–329. MR1889227 https://doi.org/10.1007/s002200100583

[66] ROTSKOFF, G. and VANDEN-EIJNDEN, E. (2018). Neural networks as interacting particle systems: Asymptotic convexity of the loss landscape and universal scaling of the approximation error. Available at arXiv:1805.00915.

[67] SCHNEIDER, T. and STOLL, E. (1978). Molecular-dynamics study of a three-dimensional one-component model for distortive phase transitions. *Phys*. *Rev. B* **17** 1302–1322.

[68] SCHUH, K. (2022). Global contractivity for Langevin dynamics with distribution-dependent forces and uniform in time propagation of chaos. Preprint. Available at arXiv:2206.03082.

[69] ŠIŠKA, D. and SZPRUCH, L. (2020). Gradient flows for regularized stochastic control problems. Preprint. Available at arXiv:2006.05956.

[70] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[71] TALAY, D. (2002). Stochastic Hamiltonian systems: Exponential convergence to the invariant measure, and discretization by the implicit Euler scheme. In *Markov Process*. *Related Fields*, 2, *Inhomogeneous Random Systems* (*Cergy-Pontoise*, 2001). **8** 163–198. MR1924934

[72] ÜSTÜNEL, A. S. and ZAKAI, M. (2000). *Transformation of Measure on Wiener Space*. *Springer Monographs in Mathematics*. Springer, Berlin. MR1736980 https://doi.org/10.1007/978-3-662-13225-8

[73] VILLANI, C. (2007). Hypocoercive diffusion operators. *Boll*. *Unione Mat*. *Ital*., *B* **10** 257–275. MR2339441

[74] VILLANI, C. (2009). Hypocoercivity. *Mem*. *Amer. Math*. *Soc*. **202** iv+141. MR2562709 https://doi.org/10.1090/S0065-9266-09-00567-5

[75] WU, L. (2001). Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. *Stochastic Process*. *Appl*. **91** 205–238. MR1807683 https://doi.org/10.1016/S0304-4149(00)00061-2

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3227–3250
https://doi.org/10.1214/23-AAP2037

CLUSTERING OF LARGE DEVIATIONS IN MOVING AVERAGE PROCESSES: THE SHORT MEMORY REGIME

BY ARIJIT CHAKRABARTY[1,a] AND GENNADY SAMORODNITSKY[2,b]

[1]*Theoretical Statistics and Mathematics Unit, Indian Statistical Institute,* [a]*arijit.isi@gmail.com*
[2]*School of Operations Research and Information Engineering, Cornell University,* [b]*gs18@cornell.edu*

We describe the cluster of large deviations events that arise when one such large deviations event occurs. We work in the framework of an infinite moving average process with a noise that has finite exponential moments.

REFERENCES

BASRAK, B. and SEGERS, J. (2009). Regularly varying multivariate time series. *Stochastic Process*. *Appl*. **119** 1055–1080. MR2508565 https://doi.org/10.1016/j.spa.2008.05.004

BILLINGSLEY, P. (1995). *Probability and Measure*, 3rd ed. *Wiley Series in Probability and Mathematical Statistics*. Wiley, New York. MR1324786

BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962

CHAGANTY, N. R. and SETHURAMAN, J. (1993). Strong large deviation and local limit theorems. *Ann*. *Probab*. **21** 1671–1690. MR1235434

CHAKRABARTY, A. and SAMORODNITSKY, G. (2023). Clustering of large deviations in moving average processes: The long memory regime. *Stochastic Process*. *Appl*. **163** 387–423. MR4612303 https://doi.org/10.1016/j.spa.2023.06.009

DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics* (*New York*) **38**. Springer, New York. MR1619036 https://doi.org/10.1007/978-1-4612-5320-4

DEUSCHEL, J.-D. and STROOCK, D. W. (1989). *Large Deviations*. *Pure and Applied Mathematics* **137**. Academic Press, Boston, MA. MR0997938

DJELLOUT, H. and GUILLIN, A. (2001). Large and moderate deviations for moving average processes. *Ann*. *Fac*. *Sci*. *Toulouse Math*. (6) **10** 23–31. MR1928987

EMBRECHTS, P., KLÜPPELBERG, C. and MIKOSCH, T. (2003). *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin. 4th corrected printing.

GOYAL, A. and WELCH, I. (2003). Predicting the equity premium with dividend ratios. *Manage*. *Sci*. **49** 639–654.

JIANG, T. F., WANG, X. C. and RAO, M. B. (1992). Moderate deviations for some weakly dependent random processes. *Statist*. *Probab*. *Lett*. **15** 71–76. MR1190249 https://doi.org/10.1016/0167-7152(92)90287-F

KULIK, R. and SOULIER, P. (2020). *Heavy-Tailed Time Series*. *Springer Series in Operations Research and Financial Engineering*. Springer, New York. MR4174389 https://doi.org/10.1007/978-1-0716-0737-4

MIKOSCH, T. and NAGAEV, A. V. (1998). Large deviations of heavy-tailed sums with applications in insurance. *Extremes* **1** 81–110. MR1652936 https://doi.org/10.1023/A:1009913901219

RYBSKI, D., BUNDE, A., HAVLIN, S. and VON STORCH, H. (2006). Long-term persistence in climate and detection problem. *Geophys*. *Res*. *Lett*. **33**. https://doi.org/10.1029/2005GL02559

SAMORODNITSKY, G. (2016). *Stochastic Processes and Long Range Dependence*. *Springer Series in Operations Research and Financial Engineering*. Springer, Cham. MR3561100 https://doi.org/10.1007/978-3-319-45575-4

MSC2020 subject classifications. 60F10.
Key words and phrases. Large deviations, clustering, infinite moving average.

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3251–3287
https://doi.org/10.1214/23-AAP2038

NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMAL CONTROL OF SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

BY WILHELM STANNAT[1,a] AND LUKAS WESSELS[2,b]

[1]*Institut für Mathematik, Technische Universität Berlin,* [a]*stannat@math.tu-berlin.de*
[2]*School of Mathematics, Georgia Institute of Technology,* [b]*wessels@gatech.edu*

Using a recently introduced representation of the second order adjoint state as the solution of a function-valued backward stochastic partial differential equation (SPDE), we calculate the viscosity super- and subdifferential of the value function evaluated along an optimal trajectory for controlled semilinear SPDEs. This establishes the well-known connection between Pontryagin's maximum principle and dynamic programming within the framework of viscosity solutions. As a corollary, we derive that the correction term in the stochastic Hamiltonian arising in nonsmooth stochastic control problems is nonpositive. These results directly lead us to a stochastic verification theorem for fully nonlinear Hamilton–Jacobi–Bellman equations in the framework of viscosity solutions.

REFERENCES

[1] BELLMAN, R. (1957). *Dynamic Programming*. Princeton Univ. Press, Princeton, NJ. MR0090477
[2] BENSOUSSAN, A. (1982). Lectures on stochastic control. In *Nonlinear Filtering and Stochastic Control* (*Cortona*, 1981). *Lecture Notes in Math*. **972** 1–62. Springer, Berlin. MR0705931
[3] BENSOUSSAN, A. (1983). Stochastic maximum principle for distributed parameter systems. *J. Franklin Inst*. **315** 387–406. MR0713370 https://doi.org/10.1016/0016-0032(83)90059-5
[4] BISMUT, J.-M. (1978). An introductory approach to duality in optimal stochastic control. *SIAM Rev.* **20** 62–78. MR0469466 https://doi.org/10.1137/1020004
[5] CANNARSA, P. and FRANKOWSKA, H. (1991). Some characterizations of optimal trajectories in control theory. *SIAM J. Control Optim*. **29** 1322–1347. MR1132185 https://doi.org/10.1137/0329068
[6] CANNARSA, P. and FRANKOWSKA, H. (1992). Value function and optimality conditions for semilinear control problems. *Appl. Math. Optim*. **26** 139–169. MR1166210 https://doi.org/10.1007/BF01189028
[7] CHEN, L. and LÜ, P. (2022). Stochastic verification theorem for infinite dimensional stochastic control systems. Preprint. Available at https://arxiv.org/abs/2209.09576.
[8] DEMENGEL, F. and DEMENGEL, G. (2012). *Functional Spaces for the Theory of Elliptic Partial Differential Equations*. *Universitext*. Springer, London. MR2895178 https://doi.org/10.1007/978-1-4471-2807-6
[9] FABBRI, G. (2006). First order HJB equations in Hilbert spaces and applications. PhD Thesis, Università degli Studi di Roma – La Sapienza, Rome, Italy.
[10] FABBRI, G., GOZZI, F. and ŚWIĘCH, A. (2010). Verification theorem and construction of ϵ-optimal controls for control of abstract evolution equations. *J. Convex Anal*. **17** 611–642. MR2675666
[11] FABBRI, G., GOZZI, F. and ŚWIĘCH, A. (2017). *Stochastic Optimal Control in Infinite Dimension*: *Dynamic Programming and HJB Equations*. *Probability Theory and Stochastic Modelling* **82**. Springer, Cham. MR3674558 https://doi.org/10.1007/978-3-319-53067-3
[12] FABBRI, G. and RUSSO, F. (2017). HJB equations in infinite dimension and optimal control of stochastic evolution equations via generalized Fukushima decomposition. *SIAM J. Control Optim*. **55** 4072–4091. MR3738303 https://doi.org/10.1137/17M1113801
[13] FEDERICO, S., GOLDYS, B. and GOZZI, F. (2011). HJB equations for the optimal control of differential equations with delays and state constraints, II: Verification and optimal feedbacks. *SIAM J. Control Optim*. **49** 2378–2414. MR2854622 https://doi.org/10.1137/100804292
[14] FEDERICO, S. and GOZZI, F. (2018). Verification theorems for stochastic optimal control problems in Hilbert spaces by means of a generalized Dynkin formula. *Ann. Appl. Probab.* **28** 3558–3599. MR3861820 https://doi.org/10.1214/18-AAP1397

MSC2020 subject classifications. Primary 93E20, 49K45; secondary 60H15, 49L20.
Key words and phrases. Stochastic maximum principle, Pontryagin maximum principle, dynamic programming, verification theorem, stochastic optimal control, Hamilton–Jacobi–Bellman equation.

[15] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[16] GOZZI, F., ŚWIĘCH, A. and ZHOU, X. Y. (2005). A corrected proof of the stochastic verification theorem within the framework of viscosity solutions. *SIAM J. Control Optim*. **43** 2009–2019. MR2179474 https://doi.org/10.1137/S0363012903428184

[17] GOZZI, F., ŚWIĘCH, A. and ZHOU, X. Y. (2010). Erratum: "A corrected proof of the stochastic verification theorem within the framework of viscosity solutions". *SIAM J. Control Optim*. **48** 4177–4179. MR2652076 https://doi.org/10.1137/090775567

[18] KRYLOV, N. V. (2013). A relatively short proof of Itô's formula for SPDEs and its applications. *Stoch. Partial Differ. Equ. Anal. Comput*. **1** 152–174. MR3327504 https://doi.org/10.1007/s40072-013-0003-5

[19] LIU, W. and RÖCKNER, M. (2015). *Stochastic Partial Differential Equations*: *An Introduction*. *Universitext*. Springer, Cham. MR3410409 https://doi.org/10.1007/978-3-319-22354-4

[20] PARDOUX, É. (2021). *Stochastic Partial Differential Equations—an Introduction*. *SpringerBriefs in Mathematics*. Springer, Cham. MR4385528 https://doi.org/10.1007/978-3-030-89003-2

[21] PAZY, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. *Applied Mathematical Sciences* **44**. Springer, New York. MR0710486 https://doi.org/10.1007/978-1-4612-5561-1

[22] PONTRYAGIN, L. S., BOLTYANSKII, V. G., GAMKRELIDZE, R. V. and MISHCHENKO, E. F. (1962). *The Mathematical Theory of Optimal Processes*. Interscience Publishers Wiley, New York. MR0166037

[23] STANNAT, W. and WESSELS, L. (2021). Peng's maximum principle for stochastic partial differential equations. *SIAM J. Control Optim*. **59** 3552–3573. MR4321242 https://doi.org/10.1137/20M1368057

[24] YONG, J. and ZHOU, X. Y. (1999). *Stochastic Controls*: *Hamiltonian Systems and HJB Equations*. *Applications of Mathematics* (*New York*) **43**. Springer, New York. MR1696772 https://doi.org/10.1007/978-1-4612-1466-3

[25] ZEIDLER, E. (1986). *Nonlinear Functional Analysis and Its Applications*. *I*. *Fixed-Point Theorems*. Springer, New York. MR0816732 https://doi.org/10.1007/978-1-4612-4838-5

[26] ZHOU, X. Y. (1991). A unified treatment of maximum principle and dynamic programming in stochastic controls. *Stoch*. *Stoch*. *Rep*. **36** 137–161. MR1128491

[27] ZHOU, X. Y. (1991). Remarks on optimal controls of stochastic partial differential equations. *Systems Control Lett*. **16** 465–472. MR1112763 https://doi.org/10.1016/0167-6911(91)90118-X

[28] ZHOU, X. Y., YONG, J. and LI, X. (1997). Stochastic verification theorems within the framework of viscosity solutions. *SIAM J. Control Optim*. **35** 243–253. MR1430292 https://doi.org/10.1137/S0363012995279973

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3288–3319
https://doi.org/10.1214/23-AAP2039

MAXIMUM LIKELIHOOD THRESHOLDS VIA GRAPH RIGIDITY

BY DANIEL IRVING BERNSTEIN[1,a], SEAN DEWAR[2,b], STEVEN J. GORTLER[3,c], ANTHONY NIXON[4,d], MEERA SITHARAM[5,e] AND LOUIS THERAN[6,f]

[1]*Department of Mathematics, Tulane University,* [a]*dbernstein1@tulane.edu*
[2]*School of Mathematics, University of Bristol,* [b]*sean.dewar@bristol.ac.uk*
[3]*School of Engineering and Applied Sciences, Harvard University,* [c]*sjg@cs.harvard.edu*
[4]*Department of Mathematics and Statistics, Lancaster University,* [d]*a.nixon@lancaster.ac.uk*
[5]*Department of Computer Science, University of Florida,* [e]*sitharam@cise.ufl.edu*
[6]*School of Mathematics and Statistics, University of St Andrews,* [f]*lst6@st-andrews.ac.uk*

The maximum likelihood threshold (MLT) of a graph G is the minimum number of samples to almost surely guarantee existence of the maximum likelihood estimate in the corresponding Gaussian graphical model. We give a new characterization of the MLT in terms of rigidity-theoretic properties of G and use this characterization to give new combinatorial lower bounds on the MLT of any graph.

We use the new lower bounds to give high-probability guarantees on the maximum likelihood thresholds of sparse Erdős–Rényi random graphs in terms of their average density. These examples show that the new lower bounds are within a polylog factor of tight, where, on the same graph families, all known lower bounds are trivial.

Based on computational experiments made possible by our methods, we conjecture that the MLT of an Erdős–Rényi random graph is equal to its generic completion rank with high probability. Using structural results on rigid graphs in low dimension, we can prove the conjecture for graphs with MLT at most 4 and describe the threshold probability for the MLT to switch from 3 to 4.

We also give a geometric characterization of the MLT of a graph in terms of a new "lifting" problem for frameworks that is interesting in its own right. The lifting perspective yields a new connection between the weak MLT (where the maximum likelihood estimate exists only with positive probability) and the classical Hadwiger–Nelson problem.

REFERENCES

[1] ABBOT, T. G. (2008). Generalizations of Kempe's Universality Theorem. Master's thesis, Massachusetts Institute of Technology Cambridge, MA.

[2] ALFAKIH, A. Y. (2011). On bar frameworks, stress matrices and semidefinite programming. *Math. Program.* **129** 113–128. MR2831405 https://doi.org/10.1007/s10107-010-0389-z

[3] AMÉNDOLA, C., KOHN, K., REICHENBACH, P. and SEIGAL, A. (2021). Invariant theory and scaling algorithms for maximum likelihood estimation. *SIAM J. Appl. Algebra Geom.* **5** 304–337. MR4274830 https://doi.org/10.1137/20M1328932

[4] ASIMOW, L. and ROTH, B. (1978). The rigidity of graphs. *Trans. Amer. Math. Soc.* **245** 279–289. MR0511410 https://doi.org/10.2307/1998867

[5] BEN-DAVID, E. (2015). Sharp lower and upper bounds for the Gaussian rank of a graph. *J. Multivariate Anal.* **139** 207–218. MR3349487 https://doi.org/10.1016/j.jmva.2015.03.004

[6] BERG, A. R. and JORDÁN, T. (2003). A proof of Connelly's conjecture on 3-connected circuits of the rigidity matroid. *J. Combin. Theory Ser. B* **88** 77–97. MR1973261 https://doi.org/10.1016/S0095-8956(02)00037-0

MSC2020 subject classifications. Primary 62H12, 52C25; secondary 90C25.
Key words and phrases. Gaussian graphical models, number of observations, maximum likelihood threshold, combinatorial rigidity, algebraic statistics.

[7] BERNSTEIN, D. I., BLEKHERMAN, G. and SINN, R. (2020). Typical and generic ranks in matrix completion. *Linear Algebra Appl*. **585** 71–104. MR4016092 https://doi.org/10.1016/j.laa.2019.09.001

[8] BLEKHERMAN, G. and SINN, R. (2019). Maximum likelihood threshold and generic completion rank of graphs. *Discrete Comput. Geom.* **61** 303–324. MR3903791 https://doi.org/10.1007/s00454-018-9990-3

[9] BOCHNAK, J., COSTE, M. and ROY, M.-F. (1998). *Real Algebraic Geometry*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete* (3) [*Results in Mathematics and Related Areas* (3)] **36**. Springer, Berlin. MR1659509 https://doi.org/10.1007/978-3-662-03718-8

[10] BOLKER, E. D. and ROTH, B. (1980). When is a bipartite graph a rigid framework? *Pacific J. Math*. **90** 27–44. MR0599317

[11] BRIGHTWELL, G. (1993). On the complexity of diagram testing. *Order* **10** 297–303. MR1269267 https://doi.org/10.1007/BF01108825

[12] BUHL, S. L. (1993). On the existence of maximum likelihood estimators for graphical Gaussian models. *Scand. J. Stat*. **20** 263–270. MR1241392

[13] CANDÈS, E. J. and TAO, T. (2010). The power of convex relaxation: Near-optimal matrix completion. *IEEE Trans. Inf. Theory* **56** 2053–2080. MR2723472 https://doi.org/10.1109/TIT.2010.2044061

[14] CONNELLY, R. (1982). Rigidity and energy. *Invent. Math*. **66** 11–33. MR0652643 https://doi.org/10.1007/BF01404753

[15] CONNELLY, R. (2005). Generic global rigidity. *Discrete Comput. Geom*. **33** 549–563. MR2132290 https://doi.org/10.1007/s00454-004-1124-4

[16] CONNELLY, R. and GORTLER, S. J. (2017). Universal rigidity of complete bipartite graphs. *Discrete Comput. Geom*. **57** 281–304. MR3602854 https://doi.org/10.1007/s00454-016-9836-9

[17] CONNELLY, R. and GORTLER, S. J. (2017). Universal rigidity of complete bipartite graphs. *Discrete Comput. Geom*. **57** 281–304. MR3602854 https://doi.org/10.1007/s00454-016-9836-9

[18] CONNELLY, R., GORTLER, S. J. and THERAN, L. (2018). Affine rigidity and conics at infinity. *Int. Math. Res. Not. IMRN* **13** 4084–4102. MR3829178 https://doi.org/10.1093/imrn/rnx014

[19] CONNELLY, R., GORTLER, S. J. and THERAN, L. (2020). Generically globally rigid graphs have generic universally rigid frameworks. *Combinatorica* **40** 1–37. MR4078810 https://doi.org/10.1007/s00493-018-3694-4

[20] CONNELLY, R. and WHITELEY, W. J. (2010). Global rigidity: The effect of coning. *Discrete Comput. Geom*. **43** 717–735. MR2610468 https://doi.org/10.1007/s00454-009-9220-0

[21] DE GREY, A. D. N. J. (2018). The chromatic number of the plane is at least 5. *Geombinatorics* **28** 18–31. MR3820926

[22] DEMPSTER, A. P. (1972). Covariance selection. *Biometrics* **28** 157–175. MR3931974

[23] DOBRA, A., HANS, C., JONES, B., NEVINS, J. R., YAO, G. and WEST, M. (2004). Sparse graphical models for exploring gene expression data. *J. Multivariate Anal*. **90** 196–212. MR2064941 https://doi.org/10.1016/j.jmva.2004.02.009

[24] DRTON, M., FOX, C., KÄUFL, A. and POULIOT, G. (2019). The maximum likelihood threshold of a path diagram. *Ann. Statist*. **47** 1536–1553. MR3911121 https://doi.org/10.1214/18-AOS1724

[25] ERDŐS, P., HARARY, F. and TUTTE, W. T. (1965). On the dimension of a graph. *Mathematika* **12** 118–122. MR0188096 https://doi.org/10.1112/S0025579300005222

[26] GLUCK, H. (1975). Almost all simply connected closed surfaces are rigid. In *Geometric Topology* (*Proc. Conf., Park City, Utah*, 1974). *Lecture Notes in Math., Vol*. 438 225–239. Springer, Berlin. MR0400239

[27] GORTLER, S. J., HEALY, A. D. and THURSTON, D. P. (2010). Characterizing generic global rigidity. *Amer. J. Math*. **132** 897–939. MR2663644 https://doi.org/10.1353/ajm.0.0132

[28] GORTLER, S. J. and THURSTON, D. P. (2014). Characterizing the universal rigidity of generic frameworks. *Discrete Comput. Geom*. **51** 1017–1036. MR3216675 https://doi.org/10.1007/s00454-014-9590-9

[29] GORTLER, S. J. and THURSTON, D. P. (2014). Generic global rigidity in complex and pseudo-Euclidean spaces. In *Rigidity and Symmetry*. *Fields Inst. Commun*. **70** 131–154. Springer, New York. MR3329273 https://doi.org/10.1007/978-1-4939-0781-6_8

[30] GROSS, E. and SULLIVANT, S. (2018). The maximum likelihood threshold of a graph. *Bernoulli* **24** 386–407. MR3706762 https://doi.org/10.3150/16-BEJ881

[31] HASTIE, T., TIBSHIRANI, R. and FRIEDMAN, J. (2009). *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, 2nd ed. *Springer Series in Statistics*. Springer, New York. MR2722294 https://doi.org/10.1007/978-0-387-84858-7

[32] HENDRICKSON, B. (1992). Conditions for unique graph realizations. *SIAM J. Comput*. **21** 65–84. MR1148818 https://doi.org/10.1137/0221008

[33] JACKSON, B. and JORDÁN, T. (2005). The d-dimensional rigidity matroid of sparse graphs. *J. Combin. Theory Ser. B* **95** 118–133. MR2156343 https://doi.org/10.1016/j.jctb.2005.03.004

[34] JORDÁN, T. (2017). Extremal problems and results in combinatorial rigidity. In *Proc. of the* 10*th Japanese Hungarian Symposium on Discrete Mathematics and Its Applications* 297–304.

[35] KAPOVICH, M. and MILLSON, J. J. (1997). Hodge theory and the art of paper folding. *Publ. Res. Inst. Math. Sci.* **33** 1–31. MR1442490 https://doi.org/10.2977/prims/1195145531

[36] KASIVISWANATHAN, S. P., MOORE, C. and THERAN, L. (2011). The rigidity transition in random graphs. In *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms* 1237–1252. SIAM, Philadelphia, PA. MR2858396

[37] KIRÁLY, F. J. and THERAN, L. (2013). Coherence and sufficient sampling densities for reconstruction in compressed sensing. Preprint. Available at arXiv:1302.2767.

[38] KIRÁLY, F. J., THERAN, L. and TOMIOKA, R. (2015). The algebraic combinatorial approach for low-rank matrix completion. *J. Mach. Learn. Res.* **16** 1391–1436. MR3417786

[39] KLOKS, T. (1994). *Treewidth: Computations and Approximations. Lecture Notes in Computer Science* **842**. Springer, Berlin. MR1312164 https://doi.org/10.1007/BFb0045375

[40] KRUMSIEK, J., SUHRE, K., ILLIG, T., ADAMSKI, J. and THEIS, F. J. (2011). Gaussian graphical modeling reconstructs pathway reactions from high-throughput metabolomics data. *BMC Syst. Biol.* **5** 1–16.

[41] LAMAN, G. (1970). On graphs and rigidity of plane skeletal structures. *J. Engrg. Math.* **4** 331–340. MR0269535 https://doi.org/10.1007/BF01534980

[42] LEE, A. and STREINU, I. (2008). Pebble game algorithms and sparse graphs. *Discrete Math.* **308** 1425–1437. MR2392060 https://doi.org/10.1016/j.disc.2007.07.104

[43] LEW, A., NEVO, E., PELED, Y. and RAZ, O. E. (2023). Sharp threshold for rigidity of random graphs. *Bull. Lond. Math. Soc.* **55** 490–501. MR4568355 https://doi.org/10.1112/blms.12740

[44] LINIAL, N., LOVÁSZ, L. and WIGDERSON, A. (1988). Rubber bands, convex embeddings and graph connectivity. *Combinatorica* **8** 91–102. MR0951998 https://doi.org/10.1007/BF02122557

[45] MAKAM, V., REICHENBACH, P. and SEIGAL, A. (2023). Symmetries in directed Gaussian graphical models. *Electron. J. Stat.* **17** 3969–4010. MR4677191 https://doi.org/10.1214/23-ejs2192

[46] MAXWELL, J. C. (1864). On the calculation of the equilibrium and stiffness of frames. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **27** 294–299.

[47] NEVO, E. (2007). On embeddability and stresses of graphs. *Combinatorica* **27** 465–472. MR2359827 https://doi.org/10.1007/s00493-007-2168-x

[48] NIXON, A. and WHITELEY, W. (2019). Change of metrics in rigidity theory. In *Handbook of Geometric Constraint Systems Principles. Discrete Math. Appl.* (*Boca Raton*) 351–374. CRC Press, Boca Raton, FL. MR3837062

[49] OPGEN-RHEIN, R. and STRIMMER, K. (2007). From correlation to causation networks: A simple approximate learning algorithm and its application to high-dimensional plant gene expression data. *BMC Syst. Biol.* **1** 1–10.

[50] PITTEL, B., SPENCER, J. and WORMALD, N. (1996). Sudden emergence of a giant k-core in a random graph. *J. Combin. Theory Ser. B* **67** 111–151. MR1385386 https://doi.org/10.1006/jctb.1996.0036

[51] POLLACZEK-GEIRINGER, H. (1927). Über die Gliederung ebener Fachwerke. *ZAMM Z. Angew. Math. Mech.* **7** 58–72.

[52] ROSEN, Z., SIDMAN, J. and THERAN, L. (2020). Algebraic matroids in action. *Amer. Math. Monthly* **127** 199–216. MR4067892 https://doi.org/10.1080/00029890.2020.1689781

[53] SALIOLA, F. V. and WHITELEY, W. (2007). Some notes on the equivalence of first-order rigidity in various geometries. Preprint. Available at arXiv:0709.3354.

[54] SCHÄFER, J. and STRIMMER, K. (2005). An empirical Bayes approach to inferring large-scale gene association networks. *Bioinformatics* **21** 754–764. https://doi.org/10.1093/bioinformatics/bti062

[55] SCHÄFER, J. and STRIMMER, K. (2005). A shrinkage approach to large-scale covariance matrix estimation and implications for functional genomics. *Stat. Appl. Genet. Mol. Biol.* **4** Art. 32. MR2183942 https://doi.org/10.2202/1544-6115.1175

[56] SIMON, B. (2011). *Convexity: An Analytic Viewpoint. Cambridge Tracts in Mathematics* **187**. Cambridge Univ. Press, Cambridge. MR2814377 https://doi.org/10.1017/CBO9780511910135

[57] SINGER, A. and CUCURINGU, M. (2009/10). Uniqueness of low-rank matrix completion by rigidity theory. *SIAM J. Matrix Anal. Appl.* **31** 1621–1641. MR2595541 https://doi.org/10.1137/090750688

[58] STASICA, J. (2003). Smooth points of a semialgebraic set. *Ann. Polon. Math.* **82** 149–153. MR2041841 https://doi.org/10.4064/ap82-2-5

[59] TUTTE, W. T. (1963). How to draw a graph. *Proc. Lond. Math. Soc.* (3) **13** 743–767. MR0158387 https://doi.org/10.1112/plms/s3-13.1.743

[60] UHLER, C. (2012). Geometry of maximum likelihood estimation in Gaussian graphical models. *Ann. Statist.* **40** 238–261. MR3014306 https://doi.org/10.1214/11-AOS957

[61] VANDENBERGHE, L., BOYD, S. and WU, S.-P. (1998). Determinant maximization with linear matrix inequality constraints. *SIAM J. Matrix Anal. Appl.* **19** 499–533. MR1614078 https://doi.org/10.1137/S0895479896303430

[62] WHITELEY, W. (1983). Cones, infinity and 1-story buildings. *Struct. Topol.* **8** 53–70. MR0721956

[63] WORMALD, N. C. (1999). Models of random regular graphs. In *Surveys in Combinatorics*, 1999 (*Canterbury*). *London Mathematical Society Lecture Note Series* **267** 239–298. Cambridge Univ. Press, Cambridge. MR1725006

[64] WU, X., YE, Y. and SUBRAMANIAN, K. R. (2003). Interactive analysis of gene interactions using graphical Gaussian model. In *Proceedings of the* 3*rd International Conference on Data Mining in Bioinformatics* 63–69.

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3320–3347
https://doi.org/10.1214/23-AAP2040

WASSERSTEIN CONVERGENCE RATES OF INCREASINGLY CONCENTRATING PROBABILITY MEASURES

BY MAREIKE HASENPFLUG[1,a], DANIEL RUDOLF[1,b] AND BJÖRN SPRUNGK[2,c]

[1]*Fakultät für Informatik und Mathematik, Universität Passau,* [a]*mareike.hasenpflug@uni-passau.de,* [b]*daniel.rudolf@uni-passau.de*

[2]*Fakultät für Mathematik und Informatik, Technische Universität Bergakademie Freiberg,* [c]*bjoern.sprungk@math.tu-freiberg.de*

For $\ell\colon \mathbb{R}^d \to [0,\infty)$ we consider the sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$, where μ_n is determined by a density that is proportional to $\exp(-n\ell)$. We allow for infinitely many global minimal points of ℓ, as long as they form a finite union of compact manifolds. In this scenario, we show estimates for the p-Wasserstein convergence of $(\mu_n)_{n\in\mathbb{N}}$ to its limit measure. Imposing regularity conditions we obtain a speed of convergence of $n^{-1/(2p)}$ and adding a further technical assumption, we can improve this to a p-independent rate of $1/2$ for all orders $p \in \mathbb{N}$ of the Wasserstein distance.

REFERENCES

[1] ATHREYA, K. B. and HWANG, C.-R. (2010). Gibbs measures asymptotics. *Sankhya A* **72** 191–207. MR2658170 https://doi.org/10.1007/s13171-010-0006-5

[2] BOOTHBY, W. M. (1986). *An Introduction to Differentiable Manifolds and Riemannian Geometry*, 2nd ed. *Pure and Applied Mathematics* **120**. Academic Press, Orlando, FL. MR0861409

[3] BRAS, P. (2022). Convergence rates of Gibbs measures with degenerate minimum. *Bernoulli* **28** 2431–2458. MR4474549 https://doi.org/10.3150/21-bej1424

[4] BRAS, P. and PAGÈS, G. (2022). Convergence of Langevin-simulated annealing algorithms with multiplicative noise. Preprint. Available at arXiv:2109.11669v2.

[5] DE BORTOLI, V. and DESOLNEUX, A. (2021). On quantitative Laplace-type convergence results for some exponential probability measures, with two applications. Available at arXiv:2110.12922v1.

[6] DOUC, R., MOULINES, E., PRIOURET, P. and SOULIER, P. (2018). *Markov Chains*. *Springer Series in Operations Research and Financial Engineering*. Springer, Cham. MR3889011 https://doi.org/10.1007/978-3-319-97704-1

[7] EBERLE, A. (2021). Markov processes. Lecture notes at University of Bonn. Available at https://uni-bonn.sciebo.de/s/kzTUFff5FrWGAay.

[8] HARDY, M. (2006). Combinatorics of partial derivatives. *Electron*. *J*. *Combin*. **13** R1. MR2200529 https://doi.org/10.37236/1027

[9] HELIN, T. and KRETSCHMANN, R. (2022). Non-asymptotic error estimates for the Laplace approximation in Bayesian inverse problems. *Numer. Math.* **150** 521–549. MR4382587 https://doi.org/10.1007/s00211-021-01266-9

[10] HWANG, C.-R. (1980). Laplace's method revisited: Weak convergence of probability measures. *Ann. Probab.* **8** 1177–1182. MR0602391

[11] KORSHUNOV, D. A., PITERBARG, V. I. and KHASHORVA, E. (2015). On the asymptotic Laplace method and its application to random chaos. *Mat. Zametki* **97** 868–883. MR3399144 https://doi.org/10.4213/mzm10487

[12] MILNOR, J. W. and STASHEFF, J. D. (1974). *Characteristic Classes*. *Annals of Mathematics Studies* **76**. Princeton Univ. Press, Princeton, NJ, Univ. Tokyo Press, Tokyo. MR0440554

[13] PFLUG, G. C. and PICHLER, A. (2014). *Multistage Stochastic Optimization*. *Springer Series in Operations Research and Financial Engineering*. Springer, Cham. MR3288310 https://doi.org/10.1007/978-3-319-08843-3

[14] RUDOLF, D. and SPRUNGK, B. (2022). Robust random walk-like Metropolis–Hastings algorithms for concentrating posteriors. Available at arXiv:2202.12127.

MSC2020 subject classifications. 60B10, 58C99.
Key words and phrases. Wasserstein distance, coupling construction.

[15] SCHILLINGS, C., SPRUNGK, B. and WACKER, P. (2020). On the convergence of the Laplace approximation and noise-level-robustness of Laplace-based Monte Carlo methods for Bayesian inverse problems. *Numer. Math.* **145** 915–971. MR4125981 https://doi.org/10.1007/s00211-020-01131-1

[16] VILLANI, C. (2009). *Optimal Transport*: *Old and New*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

[17] WEYL, H. (1939). On the Volume of Tubes. *Amer. J. Math.* **61** 461–472. MR1507388 https://doi.org/10.2307/2371513

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3348–3369
https://doi.org/10.1214/23-AAP2042

THE MULTIVARIATE RATE OF CONVERGENCE FOR SELBERG'S CENTRAL LIMIT THEOREM

BY ASHER ROBERTS[a]

Graduate Center, City University of New York, [a]*aroberts@gradcenter.cuny.edu*

In this paper we quantify the rate of convergence in Selberg's central limit theorem for $\log|\zeta(1/2+it)|$ based on the method of proof given by Radziwiłł and Soundararajan in (*Enseign. Math.* **63** (2017) 1–19). We achieve the same rate of convergence of $(\log\log\log T)^2/\sqrt{\log\log T}$ as Selberg in (In *Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989) Univ* (1992) 367–385) in the Kolmogorov distance by using the Dudley distance instead. We also prove the theorem for the multivariate case given by Bourgade in (*Probab. Theory Related Fields* **148** (2010) 479–500) with the same rate of convergence as in the single variable case.

REFERENCES

[1] ARGUIN, L.-P., BELIUS, D., BOURGADE, P., RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2017). Maximum of the Riemann zeta function on a short interval of the critical line. Preprint. Available at arXiv:1612.08575v2.

[2] ARGUIN, L.-P., BELIUS, D. and HARPER, A. J. (2017). Maxima of a randomized Riemann zeta function, and branching random walks. *Ann. Appl. Probab.* **27** 178–215. MR3619786 https://doi.org/10.1214/16-AAP1201

[3] ARGUIN, L.-P., BOURGADE, P. and RADZIWIŁŁ, M. (2020). The Fyodorov–Hiary–Keating conjecture. I. Preprint. Available at arXiv:2007.00988.

[4] BOURGADE, P. (2010). Mesoscopic fluctuations of the zeta zeros. *Probab. Theory Related Fields* **148** 479–500. MR2678896 https://doi.org/10.1007/s00440-009-0237-3

[5] BOURGADE, P., HUGHES, C. P., NIKEGHBALI, A. and YOR, M. (2008). The characteristic polynomial of a random unitary matrix: A probabilistic approach. *Duke Math. J.* **145** 45–69. MR2451289 https://doi.org/10.1215/00127094-2008-046

[6] DUDLEY, R. M. (1968). Distances of probability measures and random variables. *Ann. Math. Stat.* **39** 1563–1572. MR0230338 https://doi.org/10.1007/978-1-4419-5821-1_4

[7] GAUNT, R. E. and LI, S. (2023). Bounding Kolmogorov distances through Wasserstein and related integral probability metrics. *J. Math. Anal. Appl.* **522** Paper No. 126985, 24 pp. MR4533915 https://doi.org/10.1016/j.jmaa.2022.126985

[8] HSU, P.-H. and WONG, P.-J. (2020). On Selberg's central limit theorem for Dirichlet L-functions. *J. Théor. Nombres Bordeaux* **32** 685–710. MR4206427

[9] IVIĆ, A. (1985). *The Riemann Zeta-Function: The Theory of the Riemann Zeta-Function with Applications. A Wiley-Interscience Publication*. Wiley, New York. MR0792089

[10] IWANIEC, H. and KOWALSKI, E. (2004). *Analytic Number Theory. American Mathematical Society Colloquium Publications* **53**. Amer. Math. Soc., Providence, RI. MR2061214 https://doi.org/10.1090/coll/053

[11] KEATING, J. P. and SNAITH, N. C. (2000). Random matrix theory and $\zeta(1/2+it)$. *Comm. Math. Phys.* **214** 57–89. MR1794265 https://doi.org/10.1007/s002200000261

[12] KOWALSKI, E. and NIKEGHBALI, A. (2012). Mod-Gaussian convergence and the value distribution of $\zeta(\frac{1}{2}+it)$ and related quantities. *J. Lond. Math. Soc.* (2) **86** 291–319. MR2959306 https://doi.org/10.1112/jlms/jds003

[13] MONTGOMERY, H. L. and VAUGHAN, R. C. (2007). *Multiplicative Number Theory. I. Classical Theory. Cambridge Studies in Advanced Mathematics* **97**. Cambridge Univ. Press, Cambridge. MR2378655

[14] RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2017). Selberg's central limit theorem for $\log|\zeta(1/2+it)|$. *Enseign. Math.* **63** 1–19. MR3832861 https://doi.org/10.4171/LEM/63-1/2-1

MSC2020 subject classifications. Primary 11M06; secondary 60F05.
Key words and phrases. Riemann Zeta function, central limit theorem.

[15] SELBERG, A. (1946). Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvidensk.* **48** 89–155. MR0020594
[16] SELBERG, A. (1992). Old and new conjectures and results about a class of Dirichlet series. In *Proceedings of the Amalfi Conference on Analytic Number Theory* (*Maiori*, 1989) 367–385. Univ. Salerno, Salerno. MR1220477
[17] TSANG, K.-M. (1984). The distribution of the values of the Riemann zeta-function. Thesis. MR2633927
[18] WAHL, M. (2014). On the mod-Gaussian convergence of a sum over primes. *Math. Z.* **276** 635–654. MR3175154 https://doi.org/10.1007/s00209-013-1216-z

The Annals of Applied Probability
2024, Vol. 34, No. 3, 3370–3374
https://doi.org/10.1214/23-AAP2043
Main article: https://doi.org/10.1214/16-AAP1241

CORRECTIONS: ON THE CAPACITY FUNCTIONAL OF THE INFINITE CLUSTER OF A BOOLEAN MODEL

BY GÜNTER LAST[1,a], MATHEW D. PENROSE[2,b] AND SERGEI ZUYEV[3,c]

[1]*Institut für Stochastik, Karlsruhe Institute of Technology,* [a]*guenter.last@kit.edu*
[2]*Department of Mathematical Sciences, University of Bath,* [b]*m.d.penrose@bath.ac.uk*
[3]*Department of Mathematical Sciences, Chalmers University of Technology,* [c]*sergei.zuyev@chalmers.se*

We fix a major gap in our 2017 paper, along with related issues.

REFERENCES

[1] LAST, G., PENROSE, M. D. and ZUYEV, S. (2017). On the capacity functional of the infinite cluster of a Boolean model. *Ann. Appl. Probab.* **27** 1678–1701. MR3678482 https://doi.org/10.1214/16-AAP1241
[2] SCHMITZ, E. (2022). Analyticity of the capacity functional of the infinite cluster in the Boolean model. Master's thesis, Institut für Mathematik, TU Berlin.

MSC2020 subject classifications. Primary 60K35; secondary 60D05.
Key words and phrases. Continuum percolation, Boolean model, infinite cluster, capacity functional, percolation function, Reimer inequality, Margulis–Russo-type formula.