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RATE CONTROL UNDER HEAVY TRAFFIC WITH STRATEGIC SERVERS

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We consider a large queueing system that consists of many strategic servers that are weakly interacting. Each server processes jobs from its unique critically loaded buffer and controls the rate of arrivals and departures associated with its queue to minimize its expected cost. The rates and the cost functions in addition to depending on the control action, can depend, in a symmetric fashion, on the size of the individual queue and the empirical measure of the states of all queues in the system. In order to determine an approximate Nash equilibrium for this finite player game, we construct a Lasry–Lions-type mean-field game (MFG) for certain reflected diffusions that governs the limiting behavior. Under conditions, we establish the convergence of the Nash-equilibrium value for the finite size queuing system to the value of the MFG.

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NONCONVEX HOMOGENIZATION FOR ONE-DIMENSIONAL CONTROLLED RANDOM WALKS IN RANDOM POTENTIAL

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We consider a finite horizon stochastic optimal control problem for nearest-neighbor random walk $\{X_i\}$ on the set of integers. The cost function is the expectation of the exponential of the path sum of a random stationary and ergodic bounded potential plus θX_n. The random walk policies are measurable with respect to the random potential, and are adapted, with their drifts uniformly bounded in magnitude by a parameter $\delta \in [0, 1]$. Under natural conditions on the potential, we prove that the normalized logarithm of the optimal cost function converges. The proof is constructive in the sense that we identify asymptotically optimal policies given the value of the parameter δ, as well as the law of the potential. It relies on correctors from large deviation theory as opposed to arguments based on subadditivity which do not seem to work except when $\delta = 0$.

The Bellman equation associated to this control problem is a second-order Hamilton–Jacobi (HJ) partial difference equation with a separable random Hamiltonian which is nonconvex in θ unless $\delta = 0$. We prove that this equation homogenizes under linear initial data to a first-order HJ equation with a deterministic effective Hamiltonian. When $\delta = 0$, the effective Hamiltonian is the tilted free energy of random walk in random potential and it is convex in θ. In contrast, when $\delta = 1$, the effective Hamiltonian is piecewise linear and nonconvex in θ. Finally, when $\delta \in (0, 1)$, the effective Hamiltonian is expressed completely in terms of the tilted free energy for the $\delta = 0$ case and its convexity/nonconvexity in θ is characterized by a simple inequality involving δ and the magnitude of the potential, thereby marking two qualitatively distinct control regimes.

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PARTICLE SYSTEMS WITH SINGULAR INTERACTION THROUGH HITTING TIMES: APPLICATION IN SYSTEMIC RISK MODELING

BY SERGEY NADTOCHIY[1] AND MYKHAYLO SHKOLNIKOV[2]

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We propose an interacting particle system to model the evolution of a system of banks with mutual exposures. In this model, a bank defaults when its normalized asset value hits a lower threshold, and its default causes instantaneous losses to other banks, possibly triggering a cascade of defaults. The strength of this interaction is determined by the level of the so-called *noncore exposure*. We show that, when the size of the system becomes large, the cumulative loss process of a bank resulting from the defaults of other banks exhibits discontinuities. These discontinuities are naturally interpreted as *systemic events*, and we characterize them explicitly in terms of the level of noncore exposure and the fraction of banks that are "about to default." The main mathematical challenges of our work stem from the very singular nature of the interaction between the particles, which is inherited by the limiting system. A similar particle system is analyzed in [*Ann. Appl. Probab.* **25** (2015) 2096–2133] and [*Stochastic Process. Appl.* **125** (2015) 2451–2492], and we build on and extend their results. In particular, we characterize the large-population limit of the system and analyze the jump times, the regularity between jumps, and the local uniqueness of the limiting process.

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CUBATURE ON WIENER SPACE FOR MCKEAN–VLASOV SDES WITH SMOOTH SCALAR INTERACTION[1]

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We present two cubature on Wiener space algorithms for the numerical solution of McKean–Vlasov SDEs with smooth scalar interaction. First, we consider a method introduced in [*Stochastic Process. Appl.* **125** (2015) 2206–2255] under a uniformly elliptic assumption and extend the analysis to a uniform strong Hörmander assumption. Then we introduce a new method based on Lagrange polynomial interpolation. The analysis hinges on sharp gradient to time-inhomogeneous parabolic PDEs bounds. These bounds may be of independent interest. They extend the classical results of Kusuoka and Stroock [*J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math.* **32** (1985) 1–76] and Kusuoka [*J. Math. Sci. Univ. Tokyo* **10** (2003) 261–277] further developed in [*J. Funct. Anal.* **263** (2012) 3024–3101; *J. Funct. Anal.* **268** (2015) 1928–1971; *Cubature Methods and Applications* (2013), Springer, Cham] and, more recently, in [*Probab. Theory Related Fields* **171** (2016) 97–148]. Both algorithms are tested through two numerical examples.

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LOWER ERROR BOUNDS FOR STRONG APPROXIMATION OF SCALAR SDES WITH NON-LIPSCHITZIAN COEFFICIENTS

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We study pathwise approximation of scalar stochastic differential equations at a single time point or globally in time by means of methods that are based on finitely many observations of the driving Brownian motion. We prove lower error bounds in terms of the average number of evaluations of the driving Brownian motion that hold for every such method under rather mild assumptions on the coefficients of the equation. The underlying simple idea of our analysis is as follows: the lower error bounds known for equations with coefficients that have sufficient regularity globally in space should still apply in the case of coefficients that have this regularity in space only locally, in a small neighborhood of the initial value. Our results apply to a huge variety of equations with coefficients that are not globally Lipschitz continuous in space including Cox–Ingersoll–Ross processes, equations with superlinearly growing coefficients, and equations with discontinuous coefficients. In many of these cases, the resulting lower error bounds even turn out to be sharp.

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KNUDSEN GAS IN FLAT TIRE

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We consider random reflections (according to the Lambertian distribution) of a light ray in a thin variable width (but almost circular) tube. As the width of the tube goes to zero, properly rescaled angular component of the light ray position converges in distribution to a diffusion whose parameters (diffusivity and drift) are given explicitly in terms of the tube width.

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Key words and phrases. Stochastic billiard, invariance principle, Knudsen random walk, cosine distribution.

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THE BOUCHAUD–ANDERSON MODEL WITH DOUBLE-EXPONENTIAL POTENTIAL

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The Bouchaud–Anderson model (BAM) is a generalisation of the parabolic Anderson model (PAM) in which the driving simple random walk is replaced by a random walk in an inhomogeneous trapping landscape; the BAM reduces to the PAM in the case of constant traps. In this paper, we study the BAM with double-exponential potential. We prove the complete localisation of the model whenever the distribution of the traps is unbounded. This may be contrasted with the case of constant traps (i.e., the PAM), for which it is known that complete localisation fails. This shows that the presence of an inhomogeneous trapping landscape may cause a system of branching particles to exhibit qualitatively distinct concentration behaviour.

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Key words and phrases. Parabolic Anderson model, Bouchaud trap model, intermittency, localisation.

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RANDOM SWITCHING BETWEEN VECTOR FIELDS HAVING A COMMON ZERO[1]

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Let E be a finite set, $\{F^i\}_{i\in E}$ a family of vector fields on $\mathbb{R}^d$ leaving positively invariant a compact set M and having a common zero $p \in M$. We consider a piecewise deterministic Markov process (X, I) on $M \times E$ defined by $\dot{X}_t = F^{I_t}(X_t)$ where I is a jump process controlled by X: $\mathsf{P}(I_{t+s} = j|(X_u, I_u)_{u\le t}) = a_{ij}(X_t)s + o(s)$ for $i \neq j$ on $\{I_t = i\}$.

We show that the behaviour of (X, I) is mainly determined by the behaviour of the linearized process (Y, J) where $\dot{Y}_t = A^{J_t} Y_t$, A^i is the Jacobian matrix of F^i at p and J is the jump process with rates $(a_{ij}(p))$. We introduce two quantities Λ^- and Λ^+, respectively, defined as the *minimal* (resp., *maximal*) *growth rate* of $\|Y_t\|$, where the minimum (resp., maximum) is taken over all the ergodic measures of the angular process (Θ, J) with $\Theta_t = \frac{Y_t}{\|Y_t\|}$. It is shown that Λ^+ coincides with the top Lyapunov exponent (in the sense of ergodic theory) of (Y, J) and that under general assumptions $\Lambda^- = \Lambda^+$. We then prove that, under certain irreducibility conditions, $X_t \to p$ exponentially fast when $\Lambda^+ < 0$ and (X, I) converges in distribution at an exponential rate toward a (unique) invariant measure supported by $M \setminus \{p\} \times E$ when $\Lambda^- > 0$. Some applications to certain epidemic models in a fluctuating environment are discussed and illustrate our results.

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Key words and phrases. Piecewise deterministic Markov processes, random switching, Lyapunov exponents, stochastic persistence, Hörmander-bracket conditions, epidemic models, SIS.

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MULTI-SCALE LIPSCHITZ PERCOLATION OF INCREASING EVENTS FOR POISSON RANDOM WALKS

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Consider the graph induced by $\mathbb{Z}^d$, equipped with *uniformly elliptic* random conductances. At time 0, place a Poisson point process of particles on $\mathbb{Z}^d$ and let them perform independent simple random walks. Tessellate the graph into cubes indexed by $i \in \mathbb{Z}^d$ and tessellate time into intervals indexed by τ. Given a local event $E(i, \tau)$ that depends only on the particles inside the space time region given by the cube i and the time interval τ, we prove the existence of a Lipschitz connected surface of *cells* (i, τ) that separates the origin from infinity on which $E(i, \tau)$ holds. This gives a directly applicable and robust framework for proving results in this setting that need a multi-scale argument. For example, this allows us to prove that an infection spreads with positive speed among the particles.

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Key words and phrases. Multi-scale percolation, Lipschitz surface, spread of infection.

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THEORETICAL PROPERTIES OF QUASI-STATIONARY MONTE CARLO METHODS

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This paper gives foundational results for the application of quasi-stationarity to Monte Carlo inference problems. We prove natural sufficient conditions for the quasi-limiting distribution of a killed diffusion to coincide with a target density of interest. We also quantify the rate of convergence to quasi-stationarity by relating the killed diffusion to an appropriate Langevin diffusion. As an example, we consider in detail a killed Ornstein–Uhlenbeck process with Gaussian quasi-stationary distribution.

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APPROXIMATION OF STABLE LAW IN WASSERSTEIN-1 DISTANCE BY STEIN'S METHOD[1]

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Let $n \in \mathbb{N}$, let $\zeta_{n,1}, \dots, \zeta_{n,n}$ be a sequence of independent random variables with $\mathbb{E}\zeta_{n,i} = 0$ and $\mathbb{E}|\zeta_{n,i}| < \infty$ for each i, and let μ be an α-stable distribution having characteristic function $e^{-|\lambda|^\alpha}$ with $\alpha \in (1, 2)$. Denote $S_n = \zeta_{n,1} + \cdots + \zeta_{n,n}$ and its distribution by $\mathcal{L}(S_n)$, we bound the Wasserstein-1 distance of $\mathcal{L}(S_n)$ and μ essentially by an L^1 discrepancy between two kernels. More precisely, we prove the following inequality:

$$d_W(\mathcal{L}(S_n), \mu) \le C\left[\sum_{i=1}^n \int_{-N}^N \left|\frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha}\right| \mathrm{d}t + \mathcal{R}_{N,n}\right],$$

where d_W is the Wasserstein-1 distance of probability measures, $\mathcal{K}_\alpha(t, N)$ is the kernel of a decomposition of the fractional Laplacian $\Delta^{\frac{\alpha}{2}}$, $K_i(t, N)$ is a K function (*Normal Approximation by Stein's Method* (2011) Springer) with a truncation and $\mathcal{R}_{N,n}$ is a small remainder. The integral term

$$\sum_{i=1}^n \int_{-N}^N \left|\frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha}\right| \mathrm{d}t$$

can be interpreted as an L^1 discrepancy.

As an application, we prove a general theorem of stable law convergence rate when $\zeta_{n,i}$ are i.i.d. and the distribution falls in the normal domain of attraction of μ. To test our results, we compare our convergence rates with those known in the literature for four given examples, among which the distribution in the fourth example is not in the normal domain of attraction of μ.

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CONTINUITY OF THE OPTIMAL STOPPING BOUNDARY FOR TWO-DIMENSIONAL DIFFUSIONS

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We first show that a smooth fit between the value function and the gain function at the optimal stopping boundary for a two-dimensional diffusion process implies the absence of boundary's discontinuities of the first kind (the right-hand and left-hand limits exist but differ). We then show that the smooth fit itself is satisfied over the flat portion of the optimal stopping boundary arising from any of its hypothesised jumps. Combining the two facts we obtain that the optimal stopping boundary is continuous whenever it has no discontinuities of the second kind. The derived fact holds both in the parabolic and elliptic case under the sole hypothesis of Hölder continuous coefficients, thus improving upon all known results in the parabolic case, and establishing the fact for the first time in the elliptic case. The method of proof relies upon regularity results for the second-order parabolic/elliptic PDEs and makes use of the local time-space calculus techniques.

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ROBUST HEDGING OF OPTIONS ON A LEVERAGED EXCHANGE TRADED FUND

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A leveraged exchange traded fund (LETF) is an exchange traded fund that uses financial derivatives to amplify the price changes of a basket of goods. In this paper, we consider the robust hedging of European options on a LETF, finding model-free bounds on the price of these options.

To obtain an upper bound, we establish a new optimal solution to the Skorokhod embedding problem (SEP) using methods introduced in Beiglböck–Cox–Huesmann. This stopping time can be represented as the hitting time of some region by a Brownian motion, but unlike other solutions of, for example, Root, this region is not unique. Much of this paper is dedicated to characterising the choice of the embedding region that gives the required optimality property. Notably, this appears to be the first solution to the SEP where the solution is not uniquely characterised by its geometric structure, and an additional condition is needed on the stopping region to guarantee that it is the optimiser. An important part of determining the optimal region is identifying the correct form of the dual solution, which has a financial interpretation as a model-independent superhedging strategy.

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EXPONENTIAL UTILITY MAXIMIZATION UNDER MODEL UNCERTAINTY FOR UNBOUNDED ENDOWMENTS

BY DANIEL BARTL[1]

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We consider the robust exponential utility maximization problem in discrete time: An investor maximizes the worst case expected exponential utility with respect to a family of nondominated probabilistic models of her endowment by dynamically investing in a financial market, and statically in available options.

We show that, for any measurable random endowment (regardless of whether the problem is finite or not) an optimal strategy exists, a dual representation in terms of (calibrated) martingale measures holds true, and that the problem satisfies the dynamic programming principle (in case of no options). Further, it is shown that the value of the utility maximization problem converges to the robust superhedging price as the risk aversion parameter gets large, and examples of nondominated probabilistic models are discussed.

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SERVE THE SHORTEST QUEUE AND WALSH BROWNIAN MOTION

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We study a single-server Markovian queueing model with N customer classes in which priority is given to the shortest queue. Under a critical load condition, we establish the diffusion limit of the nominal workload and queue length processes in the form of a Walsh Brownian motion (WBM) living in the union of the N nonnegative coordinate axes in $\mathbb{R}^N$ and a linear transformation thereof. This reveals the following asymptotic behavior. Each time that queues begin to build starting from an empty system, one of them becomes dominant in the sense that it contains nearly all the workload in the system, and it remains so until the system becomes (nearly) empty again. The radial part of the WBM, given as a reflected Brownian motion (RBM) on the half-line, captures the total workload asymptotics, whereas its angular distribution expresses how likely it is for each class to become dominant on excursions.

As a heavy traffic result, it is nonstandard in three ways: (i) In the terminology of Harrison (In *Stochastic Networks* (1995) 1–20 Springer), it is *unconventional*, in that the limit is not a RBM. (ii) It does not constitute an *invariance principle*, in that the limit law (specifically, the angular distribution) is not determined solely by the first two moments of the data, and is sensitive even to tie breaking rules. (iii) The proof method does not fully characterize the limit law (specifically, it gives no information on the angular distribution).

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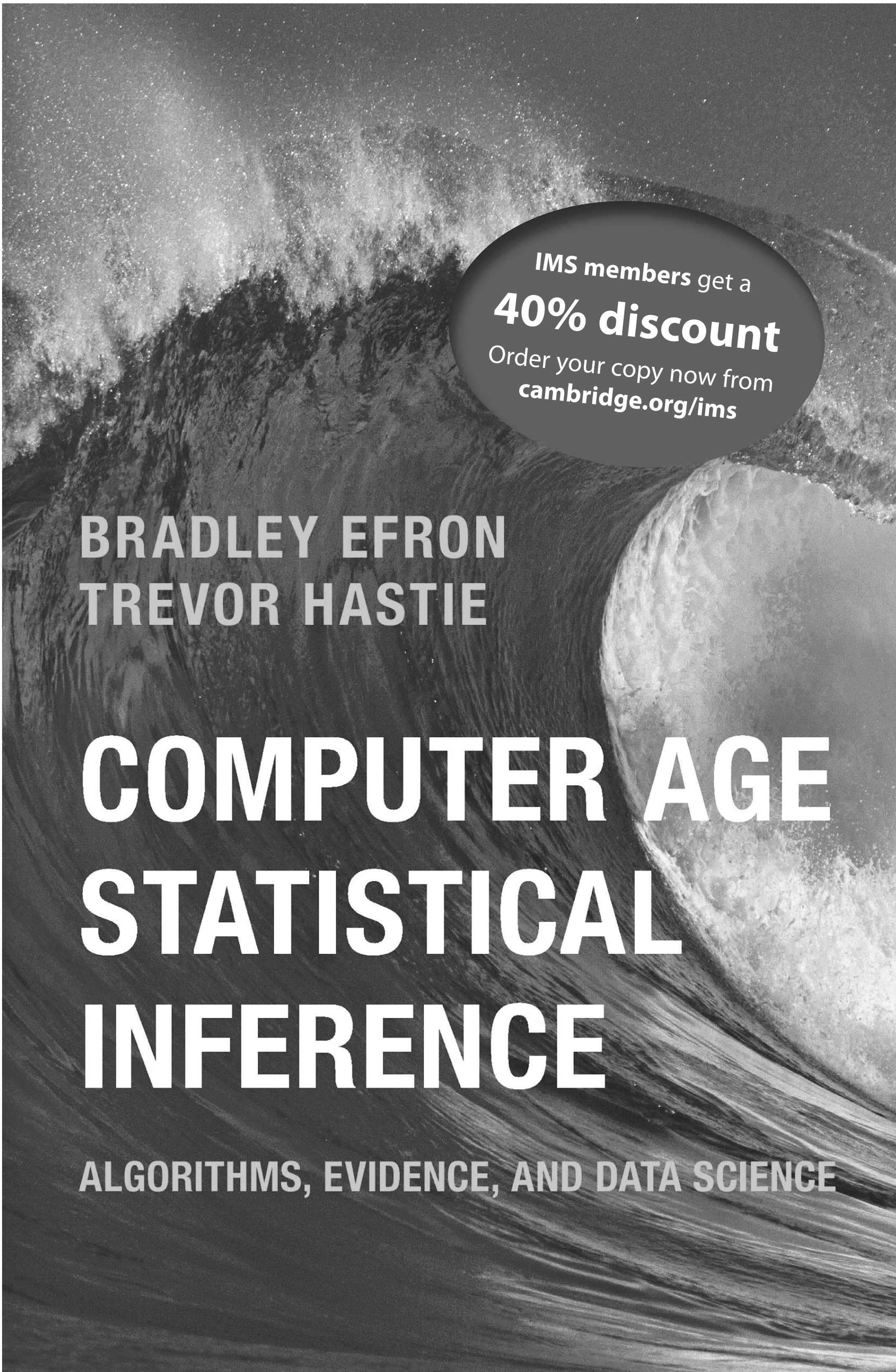
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