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THRESHOLDS FOR DETECTING AN ANOMALOUS PATH FROM NOISY ENVIRONMENTS

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We consider the "searching for a trail in a maze" composite hypothesis testing problem, in which one attempts to detect an anomalous directed path in a lattice 2D box of side n based on observations on the nodes of the box. Under the signal hypothesis, one observes independent Gaussian variables of unit variance at all nodes, with zero mean off the anomalous path and mean μ_n on it. Under the null hypothesis, one observes i.i.d. standard Gaussians on all nodes. Arias-Castro et al. [*Ann. Statist.* **36** (2008) 1726–1757] showed that if the unknown directed path under the signal hypothesis has known initial location, then detection is possible (in the minimax sense) if $\mu_n \gg 1/\sqrt{\log n}$, while it is not possible if $\mu_n \ll 1/\log n\sqrt{\log\log n}$. In this paper, we show that this result continues to hold even when the initial location of the unknown path is not known. As is the case with Arias-Castro et al. [*Ann. Statist.* **36** (2008) 1726–1757], the upper bound here also applies when the path is undirected. The improvement is achieved by replacing the linear detection statistic used in Arias-Castro et al. [*Ann. Statist.* **36** (2008) 1726–1757] with a polynomial statistic, which is obtained by employing a multiscale analysis on a quadratic statistic to bootstrap its performance. Our analysis is motivated by ideas developed in the context of the analysis of random polymers in Lacoin [*Comm. Math. Phys.* **294** (2010) 471–503].

REFERENCES

[1] ARIAS-CASTRO, E., CANDÈS, E. J., HELGASON, H. and ZEITOUNI, O. (2008). Searching for a trail of evidence in a maze. *Ann. Statist.* **36** 1726–1757. MR2435454
[2] BERGER, Q. and LACOIN, H. (2017). The high-temperature behavior for the directed polymer in dimension $1+2$. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 430–450. MR3606747
[3] GAMKRELIDZE, N. G. and ROTAR', V. I. (1978). The rate of convergence in a limit theorem for quadratic forms. *Theory Probab. Appl.* **22** 394–397. MR0443040
[4] GÖTZE, F. and TIKHOMIROV, A. N. (1999). Asymptotic distribution of quadratic forms. *Ann. Probab.* **27** 1072–1098. MR1699003
[5] LACOIN, H. (2010). New bounds for the free energy of directed polymers in dimension $1+1$ and $1+2$. *Comm. Math. Phys.* **294** 471–503. MR2579463
[6] ROTAR', V. I. (1974). Certain limit theorems for polynomials of degree two. *Theory Probab. Appl.* **18** 499–507. MR0326803
[7] ROTAR', V. I. and SHERVASHIDZE, T. L. (1986). Some estimates for distributions of quadratic forms. *Theory Probab. Appl.* **30** 585–590. MR0805309

MSC2010 subject classifications. Primary 62C20, 62G10; secondary 82B20, 60K37.
Key words and phrases. Detecting a chain of nodes in a network, minimax detection, random scenery.

[8] WHITTLE, P. (1960). Bounds for the moments of linear and quadratic forms in independent variables. *Theory Probab. Appl.* **5** 302–305. MR0133849

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OPTION PRICING WITH LINEAR MARKET IMPACT AND NONLINEAR BLACK–SCHOLES EQUATIONS

BY GREGOIRE LOEPER

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We consider a model of linear market impact, and address the problem of replicating a contingent claim in this framework. We derive a nonlinear Black–Scholes equation that provides an exact replication strategy.

This equation is fully nonlinear and singular, but we show that it is well posed, and we prove existence of smooth solutions for a large class of final payoffs, both for constant and local volatility. To obtain regularity of the solutions, we develop an original method based on Legendre transforms.

The close connections with the problem of hedging with *gamma constraints* [*SIAM J. Control Optim.* **39** (2000) 73–96, *Math. Finance* **17** (2007) 59–80, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005) 633–666], with the problem of hedging under *liquidity costs* [*Finance Stoch.* **14** (2010) 317–341] are discussed. The optimal strategy and associated diffusion are related with the *second-order target problems* of [*Ann. Appl. Probab.* **23** (2013) 308–347], and with the solutions of *optimal transport problems by diffusions* of [*Ann. Probab.* **41** (2013) 3201–3240].

We also derive a modified Black–Scholes formula valid for asymptotically small impact parameter, and finally provide numerical simulations as an illustration.

REFERENCES

[1] ABERGEL, F. and LOEPER, G. (2016). Option pricing and hedging with liquidity costs and market impact. In *Econophysics and Sociophysics*: *Recent Progress and Future Directions* (F. Abergel, H. Aoyama, B. K. Chakrabarti, A. Chakraborti, N. Deo, D. Raina and I. Vodenska, eds.) 19–40. Springer, Cham.

[2] ALMGREN, R. A. (2005). Equity market impact. *RISK* 57–62.

[3] BARLES, G. (1994). *Solutions de Viscosité des équations de Hamilton–Jacobi*. *Mathématiques & Applications* [*Mathematics & Applications*] **17**. Springer, Paris.

[4] BOUCHARD, B., LOEPER, G. and ZOU, Y. (2016). Almost-sure hedging with permanent price impact. *Finance Stoch*. **20** 741–771. MR3519167

[5] BOUCHARD, B., LOEPER, G. and ZOU, Y. (2017). Hedging of covered options with linear market impact and gamma constraint. *SIAM J. Control Optim*. **55** 3319–3348. MR3715374

[6] BRENIER, Y. (1991). Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math*. **44** 375–417. MR1100809

[7] ÇETIN, U., JARROW, R. A. and PROTTER, P. (2004). Liquidity risk and arbitrage pricing theory. *Finance Stoch*. **8** 311–341. MR2213255

MSC2010 subject classifications. 91G20, 93E20, 35K55, 49L20.
Key words and phrases. Hedging, price impact, fully nonlinear parabolic equations.

[8] ÇETIN, U., SONER, H. M. and TOUZI, N. (2010). Option hedging for small investors under liquidity costs. *Finance Stoch*. **14** 317–341. MR2670416

[9] CHERIDITO, P., METE SONER, H., TOUZI, N. and VICTOIR, N. (2007). Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. *Comm. Pure Appl. Math*. **60** 1081–1110.

[10] CHERIDITO, P., SONER, H. M. and TOUZI, N. (2005). The multi-dimensional super-replication problem under gamma constraints. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** 633–666. MR2171995

[11] CRANDALL, M. and PIERRE, M. (1982). Regularizing effects for $u_t + A\varphi(u) = 0$ in L^1. *J. Funct. Anal*. **45** 194–212.

[12] CRANDALL, M. G., ISHII, H. and LIONS, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc*. (*N.S.*) **27** 1–67. MR1118699

[13] CRANDALL, M. G. and PIERRE, M. (1982). Regularizing effects for $u_t = \Delta\varphi(u)$. *Trans. Amer. Math. Soc*. **274** 159–168. MR0670925

[14] FREY, R. Perfect option hedging for a large trader. *Finance Stoch*. **2** 115–141.

[15] FREY, R. and STREMME, A. (1997). Market volatility and feedback effects from dynamic hedging. *Math. Finance* **7** 351–374. MR1482708

[16] FRIEDMAN, A. (1964). *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs, NJ.

[17] FRIEDMAN, A. (1975). *Stochastic Differential Equations and Applications*, *Vol*. 1. *Probability and Mathematical Statistics* **28**. Academic Press, New York. MR0494490

[18] GILBARG, D. and TRUDINGER, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **224**. Springer, Berlin.

[19] HSU, E. P. (2002). *Stochastic Analysis on Manifolds*. *Graduate Studies in Mathematics* **38**. Amer. Math. Soc., Providence, RI.

[20] KANTOROVICH, L. V. (1948). On a problem of Monge. *Uspekhi Mat. Nauk* **3** 225–226.

[21] KANTOROVICH, L. V. (1942). On mass transportation. *Dokl. Akad. Nauk SSSR* **37** 227–229.

[22] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

[23] KOHN, R. V. and SERFATY, S. (2010). A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations. *Comm. Pure Appl. Math*. **63** 1298–1350.

[24] LAMBERTON, D., PHAM, H. and SCHWEIZER, M. (1998). Local risk-minimization under transaction costs. *Math. Oper. Res*. **23** 585–612. MR1653777

[25] LIEBERMAN, G. M. (1996). *Second Order Parabolic Differential Equations*. World Scientific Co., Inc., River Edge, NJ. MR1465184

[26] LIU, H. and YONG, J. M. (2005). Option pricing with an illiquid underlying asset market. *J. Econom. Dynam. Control* **29** 2125–2156.

[27] LOEPER, G. (2006). The reconstruction problem for the Euler–Poisson system in cosmology. *Arch. Ration. Mech. Anal*. **179** 153–216.

[28] PASCUCCI, A. (2011). *PDE and Martingale Methods in Option Pricing*, 1st ed. Springer, Milan. MR2791231

[29] PLATEN, E. and SCHWEIZER, M. (1998). On feedback effects from hedging derivatives. *Math. Finance* **8** 67–84.

[30] ROCKAFELLAR, R. T. (1997). *Convex Analysis*. Princeton Univ. Press, Princeton, NJ. Reprint of the 1970 original. MR1451876

[31] SCHÖNBUCHER, P. J. and WILMOTT, P. (2000). The feedback effect of hedging in illiquid markets. *SIAM J. Appl. Math*. **61** 232–272. MR1776395

[32] SONER, H. M. and TOUZI, N. (2000). Superreplication under gamma constraints. *SIAM J. Control Optim*. **39** 73–96.

[33] SONER, H. M. and TOUZI, N. (2002). Dynamic programming for stochastic target problems and geometric flows. *J. Eur. Math. Soc.* (*JEMS*) **4** 201–236.

[34] SONER, H. M. and TOUZI, N. (2007). Hedging under gamma constraints by optimal stopping and face-lifting. *Math. Finance* **17** 59–80.

[35] SONER, H. M., TOUZI, N. and ZHANG, J. (2013). Dual formulation of second order target problems. *Ann. Appl. Probab.* **23** 308–347. MR3059237

[36] TAN, X. and TOUZI, N. (2013). Optimal transportation under controlled stochastic dynamics. *Ann. Probab.* **41** 3201–3240. MR3127880

[37] TÓTH, B., EISLER, Z., LILLO, F., KOCKELKOREN, J., BOUCHAUD, J.-P. and FARMER, J. D. (2012). How does the market react to your order flow? *Quant. Finance* **12** 1015–1024. MR2949509

[38] TYCHONOFF, A. (1935). Théorèmes d'unicité pour l'équation de la chaleur. *Rec. Math. Moscou* **42** 199–216.

[39] VÁZQUEZ, J. L. (2006). Smoothing and decay estimates for nonlinear diffusion equations. In *Equations of Porous Medium Type*. *Oxford Lecture Series in Mathematics and Its Applications* **33**. Oxford Univ. Press, Oxford.

[40] VILLANI, C. (2003). *Topics in Optimal Transportation*. *Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483

[41] WANG, X.-J. (2006). Schauder estimates for elliptic and parabolic equations. *Chin. Ann. Math. Ser. B* **27** 637–642.

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ON THE GREEN–KUBO FORMULA AND THE GRADIENT CONDITION ON CURRENTS

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In the diffusive hydrodynamic limit for a symmetric interacting particle system (such as the exclusion process, the zero range process, the stochastic Ginzburg–Landau model, the energy exchange model), a possibly nonlinear diffusion equation is derived as the hydrodynamic equation. The bulk diffusion coefficient of the limiting equation is given by the Green–Kubo formula and it can be characterized by a variational formula. In the case the system satisfies the gradient condition, the variational problem is explicitly solved and the diffusion coefficient is given from the Green–Kubo formula through a static average only. In other words, the contribution of the dynamical part of the Green–Kubo formula is 0. In this paper, we consider the converse, namely if the contribution of the dynamical part of the Green–Kubo formula is 0, does it imply the system satisfies the gradient condition or not. We show that if the equilibrium measure μ is product and L^2 space of its single site marginal is separable, then the converse also holds. The result gives a new physical interpretation of the gradient condition.

As an application of the result, we consider a class of stochastic models for energy transport studied by Gaspard and Gilbert in [*J. Stat. Mech. Theory Exp.* **2008** (2008) P11021; *J. Stat. Mech. Theory Exp.* **2009** (2009) P08020], where the exact problem is discussed for this specific model.

REFERENCES

[1] GASPARD, P. and GILBERT, T. (2008). On the derivation of Fourier's law in stochastic energy exchange systems. *J. Stat. Mech. Theory Exp.* **2008** P11021.

[2] GASPARD, P. and GILBERT, T. (2009). Heat transport in stochastic energy exchange models of locally confined hard spheres. *J. Stat. Mech. Theory Exp.* **2009** P08020.

[3] GASPARD, P. and GILBERT, T. (2017). Dynamical contribution to the heat conductivity in stochastic energy exchanges of locally confined gases. *J. Stat. Mech. Theory Exp.* **2017** 043210. MR3652729

[4] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314

[5] LU, S. (1994). Equilibrium fluctuations of a one-dimensional nongradient Ginzburg–Landau model. *Ann. Probab.* **22** 1252–1272. MR1303644

[6] NAGAHATA, Y. (1998). The gradient condition for one-dimensional symmetric exclusion processes. *J. Stat. Phys.* **91** 587–602.

MSC2010 subject classifications. Primary 60K35; secondary 82C22.

Key words and phrases. Gradient condition, variational formula, diffusion coefficient, hydrodynamic limit.

[7] SASADA, M. (2016). Thermal conductivity for stochastic energy exchange models. Available at arXiv:1611.08866.

[8] SPOHN, H. (1991). *Large Scale Dynamics of Interacting Particles*. Springer, Berlin.

[9] VARADHAN, S. R. S. (1993). Nonlinear diffusion limit for a system with nearest neighbor interactions. II. In *Asymptotic Problems in Probability Theory*: *Stochastic Models and Diffusions on Fractals* (*Sanda/Kyoto*, 1990). *Pitman Res*. *Notes Math*. *Ser*. **283** 75–128. Longman Sci. Tech., Harlow. MR1354152

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LIMIT THEOREMS FOR PERSISTENCE DIAGRAMS[1]

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The persistent homology of a stationary point process on $\mathbf{R}^N$ is studied in this paper. As a generalization of continuum percolation theory, we study higher dimensional topological features of the point process such as loops, cavities, etc. in a multiscale way. The key ingredient is the persistence diagram, which is an expression of the persistent homology. We prove the strong law of large numbers for persistence diagrams as the window size tends to infinity and give a sufficient condition for the support of the limiting persistence diagram to coincide with the geometrically realizable region. We also discuss a central limit theorem for persistent Betti numbers.

REFERENCES

[1] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1700749

[2] BŁASZCZYSZYN, B. and YOGESHWARAN, D. (2015). Clustering comparison of point processes, with applications to random geometric models. In *Stochastic Geometry*, *Spatial Statistics and Random Fields*. *Lecture Notes in Math*. **2120** 31–71. Springer, Cham. MR3330572

[3] BOBROWSKI, O. and KAHLE, M. (2014). Topology of random simplicial complexes: A survey. Topology in Statistical Inference, the Proceedings of Symposia in Applied Mathematics. To appear. Available at arXiv:1409.4734.

[4] CARLSSON, G. (2009). Topology and data. *Bull*. *Amer. Math*. *Soc*. (*N.S.*) **46** 255–308. MR2476414

[5] CHAZAL, F., DE SILVA, V., GLISSE, M. and OUDOT, S. (2016). *The Structure and Stability of Persistence Modules*. *SpringerBriefs in Mathematics*. Springer, Cham. MR3524869

[6] CHAZAL, F., DE SILVA, V. and OUDOT, S. (2014). Persistence stability for geometric complexes. *Geom*. *Dedicata* **173** 193–214. MR3275299

[7] CHIU, S. N., STOYAN, D., KENDALL, W. S. and MECKE, J. (2013). *Stochastic Geometry and Its Applications*, 3rd ed. *Wiley Series in Probability and Statistics*. Wiley, Chichester. MR3236788

[8] COHEN-STEINER, D., EDELSBRUNNER, H. and HARER, J. (2007). Stability of persistence diagrams. *Discrete Comput*. *Geom*. **37** 103–120. MR2279866

[9] EDELSBRUNNER, H. and HARER, J. L. (2010). *Computational Topology*. Amer. Math. Soc., Providence, RI. MR2572029

[10] EDELSBRUNNER, H., LETSCHER, D. and ZOMORODIAN, A. (2002). Topological persistence and simplification. *Discrete Comput*. *Geom*. **28** 511–533. MR1949898

MSC2010 subject classifications. Primary 60K35, 60B10; secondary 55N20.

Key words and phrases. Point process, persistence diagram, persistent Betti number, random topology.

[11] GAMEIRO, M., HIRAOKA, Y., IZUMI, S., KRAMAR, M., MISCHAIKOW, K. and NANDA, V. (2015). A topological measurement of protein compressibility. *Jpn. J. Ind. Appl. Math.* **32** 1–17. MR3318898

[12] GHOSH, S. (2012). Rigidity and Tolerance in Gaussian zeroes and Ginibre eigenvalues: Quantitative estimates. Preprint. Available at arXiv:1211.3506.

[13] GHOSH, S. and PERES, Y. (2017). Rigidity and tolerance in point processes: Gaussian zeros and Ginibre eigenvalues. *Duke Math. J.* **166** 1789–1858. MR3679882

[14] GRESCHONIG, G. and SCHMIDT, K. (2000). Ergodic decomposition of quasi-invariant probability measures. *Colloq. Math.* **84/85** 495–514. MR1784210

[15] HATCHER, A. (2002). *Algebraic Topology*. Cambridge Univ. Press, Cambridge. MR1867354

[16] HIRAOKA, Y., NAKAMURA, T., HIRATA, A., ESCOLAR, E. G., MATSUE, K. and NISHIURA, Y. (2016). Hierarchical structures of amorphous solids characterized by persistent homology. *Proc. Natl. Acad. Sci. USA* **113** 7035–7040.

[17] HIRAOKA, Y. and SHIRAI, T. (2016). Tutte polynomials and random-cluster models in Bernoulli cell complexes. In *Stochastic Analysis on Large Scale Interacting Systems*. *RIMS Kôkyûroku Bessatsu*, *B*59 289–304. Res. Inst. Math. Sci. (RIMS), Kyoto. MR3675939

[18] HIRAOKA, Y. and SHIRAI, T. (2017). Minimum spanning acycle and lifetime of persistent homology in the Linial–Meshulam process. *Random Structures Algorithms* **51** 315–340. MR3683365

[19] HIRAOKA, Y. and TSUNODA, K. (2016). Limit theorems for random cubical homology. Preprint. Available at arXiv:1612.08485.

[20] HOUGH, J. B., KRISHNAPUR, M., PERES, Y. and VIRÁG, B. (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. *University Lecture Series* **51**. Amer. Math. Soc., Providence, RI. MR2552864

[21] KACZYNSKI, T., MISCHAIKOW, K. and MROZEK, M. (2004). *Computational Homology*. *Applied Mathematical Sciences* **157**. Springer, New York. MR2028588

[22] KAHLE, M. (2011). Random geometric complexes. *Discrete Comput. Geom.* **45** 553–573. MR2770552

[23] KAHLE, M. (2014). Topology of random simplicial complexes: A survey. In *Algebraic Topology: Applications and New Directions*. *Contemp. Math.* **620** 201–221. Amer. Math. Soc., Providence, RI. MR3290093

[24] KALLENBERG, O. (1986). *Random Measures*, 4th ed. Akademie-Verlag, Berlin. MR0854102

[25] KARAMCHANDANI, N., MANJUNATH, D., YOGESHWARAN, D. and IYER, S. K. (2006). Evolving random geometric graph models for mobile wireless networks. In 2006 4*th International Symposium on Modeling and Optimization in Mobile*, *Ad Hoc and Wireless Networks* 1–7. IEEE.

[26] KUSANO, G., FUKUMIZU, K. and HIRAOKA, Y. (2016). Persistence weighted Gaussian kernel for topological data analysis. In *Proceedings of the* 33*rd International Conference on Machine Learning* 2004–2013.

[27] MIYOSHI, N. and SHIRAI, T. (2014). A cellular network model with Ginibre configured base stations. *Adv. in Appl. Probab.* **46** 832–845. MR3254344

[28] NAKAMURA, T., HIRAOKA, Y., HIRATA, A., ESCOLAR, E. G. and NISHIURA, Y. (2015). Persistent homology and many-body atomic structure for medium-range order in the glass. *Nanotechnol.* **26** 304001.

[29] OWADA, T. and ADLER, R. J. (2017). Limit theorems for point processes under geometric constraints (and topological crackle). *Ann. Probab.* **45** 2004–2055. DOI:10.1214/16-AOP1106. MR3650420

[30] PENROSE, M. (2003). *Random Geometric Graphs*. *Oxford Studies in Probability* **5**. Oxford Univ. Press, Oxford. MR1986198

[31] PENROSE, M. D. and YUKICH, J. E. (2001). Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.* **11** 1005–1041. MR1878288

[32] PUGH, C. and SHUB, M. (1971). Ergodic elements of ergodic actions. *Compos. Math.* **23** 115–122. MR0283174

[33] REED, M. and SIMON, B. (1978). *Methods of Modern Mathematical Physics. IV. Analysis of Operators*. Academic Press, New York. MR0493421

[34] RILEY, S., EAMES, K., ISHAM, V., MOLLISON, D. and TRAPMAN, P. (2015). Five challenges for spatial epidemic models. *Epidemics* **10** 68–71.

[35] SAADATFAR, M., TAKEUCHI, H., ROBINS, V., FRANCOIS, N. and HIRAOKA, Y. (2017). Pore configuration landscape of granular crystallization. *Nat. Commun.* **8** 15082.

[36] SHIRAI, T. and TAKAHASHI, Y. (2003). Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes. *J. Funct. Anal.* **205** 414–463. DOI:10.1016/S0022-1236(03)00171-X. MR2018415

[37] TRINH, K. D. (2017). A remark on the convergence of Betti numbers in the thermodynamic regime. *Pac. J. Math. Ind.* **9** Art. 4, 7. DOI:10.1186/s40736-017-0029-0. MR3620264

[38] YOGESHWARAN, D. and ADLER, R. J. (2015). On the topology of random complexes built over stationary point processes. *Ann. Appl. Probab.* **25** 3338–3380. MR3404638

[39] YOGESHWARAN, D., SUBAG, E. and ADLER, R. J. (2017). Random geometric complexes in the thermodynamic regime. *Probab. Theory Related Fields* **167** 107–142. MR3602843

[40] ZOMORODIAN, A. and CARLSSON, G. (2005). Computing persistent homology. *Discrete Comput. Geom.* **33** 249–274. MR2121296

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MODERATE DEVIATION FOR RANDOM ELLIPTIC PDE WITH SMALL NOISE

BY XIAOOU LI[*,1], JINGCHEN LIU[†,2], JIANFENG LU[‡,3] AND XIANG ZHOU[§,4]

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Partial differential equations with random inputs have become popular models to characterize physical systems with uncertainty coming from imprecise measurement and intrinsic randomness. In this paper, we perform asymptotic rare-event analysis for such elliptic PDEs with random inputs. In particular, we consider the asymptotic regime that the noise level converges to zero suggesting that the system uncertainty is low, but does exist. We develop sharp approximations of the probability of a large class of rare events.

REFERENCES

ADLER, R. J. (1981). *The Geometry of Random Fields*. Wiley, Chichester. MR0611857

ADLER, R. J., BLANCHET, J. H. and LIU, J. C. (2008). Efficient simulation for tail probabilities of Gaussian random fields. In *Proceeding of Winter Simulation Conference*.

ADLER, R. J., BLANCHET, J. H. and LIU, J. (2012). Efficient Monte Carlo for high excursions of Gaussian random fields. *Ann. Appl. Probab.* **22** 1167–1214. MR2977989

ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. Springer, New York. MR2319516

ARMSTRONG, S., KUUSI, T. and MOURRAT, J.-C. (2017). Quantitative stochastic homogenization and large-scale regularity. ArXiv Preprint ArXiv:1705.05300.

AZAÏ S, J.-M. and WSCHEBOR, M. (2008). A general expression for the distribution of the maximum of a Gaussian field and the approximation of the tail. *Stochastic Process. Appl.* **118** 1190–1218. MR2428714

AZAÏS, J. M. and WSCHEBOR, M. (2009). *Level Sets and Extrema of Random Processes and Fields*. Wiley, Hoboken, NJ.

BERGLUND, N., DI GESU, G. and WEBER, H. (2017). An Eyring–Kramers law for the stochastic Allen–Cahn equation in dimension two. *Electron. J. Probab.* **22**.

BERMAN, S. M. (1985). An asymptotic formula for the distribution of the maximum of a Gaussian process with stationary increments. *J. Appl. Probab.* **22** 454–460.

BORELL, C. (1975). The Brunn–Minkowski inequality in Gauss space. *Invent. Math.* **30** 207–216. MR0399402

BORELL, C. (2003). The Ehrhard inequality. *C. R. Math. Acad. Sci. Paris* **337** 663–666. MR2030108

CHARBENEAU, R. J. (2000). *Groundwater Hydraulics and Pollutant Transport*. Prentice Hall.

CIREL'SON, B. S., IBRAGIMOV, I. A. and SUDAKOV, V. N. (1976). Norms of Gaussian sample functions. In *Proceedings of the Third Japan–USSR Symposium on Probability Theory* (G. Maruyama and J. V. Prokhorov, eds.) 20–41. Springer, Berlin.

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CIREL'SON, B. S., IBRAGIMOV, I. A. and SUDAKOV, V. N. (1976). Norms of Gaussian sample functions. In *Proceedings of the Third Japan–USSR Symposium on Probability Theory* (*Tashkent*, 1975) *Lecture Notes in Math*. **550** 20–41. Springer, Berlin. MR0458556

FREEZE, R. A. (1975). A stochastic-conceptual analysis of one-dimensional groundwater flow in nonuniform homogeneous media. *Water Resour. Res*. **11**.

GILBARG, D. and TRUDINGER, N. S. (2015). *Elliptic Partial Differential Equations of Second Order*. Springer.

LANDAU, H. J. and SHEPP, L. A. (1970). On the supremum of a Gaussian process. *Sankhya A* **32** 369–378. MR0286167

LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces*: *Isoperimetry and Processes*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete* 3. *Folge* **23**. Springer, Berlin.

LI, X., LIU, J. and XU, G. (2016). On the tail probabilities of aggregated lognormal random fields with small noise. *Math*. *Oper. Res*. **41** 236–246. MR3465750

LIU, J. (2012). Tail approximations of integrals of Gaussian random fields. *Ann*. *Probab.* **40** 1069–1104. MR2962087

LIU, J., LU, J. and ZHOU, X. (2015). Efficient rare event simulation for failure problems in random media. *SIAM J*. *Sci*. *Comput*. **37** A609–A624.

LIU, J. and XU, G. (2012). Some asymptotic results of Gaussian random fields with varying mean functions and the associated processes. *Ann*. *Statist*. **40** 262–293.

LIU, J. and ZHOU, X. (2013). On the failure probability of one dimensional random material under delta external force. *Commun*. *Math*. *Sci*. **11** 499–521. MR3002562

LIU, J. and ZHOU, X. (2014). Extreme analysis of a random ordinary differential equation. *J. Appl*. *Probab.* **51** 1021–1036.

MARCUS, M. B. and SHEPP, L. A. (1970). Continuity of Gaussian processes. *Trans*. *Amer. Math*. *Soc*. **151** 377–391.

PITERBARG, V. I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields*. Amer. Math. Soc., Providence, RI.

SUDAKOV, V. N. and TSIRELSON, B. S. (1974). Extremal properties of half spaces for spherically invariant measures. *Zap*. *Nauchn*. *Sem*. *LOMI* **45** 75–82.

SUN, J. Y. (1993). Tail probabilities of the maxima of Gaussian random-fields. *Ann*. *Probab*. **21** 34–71.

TALAGRAND, M. (1996). Majorizing measures: The generic chaining. *Ann*. *Probab.* **24** 1049–1103. MR1411488

TAYLOR, J. E. and ADLER, R. J. (2003). Euler characteristics for Gaussian fields on manifolds. *Ann*. *Probab.* **31** 533–563. MR1964940

TAYLOR, J., TAKEMURA, A. and ADLER, R. J. (2005). Validity of the expected Euler characteristic heuristic. *Ann*. *Probab.* **33** 1362–1396.

XU, G., LIN, G. and LIU, J. (2014). Rare-event simulation for the stochastic Korteweg–de Vries equation. *SIAM/ASA J. Uncertain*. *Quantificat*. **2** 698–716.

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LIMIT THEOREMS FOR BETTI NUMBERS OF EXTREME SAMPLE CLOUDS WITH APPLICATION TO PERSISTENCE BARCODES[1]

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We investigate the topological dynamics of extreme sample clouds generated by a heavy tail distribution on $\mathbb{R}^d$ by establishing various limit theorems for Betti numbers, a basic quantifier of algebraic topology. It then turns out that the growth rate of the Betti numbers and the properties of the limiting processes all depend on the distance of the region of interest from the weak core, that is, the area in which random points are placed sufficiently densely to connect with one another. If the region of interest becomes sufficiently close to the weak core, the limiting process involves a new class of Gaussian processes. We also derive the limit theorems for the sum of bar lengths in the persistence barcode plot, a graphical descriptor of persistent homology.

REFERENCES

[1] ADLER, R. J., BOBROWSKI, O., BORMAN, M. S., SUBAG, E. and WEINBERGER, S. (2010). Persistent homology for random fields and complexes. In *Borrowing Strength*: *Theory Powering Applications—a Festschrift for Lawrence D*. *Brown*. *Inst*. *Math*. *Stat*. (*IMS*) *Collect*. **6** 124–143. IMS, Beachwood, OH. MR2798515

[2] ADLER, R. J., BOBROWSKI, O. and WEINBERGER, S. (2014). Crackle: The homology of noise. *Discrete Comput*. *Geom*. **52** 680–704. MR3279544

[3] ARRATIA, R., GOLDSTEIN, L. and GORDON, L. (1989). Two moments suffice for Poisson approximations: The Chen–Stein method. *Ann*. *Probab*. **17** 9–25. MR0972770

[4] BALKEMA, G. and EMBRECHTS, P. (2007). *High Risk Scenarios and Extremes*: *A Geometric Approach*. European Mathematical Society (EMS), Zürich. MR2372552

[5] BALKEMA, G., EMBRECHTS, P. and NOLDE, N. (2010). Meta densities and the shape of their sample clouds. *J*. *Multivariate Anal*. **101** 1738–1754. MR2610743

[6] BALKEMA, G., EMBRECHTS, P. and NOLDE, N. (2013). The shape of asymptotic dependence. In *Prokhorov and Contemporary Probability Theory*. *Springer Proc*. *Math*. *Stat*. **33** 43–67. Springer, Heidelberg. MR3070466

[7] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[8] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[9] BJÖRNER, A. (1995). Topological methods. In *Handbook of Combinatorics*, *Vols* 1, 2 1819–1872. Elsevier, Amsterdam. MR1373690

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Key words and phrases. Extreme value theory, random topology, persistent homology, Betti number, central limit theorem, Poisson limit theorem.

[10] BOBROWSKI, O. and KAHLE, M. (2014). Topology of random simplicial complexes: A survey. In *Algebraic Topology*: *Applications and New Directions*. *Contemp*. *Math*. **620** 201–221. Amer. Math. Soc., Providence, RI. MR3290093

[11] BOBROWSKI, O., KAHLE, M. and SKRABA, P. (2017). Maximally persistent cycles in random geometric complexes. *Ann*. *Appl*. *Probab*. **27** 2032–2060. MR3693519

[12] BOBROWSKI, O. and MUKHERJEE, S. (2015). The topology of probability distributions on manifolds. *Probab*. *Theory Related Fields* **161** 651–686. MR3334278

[13] BUBENIK, P. (2015). Statistical topological data analysis using persistence landscapes. *J*. *Mach*. *Learn*. *Res*. **16** 77–102. MR3317230

[14] CARLSSON, G. (2014). Topological pattern recognition for point cloud data. *Acta Numer*. **23** 289–368. MR3202240

[15] COLES, S. (2001). *An Introduction to Statistical Modeling of Extreme Values*. Springer, London. MR1932132

[16] DABAGHIAN, Y., MEMOLI, F., FRANK, L. and CARLSSON, G. (2012). A topological paradigm for hippocampal spatial map formation using persistent homology. *PLoS Comput*. *Biol*. **8** e1002581.

[17] DABROWSKI, A. R., DEHLING, H. G., MIKOSCH, T. and SHARIPOV, O. (2002). Poisson limits for U-statistics. *Stochastic Process*. *Appl*. **99** 137–157. MR1894255

[18] DE HAAN, L. and FERREIRA, A. (2006). *Extreme Value Theory*: *An Introduction*. Springer, New York. MR2234156

[19] DE SILVA, V. and GHRIST, R. (2007). Coverage in sensor networks via persistent homology. *Algebr*. *Geom*. *Topol*. **7** 339–358. MR2308949

[20] DECREUSEFOND, L., SCHULTE, M. and THÄLE, C. (2016). Functional Poisson approximation in Kantorovich–Rubinstein distance with applications to U-statistics and stochastic geometry. *Ann*. *Probab*. **44** 2147–2197. MR3502603

[21] DUY, T. K., HIRAOKA, Y. and SHIRAI, T. (2016). Limit theorems for persistence diagrams. Available at arXiv:1612.08371.

[22] EDELSBRUNNER, H. and HARER, J. L. (2010). *Computational Topology*: *An Introduction*. Amer. Math. Soc., Providence, RI. MR2572029

[23] EDELSBRUNNER, H., LETSCHER, D. and ZOMORODIAN, A. (2002). Topological persistence and simplification. *Discrete Comput*. *Geom*. **28** 511–533. MR1949898

[24] EMBRECHTS, P., KLÜPPELBERG, C. and MIKOSCH, T. (1997). *Modelling Extremal Events*: *For Insurance and Finance*. *Applications of Mathematics* (*New York*) **33**. Springer, Berlin. MR1458613

[25] FASY, B. T., LECCI, F., RINALDO, A., WASSERMAN, L., BALAKRISHNAN, S. and SINGH, A. (2014). Confidence sets for persistence diagrams. *Ann*. *Statist*. **42** 2301–2339. MR3269981

[26] GALAMBOS, J. (1987). *The Asymptotic Theory of Extreme Order Statistics*, 2nd ed. Krieger, Melbourne, FL. MR0936631

[27] GHRIST, R. (2008). Barcodes: The persistent topology of data. *Bull*. *Amer*. *Math*. *Soc*. (*N.S.*) **45** 61–75. MR2358377

[28] GHRIST, R. (2014). *Elementary Applied Topology*, Createspace.

[29] HATCHER, A. (2002). *Algebraic Topology*. Cambridge Univ. Press, Cambridge. MR1867354

[30] HIRAOKA, Y. and SHIRAI, T. (2017). Minimum spanning acycle and lifetime of persistent homology in the Linial–Meshulam process. *Random Structures Algorithms* **51** 315–340. MR3683365

[31] KAHLE, M. (2011). Random geometric complexes. *Discrete Comput*. *Geom*. **45** 553–573. MR2770552

[32] KAHLE, M. and MECKES, E. (2013). Limit theorems for Betti numbers of random simplicial complexes. *Homology*, *Homotopy Appl*. **15** 343–374. MR3079211

[33] KAHLE, M. and MECKES, E. (2016). Erratum to "Limit theorems for Betti numbers of random simplicial complexes." *Homology*, *Homotopy Appl*. **18** 129–142. MR3485340

[34] KUSANO, G., FUKUMIZU, K. and HIRAOKA, Y. (2016). Persistence weighted Gaussian kernel for topological data analysis. In *ICML'*16 *Proceedings of the* 33*rd International Conference on International Conference on Machine Learning*, Vol. **48**, pp. 2004–2013.

[35] LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York–Berlin. MR0691492

[36] MARTIN, S., THOMPSON, A., COUTSIAS, E. A. and WATSON, J. (2010). Topology of cyclo-octane energy landscape. *J*. *Chem*. *Phys*. **132** 234115.

[37] MUNKRES, J. R. (1984). *Elements of Algebraic Topology*, 1st ed. Benjamin-Cummings, San Francisco, CA.

[38] NIYOGI, P., SMALE, S. and WEINBERGER, S. (2008). Finding the homology of submanifolds with high confidence from random samples. *Discrete Comput*. *Geom*. **39** 419–441. MR2383768

[39] OWADA, T. (2017). Functional central limit theorem for subgraph counting processes. *Electron*. *J*. *Probab*. **22** Paper No. 17, 38. MR3622887

[40] OWADA, T. and ADLER, R. J. (2017). Limit theorems for point processes under geometric constraints (and topological crackle). *Ann*. *Probab*. **45** 2004–2055. MR3650420

[41] PENROSE, M. (2003). *Random Geometric Graphs*. *Oxford Studies in Probability* **5**. Oxford Univ. Press, Oxford. MR1986198

[42] PORT, A., GHEORGHITA, I., GUTH, D., CLARK, J. M., LIANG, C., DASU, S. and MARCOLLI, M. (2015). Persistent topology of syntax. Available at arXiv:1507.05134.

[43] RESNICK, S. I. (1987). *Extreme Values*, *Regular Variation*, *and Point Processes*. *Applied Probability. A Series of the Applied Probability Trust* **4**. Springer, New York. MR0900810

[44] RESNICK, S. I. (2007). *Heavy-Tail Phenomena*: *Probabilistic and Statistical Modeling*. Springer, New York. MR2271424

[45] SCHULTE, M. and THÄLE, C. (2012). The scaling limit of Poisson-driven order statistics with applications in geometric probability. *Stochastic Process*. *Appl*. **122** 4096–4120. MR2971726

[46] VICK, J. W. (1994). *Homology Theory*: *An Introduction to Algebraic Topology*, 2nd ed. *Graduate Texts in Mathematics* **145**. Springer, New York. MR1254439

[47] YOGESHWARAN, D. and ADLER, R. J. (2015). On the topology of random complexes built over stationary point processes. *Ann*. *Appl*. *Probab*. **25** 3338–3380. MR3404638

[48] YOGESHWARAN, D., SUBAG, E. and ADLER, R. J. (2017). Random geometric complexes in the thermodynamic regime. *Probab. Theory Related Fields* **167** 107–142. MR3602843

[49] ZOMORODIAN, A. and CARLSSON, G. (2005). Computing persistent homology. *Discrete Comput*. *Geom*. **33** 249–274. MR2121296

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AN APPROXIMATION RESULT FOR A CLASS OF STOCHASTIC HEAT EQUATIONS WITH COLORED NOISE

BY MOHAMMUD FOONDUN, MATHEW JOSEPH[1] AND SHIU-TANG LI

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We show that a large class of stochastic heat equations can be approximated by systems of interacting stochastic differential equations. As a consequence, we prove various comparison principles extending earlier works of [*Stoch. Stoch. Rep.* **37** (1991) 225–245] and [*Ann. Probab.* **45** (2017) 377–403] among others. Among other things, our results enable us to obtain sharp estimates on the moments of the solution. A main technical ingredient of our method is a local limit theorem which is of independent interest.

REFERENCES

[1] BERTINI, L. and CANCRINI, N. (1995). The stochastic heat equation: Feynman-Kac formula and intermittence. *J. Stat. Phys.* **78** 1377–1401. MR1316109

[2] BOULANBA, L., EDDAHBI, M. and MELLOUK, M. (2010). Fractional SPDEs driven by spatially correlated noise: Existence of the solution and smoothness of its density. *Osaka J. Math.* **47** 41–65. MR2666124

[3] CARMONA, R. A. and MOLCHANOV, S. A. (1994). Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.* **108** viii+125. MR1185878

[4] CHEN, L. and DALANG, R. C. (2015). Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *Ann. Probab.* **43** 3006–3051. MR3433576

[5] CHEN, L. and KIM, K. (2017). On comparison principle and strict positivity of solutions to the nonlinear stochastic fractional heat equations. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 358–388. MR3606745

[6] CHEN, X. (2015). Precise intermittency for the parabolic Anderson equation with an $(1+1)$-dimensional time-space white noise. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 1486–1499. MR3414455

[7] CHEN, X. (2016). Spatial asymptotics for the parabolic Anderson models with generalized time-space Gaussian noise. *Ann. Probab.* **44** 1535–1598. MR3474477

[8] CONUS, D., JOSEPH, M. and KHOSHNEVISAN, D. (2013). On the chaotic character of the stochastic heat equation, before the onset of intermitttency. *Ann. Probab.* **41** 2225–2260. MR3098071

[9] CONUS, D., JOSEPH, M., KHOSHNEVISAN, D. and SHIU, S.-Y. (2013). On the chaotic character of the stochastic heat equation, II. *Probab. Theory Related Fields* **156** 483–533. MR3078278

[10] COX, J. T., FLEISCHMANN, K. and GREVEN, A. (1996). Comparison of interacting diffusions and an application to their ergodic theory. *Probab. Theory Related Fields* **105** 513–528. MR1402655

[11] DALANG, R., KHOSHNEVISAN, D., MUELLER, C., NUALART, D. and XIAO, Y. (2009). *A Minicourse on Stochastic Partial Differential Equations*. *Lecture Notes in Math.* **1962**. Springer, Berlin. MR1500166

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[12] DALANG, R. C. (1999). Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron*. *J*. *Probab*. **4** no. 6, 29 pp. MR1684157

[13] DAVAR, K. (2011). *Topics in Probability*: *Lévy Processes*. Springer, New York.

[14] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, *Vol*. *II*, 2nd ed. Wiley, New York. MR0270403

[15] FOONDUN, M. and KHOSHNEVISAN, D. (2009). Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron*. *J*. *Probab*. **14** 548–568. MR2480553

[16] FOONDUN, M. and KHOSHNEVISAN, D. (2013). On the stochastic heat equation with spatially-colored random forcing. *Trans*. *Amer*. *Math*. *Soc*. **365** 409–458. MR2984063

[17] FUNAKI, T. (1983). Random motion of strings and related stochastic evolution equations. *Nagoya Math*. *J*. **89** 129–193. MR0692348

[18] GEISS, C. and MANTHEY, R. (1994). Comparison theorems for stochastic differential equations in finite and infinite dimensions. *Stochastic Process*. *Appl*. **53** 23–35. MR1290705

[19] GEORGIOU, N., JOSEPH, M., KHOSHNEVISAN, D. and SHIU, S.-Y. (2015). Semi-discrete semi-linear parabolic SPDEs. *Ann*. *Appl*. *Probab*. **25** 2959–3006. MR3375892

[20] GRAFAKOS, L. (2008). *Classical Fourier Analysis*, 2nd ed. *Graduate Texts in Mathematics* **249**. Springer, New York. MR2445437

[21] HAIRER, M. (2013). Solving the KPZ equation. *Ann*. *of Math*. (2) **178** 559–664. MR3071506

[22] HUANG, J., LÊ, K. and NUALART, D. (2017). Large time asymptotics for the parabolic Anderson model driven by spatially correlated noise. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **53** 1305–1340. MR3689969

[23] JOSEPH, M., KHOSHNEVISAN, D. and MUELLER, C. (2017). Strong invariance and noise-comparison principles for some parabolic stochastic PDEs. *Ann*. *Probab*. **45** 377–403. MR3601652

[24] KIM, K. On the large-scale structure of the tall peaks for stochastic heat equations with fractional Laplacian. Preprint.

[25] KOLOKOLTSOV, V. (2000). Symmetric stable laws and stable-like jump-diffusions. *Proc*. *Lond*. *Math*. *Soc*. (3) **80** 725–768. MR1744782

[26] LI, S.-T. (2016). Comparison principles of solutions to stochastic heat equations. Ph.D. thesis, Univ. Utah, Salt Lake City, UT.

[27] MANTHEY, R. and ZAUSINGER, T. (1999). Stochastic evolution equations in $L^{2\nu}_{\rho}$. *Stoch*. *Stoch*. *Rep*. **66** 37–85. MR1687799

[28] MUELLER, C. (1991). On the support of solutions to the heat equation with noise. *Stoch*. *Stoch*. *Rep*. **37** 225–245. MR1149348

[29] SAMORODNITSKY, G. and TAQQU, M. S. (1994). *Stable Non-Gaussian Random Processes*: *Stochastic Models with Infinite Variance*. *Stochastic Modeling*. Chapman & Hall, New York. MR1280932

[30] SHIGA, T. and SHIMIZU, A. (1980). Infinite-dimensional stochastic differential equations and their applications. *J*. *Math*. *Kyoto Univ*. **20** 395–416. MR0591802

[31] STEIN, E. M. (1993). *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality*, *and Oscillatory Integrals*. *Princeton Mathematical Series* **43**. Princeton Univ. Press, Princeton, NJ. MR1232192

[32] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour*, *XIV*—1984. *Lecture Notes in Math*. **1180** 265–439. Springer, Berlin. MR0876085

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A SIMPLE EVOLUTIONARY GAME ARISING FROM THE STUDY OF THE ROLE OF IGF-II IN PANCREATIC CANCER

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We study an evolutionary game in which a producer at x gives birth at rate 1 to an offspring sent to a randomly chosen point in $x + \mathcal{N}_c$, while a cheater at x gives birth at rate $\lambda > 1$ times the fraction of producers in $x + \mathcal{N}_d$ and sends its offspring to a randomly chosen point in $x + \mathcal{N}_c$. We first study this game on the d-dimensional torus $(\mathbb{Z} \bmod L)^d$ with $\mathcal{N}_d = (\mathbb{Z} \bmod L)^d$ and $\mathcal{N}_c$ = the $2d$ nearest neighbors. If we let $L \to \infty$ then $t \to \infty$ the fraction of producers converges to $1/\lambda$. In $d \geq 3$ the limiting finite dimensional distributions converge as $t \to \infty$ to the voter model equilibrium with density $1/\lambda$. We next reformulate the system as an evolutionary game with "birth-death" updating and take $\mathcal{N}_c = \mathcal{N}_d = \mathcal{N}$. Using results for voter model perturbations we show that in $d = 3$ with $\mathcal{N}$ = the six nearest neighbors, the density of producers converges to $(2/\lambda) - 0.5$ for $4/3 < \lambda < 4$. Producers take over the system when $\lambda < 4/3$ and die out when $\lambda > 4$. In $d = 2$ with $\mathcal{N} = [-c\sqrt{\log N}, c\sqrt{\log N}]^2$ there are similar phase transitions, with coexistence occurring when $(1 + 2\theta)/(1 + \theta) < \lambda < (1 + 2\theta)/\theta$ where $\theta = (e^{3/(\pi c^2)} - 1)/2$.

REFERENCES

[1] ARCHETTI, M., FERRARO, D. A. and CHRISTOFORI, G. (2015). Heterogeneity for IGF-II production maintained by public goods dynamics in neuroendocrine pancreatic cancer. *Proc. Natl. Acad. Sci. USA* **112** 1833–1838.

[2] COX, J. T. (2010). Intermediate range migration in the two-dimensional stepping stone model. *Ann. Appl. Probab.* **20** 785–805. MR2680548

[3] COX, J. T. and DURRETT, R. (2016). Evolutionary games on the torus with weak selection. *Stochastic Process. Appl.* **126** 2388–2409. MR3505231

[4] COX, J. T., DURRETT, R. and PERKINS, E. A. (2013). Voter model perturbations and reaction diffusion equations. *Astérisque* **349** vi+113. MR3075759

[5] COX, J. T., MERLE, M. and PERKINS, E. (2010). Coexistence in a two-dimensional Lotka–Volterra model. *Electron. J. Probab.* **15** 1190–1266. MR2678390

[6] DURRETT, R. (2010). *Probability: Theory and Examples*, 4th ed. *Cambridge Series in Statistical and Probabilistic Mathematics* **31**. Cambridge Univ. Press, Cambridge. MR2722836

[7] DURRETT, R. (2014). Spatial evolutionary games with small selection coefficients. *Electron. J. Probab.* **19** no. 121, 64. MR3296537

[8] DURRETT, R., LIGGETT, T. and ZHANG, Y. (2014). The contact process with fast voting. *Electron. J. Probab.* **19** no. 28, 19. MR3174840

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[9] DURRETT, R. and NEUHAUSER, C. (1994). Particle systems and reaction-diffusion equations. *Ann. Probab.* **22** 289–333. MR1258879

[10] DURRETT, R. and ZÄHLE, I. (2007). On the width of hybrid zones. *Stochastic Process. Appl.* **117** 1751–1763. MR2437727

[11] DVORETZKY, A. and ERDÖS, P. (1951). Some problems on random walk in space. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950 353–367. Univ. California Press, Berkeley and Los Angeles. MR0047272

[12] GRIFFEATH, D. (1979). *Additive and Cancellative Interacting Particle Systems. Lecture Notes in Math.* **724**. Springer, Berlin. MR0538077

[13] HARRIS, T. E. (1976). On a class of set-valued Markov processes. *Ann. Probab.* **4** 175–194. MR0400468

[14] HOFBAUER, J. and SIGMUND, K. (1998). *Evolutionary Games and Population Dynamics*. Cambridge Univ. Press, Cambridge. MR1635735

[15] KESTEN, H. (1978). Erickson's conjecture on the rate of escape of d-dimensional random walk. *Trans. Amer. Math. Soc.* **240** 65–113. MR0489585

[16] LIGGETT, T. M. (1985). *Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231

[17] LIGGETT, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **324**. Springer, Berlin. MR1717346

[18] MOLOFSKY, J., DURRETT, R., DUSHOFF, J., GRIFFEATH, D. and LEVIN, S. A. (1999). Local frequency dependence and global coexistence. *Theor. Popul. Biol.* **55** 270–282.

[19] OHTSUKI, H., HAUERT, C., LIEBERMAN, E. and NOWAK, M. A. (2006). A simple rule for the evolution of cooperation on graphs and social networks. *Nature* **441** 502–505.

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EQUILIBRIUM LARGE DEVIATIONS FOR MEAN-FIELD SYSTEMS WITH TRANSLATION INVARIANCE[1]

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We consider particle systems with mean-field interactions whose distribution is invariant by translations. Under the assumption that the system seen from its centre of mass be reversible with respect to a Gibbs measure, we establish large deviation principles for its empirical measure at equilibrium. Our study covers the cases of McKean–Vlasov particle systems without external potential, and systems of rank-based interacting diffusions. Depending on the strength of the interaction, the large deviation principles are stated in the space of centered probability measures endowed with the Wasserstein topology of appropriate order, or in the orbit space of the action of translations on probability measures. An application to the study of atypical capital distribution is detailed.

REFERENCES

[1] AMBROSIO, L., GIGLI, N. and SAVARÉ, G. (2008). *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed. Birkhäuser, Basel. MR2401600

[2] BANNER, A. D., FERNHOLZ, R. and KARATZAS, I. (2005). Atlas models of equity markets. *Ann. Appl. Probab.* **15** 2296–2330. MR2187296

[3] BENACHOUR, S., ROYNETTE, B., TALAY, D. and VALLOIS, P. (1998). Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos. *Stochastic Process. Appl.* **75** 173–201. MR1632193

[4] BENACHOUR, S., ROYNETTE, B. and VALLOIS, P. (1998). Nonlinear self-stabilizing processes. II. Convergence to invariant probability. *Stochastic Process. Appl.* **75** 203–224. MR1632197

[5] BENEDETTO, D., CAGLIOTI, E., CARRILLO, J. A. and PULVIRENTI, M. (1998). A non-Maxwellian steady distribution for one-dimensional granular media. *J. Stat. Phys.* **91** 979–990. MR1637274

[6] BENEDETTO, D., CAGLIOTI, E. and PULVIRENTI, M. (1997). A kinetic equation for granular media. *RAIRO Modél. Math. Anal. Numér.* **31** 615–641. MR1471181

[7] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[8] BOBKOV, S. and LEDOUX, M. (2016). One-dimensional empirical measures, order statistics and Kantorovich transport distances. *Mem. Amer. Math. Soc.* To appear.

[9] BOSSY, M. and TALAY, D. (1996). Convergence rate for the approximation of the limit law of weakly interacting particles: Application to the Burgers equation. *Ann. Appl. Probab.* **6** 818–861. MR1410117

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[10] BOSSY, M. and TALAY, D. (1997). A stochastic particle method for the McKean–Vlasov and the Burgers equation. *Math. Comp.* **66** 157–192. MR1370849

[11] BRUGGEMAN, C. (2016). Dynamics of Large Rank-Based Systems of Interacting Diffusions. Ph.D. thesis, Columbia Univ., New York, NY. MR3503494

[12] CARRILLO, J. A., MCCANN, R. J. and VILLANI, C. (2003). Kinetic equilibration rates for granular media and related equations: Entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoam.* **19** 971–1018. MR2053570

[13] CARRILLO, J. A., MCCANN, R. J. and VILLANI, C. (2006). Contractions in the 2-Wasserstein length space and thermalization of granular media. *Arch. Ration. Mech. Anal.* **179** 217–263. MR2209130

[14] CATTIAUX, P., GUILLIN, A. and MALRIEU, F. (2008). Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields* **140** 19–40. MR2357669

[15] CATTIAUX, P. and PÉDÈCHES, L. (2016). The 2-D stochastic Keller–Segel particle model: Existence and uniqueness. *ALEA Lat. Am. J. Probab. Math. Stat.* **13** 447–463. MR3519253

[16] CHAFAÏ, D., GOZLAN, N. and ZITT, P.-A. (2014). First-order global asymptotics for confined particles with singular pair repulsion. *Ann. Appl. Probab.* **24** 2371–2413. MR3262506

[17] CHATTERJEE, S. and PAL, S. (2010). A phase transition behavior for Brownian motions interacting through their ranks. *Probab. Theory Related Fields* **147** 123–159. MR2594349

[18] DAWSON, D. A. and GÄRTNER, J. (1986). Large deviations and tunnelling for particle systems with mean field interaction. *C. R. Math. Rep. Acad. Sci. Can.* **8** 387–392. MR0866922

[19] DAWSON, D. A. and GÄRTNER, J. (1987). Large deviations from the McKean–Vlasov limit for weakly interacting diffusions. *Stochastics* **20** 247–308. MR0885876

[20] DAWSON, D. A. and GÄRTNER, J. (1989). Large deviations, free energy functional and quasi-potential for a mean field model of interacting diffusions. *Mem. Amer. Math. Soc.* **78** iv + 94. MR0989931

[21] DEMBO, A., SHKOLNIKOV, M., VARADHAN, S. R. S. and ZEITOUNI, O. (2016). Large deviations for diffusions interacting through their ranks. *Comm. Pure Appl. Math.* **69** 1259–1313. MR3503022

[22] DEMBO, A. and ZEITOUNI, O. (2010). *Large Deviations Techniques and Applications. Stochastic Modelling and Applied Probability* **38**. Springer, Berlin. Corrected reprint of the second (1998) edition. MR2571413

[23] DUPUIS, P. and ELLIS, R. S. (1997). *A Weak Convergence Approach to the Theory of Large Deviations*. Wiley, New York. MR1431744

[24] DUPUIS, P., LASCHOS, V. and RAMANAN, K. (2015). Large deviations for empirical measures generated by Gibbs measures with singular energy functionals. Preprint. Available at arXiv:1511.06928.

[25] ELLIS, R. S. (1985). *Entropy, Large Deviations, and Statistical Mechanics. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **271**. Springer, New York. MR0793553

[26] FERNHOLZ, E. R. (2002). *Stochastic Portfolio Theory. Applications of Mathematics* (*New York*) **48**. Springer, New York. MR1894767

[27] FOURNIER, N. and JOURDAIN, B. (2017). Stochastic particle approximation of the Keller–Segel equation and two-dimensional generalization of Bessel processes. *Ann. Appl. Probab.* **27** 2807–2861. MR3719947

[28] JORDAN, R., KINDERLEHRER, D. and OTTO, F. (1998). The variational formulation of the Fokker–Planck equation. *SIAM J. Math. Anal.* **29** 1–17. MR1617171

[29] JOURDAIN, B. (2000). Diffusion processes associated with nonlinear evolution equations for signed measures. *Methodol. Comput. Appl. Probab.* **2** 69–91. MR1783154

[30] JOURDAIN, B. and MALRIEU, F. (2008). Propagation of chaos and Poincaré inequalities for a system of particles interacting through their CDF. *Ann. Appl. Probab.* **18** 1706–1736. MR2462546

[31] JOURDAIN, B. and REYGNER, J. (2013). Propogation of chaos for rank-based interacting diffusions and long time behaviour of a scalar quasilinear parabolic equation. *Stoch. Partial Differ. Equ., Anal. Computat.* **1** 455–506. MR3327514

[32] JOURDAIN, B. and REYGNER, J. (2015). Capital distribution and portfolio performance in the mean-field Atlas model. *Ann. Finance* **11** 151–198. MR3340241

[33] KOLLI, P. and SHKOLNIKOV, M. (2018). SPDE limit of the global fluctuations in rank-based models. *Ann. Probab.* **46** 1042–1069. MR3773380

[34] LEBLÉ, T. and SERFATY, S. (2017). Large deviation principle for empirical fields of log and Riesz gases. *Invent. Math.* **210** 645–757. MR3735628

[35] LÉONARD, C. (1987). Large deviations and law of large numbers for a mean field type interacting particle system. *Stochastic Process. Appl.* **25** 215–235. MR0915135

[36] MALRIEU, F. (2003). Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.* **13** 540–560. MR1970276

[37] MUKHERJEE, C. and VARADHAN, S. R. S. (2016). Brownian occupation measures, compactness and large deviations. *Ann. Probab.* **44** 3934–3964. MR3572328

[38] OTTO, F. (2001). The geometry of dissipative evolution equations: The porous medium equation. *Comm. Partial Differential Equations* **26** 101–174. MR1842429

[39] PAL, S. and PITMAN, J. (2008). One-dimensional Brownian particle systems with rank-dependent drifts. *Ann. Appl. Probab.* **18** 2179–2207. MR2473654

[40] REY-BELLET, L. and SPILIOPOULOS, K. (2015). Irreversible Langevin samplers and variance reduction: A large deviations approach. *Nonlinearity* **28** 2081–2103. MR3366637

[41] REYGNER, J. (2015). Chaoticity of the stationary distribution of rank-based interacting diffusions. *Electron. Commun. Probab.* **20** no. 60, 1–20. MR3399811

[42] REYGNER, J. (2017). Long time behaviour and mean-field limit of Atlas models. *ESAIM Proc. Surv.* **60** 132–143.

[43] SHKOLNIKOV, M. (2012). Large systems of diffusions interacting through their ranks. *Stochastic Process. Appl.* **122** 1730–1747. MR2914770

[44] SUN, L.-H. (2013). Systemic risk illustrated. In *Handbook on Systemic Risk* 444–452. Cambridge Univ. Press, Cambridge.

[45] VILLANI, C. (2009). *Optimal Transport: Old and New. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454

[46] WANG, R., WANG, X. and WU, L. (2010). Sanov's theorem in the Wasserstein distance: A necessary and sufficient condition. *Statist. Probab. Lett.* **80** 505–512. MR2593592

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ASYMPTOTIC ANALYSIS OF THE RANDOM WALK METROPOLIS ALGORITHM ON RIDGED DENSITIES

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We study the asymptotic behaviour of the Random Walk Metropolis algorithm on "ridged" probability densities where most of the probability mass is distributed along some key directions. Such class of probability measures arise in various applied contexts including for instance Bayesian inverse problems where the posterior measure concentrates on a manifold when the noise variance goes to zero. When the target measure concentrates on a linear manifold, we derive analytically a diffusion limit for the Random Walk Metropolis Markov chain as the scale parameter goes to zero. In contrast to the existing works on scaling limits, our limiting stochastic differential equation does *not* in general have a constant diffusion coefficient. Our results show that in some cases, the usual practice of adapting the step-size to control the acceptance probability might be sub-optimal as the optimal acceptance probability is zero (in the limit).

REFERENCES

[1] BARKER, A. A. (1965). Monte Carlo calculations of the radial distribution functions for a proton-electron plasma. *Australian Journal of Physics* **18** 119–134.

[2] BÉDARD, M. (2007). Weak convergence of Metropolis algorithms for non-i.i.d. target distributions. *Ann. Appl. Probab.* **17** 1222–1244. MR2344305

[3] BÉDARD, M. and ROSENTHAL, J. S. (2008). Optimal scaling of Metropolis algorithms: Heading toward general target distributions. *Canad. J. Statist.* **36** 483–503. MR2532248

[4] BESKOS, A., ROBERTS, G. and STUART, A. (2009). Optimal scalings for local Metropolis–Hastings chains on nonproduct targets in high dimensions. *Ann. Appl. Probab.* **19** 863–898. MR2537193

[5] BREYER, L. A., PICCIONI, M. and SCARLATTI, S. (2004). Optimal scaling of MaLa for nonlinear regression. *Ann. Appl. Probab.* **14** 1479–1505. MR2071431

[6] COTTER, S. L., DASHTI, M., ROBINSON, J. C. and STUART, A. M. (2009). Bayesian inverse problems for functions and applications to fluid mechanics. *Inverse Probl.* **25** 115008, 43. MR2558668

[7] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR0838085

[8] FERGUSON, T. S. (1996). *A Course in Large Sample Theory*. Chapman & Hall, London. MR1699953

[9] GIROLAMI, M. and CALDERHEAD, B. (2011). Riemann manifold Langevin and Hamiltonian Monte Carlo methods. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **73** 123–214. MR2814492

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[10] KNAPIK, B. T., VAN DER VAART, A. W. and VAN ZANTEN, J. H. (2011). Bayesian inverse problems with Gaussian priors. *Ann. Statist.* **39** 2626–2657. MR2906881

[11] LIVINGSTONE, S. and GIROLAMI, M. (2014). Information-geometric Markov chain Monte Carlo methods using diffusions. *Entropy* **16** 3074–3102. MR3234224

[12] MATTINGLY, J. C., PILLAI, N. S. and STUART, A. M. (2012). Diffusion limits of the random walk Metropolis algorithm in high dimensions. *Ann. Appl. Probab.* **22** 881–930. MR2977981

[13] PASARICA, C. and GELMAN, A. (2010). Adaptively scaling the Metropolis algorithm using expected squared jumped distance. *Statist. Sinica* **20** 343–364. MR2640698

[14] PILLAI, N. S., STUART, A. M. and THIÉRY, A. H. (2012). Optimal scaling and diffusion limits for the Langevin algorithm in high dimensions. *Ann. Appl. Probab.* **22** 2320–2356. MR3024970

[15] RADDE, N. E. and OFFTERMATT, J. (2014). Convergence of posteriors for structurally non-identified problems using results from the theory of inverse problems. *J. Inverse Ill-Posed Probl.* **22** 251–276. MR3187940

[16] ROBERTS, G. O., GELMAN, A. and GILKS, W. R. (1997). Weak convergence and optimal scaling of random walk Metropolis algorithms. *Ann. Appl. Probab.* **7** 110–120. MR1428751

[17] ROBERTS, G. O. and ROSENTHAL, J. S. (2001). Optimal scaling for various Metropolis–Hastings algorithms. *Statist. Sci.* **16** 351–367. MR1888450

[18] ROBERTS, G. O. and ROSENTHAL, J. S. (2014). Minimising MCMC variance via diffusion limits, with an application to simulated tempering. *Ann. Appl. Probab.* **24** 131–149. MR3161644

[19] ROBERTS, G. O. and ROSENTHAL, J. S. (2016). Complexity bounds for Markov chain Monte Carlo algorithms via diffusion limits. *J. Appl. Probab.* **53** 410–420. MR3514287

[20] VAN DER VAART, A. W. (1998). *Asymptotic Statistics. Cambridge Series in Statistical and Probabilistic Mathematics* **3**. Cambridge Univ. Press, Cambridge. MR1652247

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AN IMPOSSIBILITY RESULT FOR RECONSTRUCTION IN THE DEGREE-CORRECTED STOCHASTIC BLOCK MODEL

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We consider the Degree-Corrected Stochastic Block Model (DC-SBM): a random graph on n nodes, having i.i.d. weights $(\phi_u)_{u=1}^n$ (possibly heavy-tailed), partitioned into $q \geq 2$ asymptotically equal-sized clusters. The model parameters are two constants $a, b > 0$ and the finite second moment of the weights $\Phi^{(2)}$. Vertices u and v are connected by an edge with probability $\frac{\phi_u \phi_v}{n} a$ when they are in the same class and with probability $\frac{\phi_u \phi_v}{n} b$ otherwise.

We prove that it is information-theoretically impossible to estimate the clusters in a way positively correlated with the true community structure when $(a-b)^2 \Phi^{(2)} \leq q(a+b)$.

As by-products of our proof we obtain (1) a precise coupling result for local neighbourhoods in DC-SBMs, that we use in Gulikers, Lelarge and Massoulié (2016) to establish a law of large numbers for local-functionals and (2) that long-range interactions are weak in (power-law) DC-SBMs.

REFERENCES

[1] ABBE, E. and SANDON, C. (2016). Detection in the stochastic block model with multiple clusters: Proof of the achievability conjectures, acyclic BP, and the information-computation gap. ArXiv Preprint ArXiv:1512.09080.

[2] AIELLO, W., CHUNG, F. and LU, L. (2001). A random graph model for power law graphs. *Exp. Math.* **10** 53–66. MR1822851

[3] BOLLOBÁS, B., JANSON, S. and RIORDAN, O. (2007). The phase transition in inhomogeneous random graphs. *Random Structures Algorithms* **31** 3–122. MR2337396

[4] BORDENAVE, C., LELARGE, M. and MASSOULIÉ, L. (2015). Non-backtracking spectrum of random graphs: Community detection and non-regular Ramanujan graphs. In 2015 *IEEE 56th Annual Symposium on Foundations of Computer Science—FOCS* 2015 1347–1357. IEEE Computer Soc., Los Alamitos, CA. MR3473374

[5] CHUNG, F., LU, L. and VU, V. (2004). The spectra of random graphs with given expected degrees. *Internet Math.* **1** 257–275. MR2111009

[6] COJA-OGHLAN, A. and LANKA, A. (2008). Partitioning random graphs with general degree distributions. In *Fifth IFIP International Conference on Theoretical Computer Science—TCS* 2008. *IFIP Int. Fed. Inf. Process.* **273** 127–141. Springer, New York. MR2757371

[7] DASGUPTA, A., HOPCROFT, J. E. and MCSHERRY, F. (2004). Spectral analysis of random graphs with skewed degree distributions. In *Foundations of Computer Science*, 2004. *Proceedings*. 45*th Annual IEEE Symposium on* 602–610.

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[8] DECELLE, A., KRZAKALA, F., MOORE, C. and ZDEBOROVÁ, L. (2011). Asymptotic analysis of the stochastic block model for modular networks and its algorithmic applications. *Phys. Rev. E* (3) **84** 066106.

[9] EVANS, W., KENYON, C., PERES, Y. and SCHULMAN, L. J. (2000). Broadcasting on trees and the Ising model. *Ann. Appl. Probab.* **10** 410–433. MR1768240

[10] GULIKERS, L., LELARGE, M. and MASSOULIÉ, L. (2016). Non-backtracking spectrum of degree-corrected stochastic block models. ArXiv Preprint ArXiv:1609.02487.

[11] GULIKERS, L., LELARGE, M. and MASSOULIÉ, L. (2017). A spectral method for community detection in moderately sparse degree-corrected stochastic block models. *Adv. in Appl. Probab.* **49** 686–721. MR3694314

[12] HOLLAND, P. W., LASKEY, K. B. and LEINHARDT, S. (1983). Stochastic blockmodels: First steps. *Soc. Netw.* **5** 109–137. MR0718088

[13] KARRER, B. and NEWMAN, M. E. J. (2011). Stochastic blockmodels and community structure in networks. *Phys. Rev. E* (3) **83** 016107, 10. MR2788206

[14] KRZAKALA, F., MOORE, C., MOSSEL, E., NEEMAN, J., SLY, A., ZDEBOROVÁ, L. and ZHANG, P. (2013). Spectral redemption in clustering sparse networks. *Proc. Natl. Acad. Sci. USA* **110** 20935–20940. MR3174850

[15] LEI, J. and RINALDO, A. (2015). Consistency of spectral clustering in stochastic block models. *Ann. Statist.* **43** 215–237. MR3285605

[16] LYONS, R. (1990). Random walks and percolation on trees. *Ann. Probab.* **18** 931–958. MR1062053

[17] MASSOULIÉ, L. (2014). Community detection thresholds and the weak Ramanujan property. In *STOC'*14—*Proceedings of the* 2014 *ACM Symposium on Theory of Computing* 694–703. ACM, New York. MR3238997

[18] MOSSEL, E. (2004). Survey: Information flow on trees. In *Graphs, Morphisms and Statistical Physics. DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* **63** 155–170. Amer. Math. Soc., Providence, RI. MR2056226

[19] MOSSEL, E., NEEMAN, J. and SLY, A. (2013). A proof of the block model threshold conjecture. ArXiv Preprint ArXiv:1311.4115.

[20] MOSSEL, E., NEEMAN, J. and SLY, A. (2015). Reconstruction and estimation in the planted partition model. *Probab. Theory Related Fields* **162** 431–461. MR3383334

[21] MOSSEL, E. and PERES, Y. (2003). Information flow on trees. *Ann. Appl. Probab.* **13** 817–844. MR1994038

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REGULARITY AND STABILITY FOR THE SEMIGROUP OF JUMP DIFFUSIONS WITH STATE-DEPENDENT INTENSITY

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We consider stochastic differential systems driven by a Brownian motion and a Poisson point measure where the intensity measure of jumps depends on the solution. This behavior is natural for several physical models (such as Boltzmann equation, piecewise deterministic Markov processes, etc.). First, we give sufficient conditions guaranteeing that the semigroup associated with such an equation preserves regularity by mapping the space of k-times differentiable bounded functions into itself. Furthermore, we give an upper estimate of the operator norm. This is the key-ingredient in a quantitative Trotter–Kato-type stability result: it allows us to give an upper estimate of the distance between two semigroups associated with different sets of coefficients in terms of the difference between the corresponding infinitesimal operators. As an application, we present a method allowing to replace "small jumps" by a Brownian motion or by a drift component. The example of the 2D Boltzmann equation is also treated in all detail.

REFERENCES

[1] ALFONSI, A., CANCÈS, E., TURINICI, G., DI VENTURA, B. and HUISINGA, W. (2004). Exact simulation of hybrid stochastic and deterministic models for biochemical systems. Research Report No. RR-5435, INRIA. Available at https://hal.inria.fr/inria-00070572.

[2] ALFONSI, A., CANCÈS, E., TURINICI, G., DI VENTURA, B. and HUISINGA, W. (2005). Adaptive simulation of hybrid stochastic and deterministic models for biochemical systems. In *CEMRACS* 2004—*Mathematics and Applications to Biology and Medicine*. *ESAIM Proc*. **14** 1–13. EDP Sci., Les Ulis. MR2226797

[3] ANGIUS, A., BALBO, G., BECCUTI, M., BIBBONA, E., HORVATH, A. and SIROVICH, R. (2015). Approximate analysis of biological systems by hybrid switching jump diffusion. *Theoret*. *Comput*. *Sci*. **587** 49–72. MR3354838

[4] ASMUSSEN, S. and ROSIŃSKI, J. (2001). Approximations of small jumps of Lévy processes with a view towards simulation. *J*. *Appl*. *Probab*. **38** 482–493. MR1834755

[5] BALL, K., KURTZ, T. G., POPOVIC, L. and REMPALA, G. (2006). Asymptotic analysis of multiscale approximations to reaction networks. *Ann*. *Appl*. *Probab*. **16** 1925–1961. MR2288709

[6] BALLY, V. and FOURNIER, N. (2011). Regularization properties of the 2D homogeneous Boltzmann equation without cutoff. *Probab*. *Theory Related Fields* **151** 659–704. MR2851696

[7] BENAÏ M, M., BOUGUET, F. and CLOEZ, B. (2017). Ergodicity of inhomogeneous Markov chains through asymptotic pseudotrajectories. *Ann*. *Appl*. *Probab*. **27** 3004–3049. MR3719952

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[8] BENAÏM, M. and HIRSCH, M. W. (1996). Asymptotic pseudotrajectories and chain recurrent flows, with applications. *J. Dynam. Differential Equations* **8** 141–176. MR1388167

[9] BOXMA, O., KASPI, H., KELLA, O. and PERRY, D. (2005). On/off storage systems with state-dependent input, output, and switching rates. *Probab. Engrg. Inform. Sci.* **19** 1–14. MR2104547

[10] BRESSLOFF, P. C. and NEWBY, J. M. (2013). Metastability in a stochastic neural network modeled as a velocity jump Markov process. *SIAM J. Appl. Dyn. Syst.* **12** 1394–1435. MR3085109

[11] BUCKWAR, E. and RIEDLER, M. G. (2011). An exact stochastic hybrid model of excitable membranes including spatio-temporal evolution. *J. Math. Biol.* **63** 1051–1093. MR2855804

[12] COCOZZA-THIEVENT, C. (2017). Renouvellement Markovien et PDMP. Available at http://perso-math.univ-mlv.fr/users/cocozza.christiane/recherche-page-perso/PresentationRMetPDMP.html.

[13] CRUDU, A., DEBUSSCHE, A., MULLER, A. and RADULESCU, O. (2012). Convergence of stochastic gene networks to hybrid piecewise deterministic processes. *Ann. Appl. Probab.* **22** 1822–1859. MR3025682

[14] DAVIS, M. H. A. (1984). Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. *J. Roy. Statist. Soc. Ser. B* **46** 353–388. MR0790622

[15] DAVIS, M. H. A. (1993). *Markov Models and Optimization. Monographs on Statistics and Applied Probability* **49**. Chapman & Hall, London. MR1283589

[16] DUFOUR, F., DUTUIT, Y., GONZALEZ, K. and ZHANG, H. (2008). Piecewise deterministic Markov processes and dynamic reliability. *J. Risk Reliab.* **222** 545–551.

[17] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR0838085

[18] FOURNIER, N. (2011). Simulation and approximation of Lévy-driven stochastic differential equations. *ESAIM Probab. Stat.* **15** 233–248. MR2870514

[19] FOURNIER, N. and GODINHO, D. (2012). Asymptotic of grazing collisions and particle approximation for the Kac equation without cutoff. *Comm. Math. Phys.* **316** 307–344. MR2993918

[20] FOURNIER, N. and MOUHOT, C. (2009). On the well-posedness of the spatially homogeneous Boltzmann equation with a moderate angular singularity. *Comm. Math. Phys.* **289** 803–824. MR2511651

[21] GANGULY, A., ALTINTAN, D. and KOEPPL, H. (2015). Jump-diffusion approximation of stochastic reaction dynamics: Error bounds and algorithms. *Multiscale Model. Simul.* **13** 1390–1419. MR3431122

[22] GILLESPIE, D. T. (1976). A general method for numerically simulating the stochastic time evolution of coupled chemical reactions. *J. Comput. Phys.* **22** 403–434. MR0503370

[23] GODINHO, D. (2013). Asymptotic of grazing collisions for the spatially homogeneous Boltzmann equation for soft and Coulomb potentials. *Stochastic Process. Appl.* **123** 3987–4039. MR3091097

[24] GRAHAM, C. (1992). McKean–Vlasov Itô–Skorohod equations, and nonlinear diffusions with discrete jump sets. *Stochastic Process. Appl.* **40** 69–82. MR1145460

[25] HE, L. (2014). Asymptotic analysis of the spatially homogeneous Boltzmann equation: Grazing collisions limit. *J. Stat. Phys.* **155** 151–210. MR3180973

[26] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. MR1011252

[27] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[28] JAHNKE, T. and KREIM, M. (2012). Error bound for piecewise deterministic processes modeling stochastic reaction systems. *Multiscale Model*. *Simul*. **10** 1119–1147. MR3022034

[29] KOLOKOLTSOV, V. N. (2011). *Markov Processes*, *Semigroups and Generators*. *De Gruyter Studies in Mathematics* **38**. de Gruyter, Berlin. MR2780345

[30] KUNITA, H. (2004). Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms. In *Real and Stochastic Analysis*. 305–373. Birkhäuser, Boston, MA. MR2090755

[31] KURTZ, T. G. (1971). Limit theorems for sequences of jump Markov processes approximating ordinary differential processes. *J*. *Appl*. *Probab*. **8** 344–356. MR0287609

[32] KURTZ, T. G. (1977/78). Strong approximation theorems for density dependent Markov chains. *Stochastic Process*. *Appl*. **6** 223–240. MR0464414

[33] LAPEYRE, B., PARDOUX, É. and SENTIS, R. (1998). *Méthodes de Monte-Carlo Pour les équations de Transport et de Diffusion*. *Mathématiques & Applications* (*Berlin*) [*Mathematics & Applications*] **29**. Springer, Berlin. MR1621249

[34] LÖCHERBACH, E. (2017). Convergence to equilibrium for time inhomogeneous jump diffusions with state dependent jump intensity. Preprint. Available at arXiv:1712.03507.

[35] PAZY, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. *Applied Mathematical Sciences* **44**. Springer, New York. MR0710486

[36] PROTTER, P. and TALAY, D. (1997). The Euler scheme for Lévy driven stochastic differential equations. *Ann*. *Probab*. **25** 393–423. MR1428514

[37] TANAKA, H. (1978/79). Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z*. *Wahrsch*. *Verw*. *Gebiete* **46** 67–105. MR0512334

[38] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour*, *XIV*—1984. *Lecture Notes in Math*. **1180** 265–439. Springer, Berlin. MR0876085

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REAL EIGENVALUES IN THE NON-HERMITIAN ANDERSON MODEL

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The eigenvalues of the Hatano–Nelson non-Hermitian Anderson matrices, in the spectral regions in which the Lyapunov exponent exceeds the non-Hermiticity parameter, are shown to be real and exponentially close to the Hermitian eigenvalues. This complements previous results, according to which the eigenvalues in the spectral regions in which the non-Hermiticity parameter exceeds the Lyapunov exponent are aligned on curves in the complex plane.

REFERENCES

[1] AIZENMAN, M. and MOLCHANOV, S. (1993). Localization at large disorder and at extreme energies: An elementary derivation. *Comm. Math. Phys.* **157** 245–278. MR1244867

[2] BERNSTEIN, S. (1924). Über eine modifikation der ungleichung von Tschebyscheff und über die abweichung der laplaceschen formel. *Charkov Ann. Sc.* **1** 38–49.

[3] BERNSTEIN, S. (1927). Sur l'extension du théoréme limite du calcul des probabilités aux sommes de quantités dépendantes. *Math. Ann.* **97** 1–59. MR1512353

[4] BOURGAIN, J. (2004). On localization for lattice Schrödinger operators involving Bernoulli variables. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math.* **1850** 77–99. Springer, Berlin. MR2087153

[5] BOURGAIN, J. (2012). On the Furstenberg measure and density of states for the Anderson-Bernoulli model at small disorder. *J. Anal. Math.* **117** 273–295. MR2944098

[6] BOURGAIN, J. (2014). On eigenvalue spacings for the 1-D Anderson model with singular site distribution. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math.* **2116** 71–83. Springer, Cham. MR3364680

[7] BRÉZIN, E. and ZEE, A. (1998). Non-Hermitean delocalization: Multiple scattering and bounds. *Nuclear Phys. B* **509** 599–614. MR1487025

[8] BROUWER, P. W., SILVESTROV, P. G. and BEENAKKER, C. WJ. (1997). Theory of directed localization in one dimension. *Phys. Rev. B* **56** Article ID R4333.

[9] CARMONA, R., KLEIN, A. and MARTINELLI, F. (1987). Anderson localization for Bernoulli and other singular potentials. *Comm. Math. Phys.* **108** 41–66. MR0872140

[10] COMBES, J.-M., GERMINET, F. and KLEIN, A. (2009). Generalized eigenvalue-counting estimates for the Anderson model. *J. Stat. Phys.* **135** 201–216. MR2505733

[11] FEINBERG, J. and ZEE, A. (1999). Spectral curves of non-Hermitian Hamiltonians. *Nuclear Phys. B* **552** 599–623. MR1704620

[12] FRÖHLICH, J. and SPENCER, T. (1983). Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Comm. Math. Phys.* **88** 151–184. MR0696803

[13] FURSTENBERG, H. (1963). Noncommuting random products. *Trans. Amer. Math. Soc.* **108** 377–428. MR0163345

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[14] FURSTENBERG, H. and KESTEN, H. (1960). Products of random matrices. *Ann. Math. Stat.* **31** 457–469. MR0121828

[15] GOLDSHEID, I. YA. and KHORUZHENKO, B. A. (1998). Distribution of eigenvalues in non-Hermitian Anderson models. *Phys. Rev. Lett.* **80** 2897.

[16] GOLDSHEID, I. JA. (1980). Asymptotic properties of the product of random matrices depending on a parameter. In *Multicomponent Random Systems*. *Adv. Probab. Related Topics* **6** 239–283. Dekker, New York. MR0599538

[17] GOLDSHEID, I. YA. and KHORUZHENKO, B. A. (2000). Eigenvalue curves of asymmetric tridiagonal random matrices. *Electron. J. Probab.* **5** Article ID 16. MR1800072

[18] GOLDSHEID, I. YA. and KHORUZHENKO, B. A. (2003). Regular spacings of complex eigenvalues in the one-dimensional non-Hermitian Anderson model. *Comm. Math. Phys.* **238** 505–524. MR1993383

[19] GRAF, G. M. and VAGHI, A. (2007). A remark on the estimate of a determinant by Minami. *Lett. Math. Phys.* **79** 17–22. MR2290333

[20] GUIVARC'H, Y. In Produits de Matrices Aléatoires et Applications aux Propriétés Géométriques des Sous-groupes du Groupe Linéaire.

[21] HATANO, N. and NELSON, D. R. (1996). Localization transitions in non-Hermitian quantum mechanics. *Phys. Rev. Lett.* **77** 570.

[22] HATANO, N. and NELSON, D. R. (1998). Non-Hermitian delocalization and eigenfunctions. *Phys. Rev. B* **58** 8384.

[23] KUWAE, T. and TANIGUCHI, N. (2001). Two-dimensional non-Hermitian delocalization transition as a probe for the localization length. *Phys. Rev. B* **64** 201321.

[24] LE PAGE, É. (1982). Théorèmes limites pour les produits de matrices aléatoires. In *Probability Measures on Groups* (*Oberwolfach*, 1981). *Lecture Notes in Math.* **928** 258–303. Springer, Berlin. MR0669072

[25] LE PAGE, É. (1984). Répartition d'état d'un opérateur de Schrödinger aléatoire. Distribution empirique des valeurs propres d'une matrice de Jacobi. In *Probability Measures on Groups*, *VII* (*Oberwolfach*, 1983). *Lecture Notes in Math.* **1064** 309–367. Springer, Berlin. MR0772418

[26] MINAMI, N. (1996). Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. *Comm. Math. Phys.* **177** 709–725. MR1385082

[27] MOLCHANOV, S. A. (1982). The local structure of the spectrum of a random one-dimensional Schrödinger operator. *Tr. Semin. im. I. G. Petrovskogo* **8** 195–210. MR0673165

[28] MOLINARI, L. G. (2009). Non-Hermitian spectra and Anderson localization. *J. Phys. A* **42** Article ID 265204. MR2515502

[29] OSELEDEC, V. I. (1968). A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems. *Tr. Mosk. Mat. Obŝ.* **19** 179–210. MR0240280

[30] REDDY, N. K. Lyapunov exponents and eigenvalues of products of random matrices. Preprint. Available at arXiv:1606.07704.

[31] SERT, C. Large deviation principle for random matrix products. Preprint. Available at arXiv:1704.00615.

[32] SHUBIN, C., VAKILIAN, R. and WOLFF, T. (1998). Some harmonic analysis questions suggested by Anderson–Bernoulli models. *Geom. Funct. Anal.* **8** 932–964. MR1650106

[33] TUTUBALIN, V. N. (1965). Limit theorems for a product of random matrices. *Teor. Veroâtn. Ee Primen.* **10** 19–32. MR0175169

[34] WEGNER, F. (1981). Bounds on the density of states in disordered systems. *Z. Phys. B* **44** 9–15. MR0639135

[35] ZEE, A. (1998). A non-Hermitean particle in a disordered world. *Phys. A* **254** 300–316.

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DICTATOR FUNCTIONS MAXIMIZE MUTUAL INFORMATION

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Let $(\mathbf{X}, \mathbf{Y})$ denote n independent, identically distributed copies of two arbitrarily correlated Rademacher random variables (X, Y). We prove that the inequality $\mathrm{I}(f(\mathbf{X}); g(\mathbf{Y})) \le \mathrm{I}(\mathsf{X}; \mathsf{Y})$ holds for any two Boolean functions: $f, g\colon \{-1, 1\}^n \to \{-1, 1\}$ [$\mathrm{I}(\cdot;\cdot)$ denotes mutual information]. We further show that equality in general is achieved only by the dictator functions $f(\boldsymbol{x}) = \pm g(\boldsymbol{x}) = \pm x_i$, $i \in \{1, 2, \dots, n\}$.

REFERENCES

[1] ANANTHARAM, V., GOHARI, A. A., KAMATH, S. and NAIR, C. (2013). On hypercontractivity and the mutual information between Boolean functions. In *Proc*. 51*st Annual Allerton Conference on Communication*, *Control*, *and Computing* 13–19.
[2] COURTADE, T. A. and KUMAR, G. R. (2014). Which Boolean functions maximize mutual information on noisy inputs? *IEEE Trans*. *Inform*. *Theory* **60** 4515–4525. MR3245339
[3] COVER, T. M. and THOMAS, J. A. (2006). *Elements of Information Theory*, 2nd ed. Wiley, Hoboken, NJ. MR2239987
[4] KLOTZ, J. G., KRACHT, D., BOSSERT, M. and SCHOBER, S. (2014). Canalizing Boolean functions maximize mutual information. *IEEE Trans*. *Inform*. *Theory* **60** 2139–2147. MR3181516
[5] KUMAR, G. R. and COURTADE, T. A. (2013). Which Boolean functions are most informative? In *Proc*. *IEEE Int*. *Symp*. *on Inform*. *Theory* 226–230. DOI:10.1109/ISIT.2013.6620221.
[6] O'DONNELL, R. (2014). *Analysis of Boolean Functions*. Cambridge Univ. Press, New York. MR3443800
[7] PICHLER, G. (2017). Clustering by mutual information. Ph.D. thesis, Vienna Univ. Technology.
[8] ROBERTS, A. W. and VARBERG, D. E. (1973). *Convex Functions*. *Pure and Applied Mathematics* **57**. Academic Press, New York. MR0442824

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DIFFUSION TRANSFORMATIONS, BLACK–SCHOLES EQUATION AND OPTIMAL STOPPING

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We develop a new class of path transformations for one-dimensional diffusions that are tailored to alter their long-run behaviour from transient to recurrent or vice versa. This immediately leads to a formula for the distribution of the first exit times of diffusions, which is recently characterised by Karatzas and Ruf [*Probab. Theory Related Fields* **164** (2016) 1027–1069] as the minimal solution of an appropriate Cauchy problem under more stringent conditions. A particular limit of these transformations also turn out to be instrumental in characterising the stochastic solutions of Cauchy problems defined by the generators of strict local martingales, which are well known for not having unique solutions even when one restricts solutions to have linear growth. Using an appropriate diffusion transformation, we show that the aforementioned stochastic solution can be written in terms of the unique classical solution of an *alternative* Cauchy problem with suitable boundary conditions. This in particular resolves the long-standing issue of non-uniqueness with the Black–Scholes equations in derivative pricing in the presence of *bubbles*. Finally, we use these path transformations to propose a unified framework for solving explicitly the optimal stopping problem for one-dimensional diffusions with discounting, which in particular is relevant for the pricing and the computation of optimal exercise boundaries of perpetual American options.

REFERENCES

BALDI, P. (1995). Exact asymptotics for the probability of exit from a domain and applications to simulation. *Ann. Probab.* **23** 1644–1670. MR1379162

BALDI, P., CARAMELLINO, L. and IOVINO, M. G. (1999). Pricing general barrier options: A numerical approach using sharp large deviations. *Math. Finance* **9** 293–322. MR1849251

BAYRAKTAR, E., KARDARAS, C. and XING, H. (2012a). Valuation equations for stochastic volatility models. *SIAM J. Financial Math.* **3** 351–373. MR2968038

BAYRAKTAR, E., KARDARAS, C. and XING, H. (2012b). Strict local martingale deflators and valuing American call-type options. *Finance Stoch.* **16** 275–291. MR2903625

BAYRAKTAR, E. and XING, H. (2010). On the uniqueness of classical solutions of Cauchy problems. *Proc. Amer. Math. Soc.* **138** 2061–2064. MR2596042

BEIBEL, M. and LERCHE, H. R. (2000). A note on optimal stopping of regular diffusions under random discounting. *Teor. Veroyatn. Primen.* **45** 657–669. MR1968720

BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory. Pure and Applied Mathematics, Vol.* 29. Academic Press, New York–London. MR0264757

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BORODIN, A. N. and SALMINEN, P. (2002). *Handbook of Brownian Motion—Facts and Formulae*, 2nd ed. *Probability and Its Applications*. Birkhäuser, Basel. MR1912205

CHUNG, K. L. and WALSH, J. B. (2005). *Markov Processes*, *Brownian Motion*, *and Time Symmetry*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **249**. Springer, New York. MR2152573

CISSÉ, M., PATIE, P. and TANRÉ, E. (2012). Optimal stopping problems for some Markov processes. *Ann*. *Appl*. *Probab*. **22** 1243–1265. MR2977991

COX, A. M. G. and HOBSON, D. G. (2005). Local martingales, bubbles and option prices. *Finance Stoch*. **9** 477–492. MR2213778

DAYANIK, S. and KARATZAS, I. (2003). On the optimal stopping problem for one-dimensional diffusions. *Stochastic Process*. *Appl*. **107** 173–212. MR1999788

DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math*. *Ann*. **300** 463–520. MR1304434

DELBAEN, F. and SHIRAKAWA, H. (2002). No arbitrage condition for positive diffusion price processes. *Asia-Pac*. *Financ*. *Mark*. **9** 159–168. DOI:10.1023/A:1024173029378.

DYNKIN, E. B. (1963). Optimal choice of the stopping moment of a Markov process. *Dokl*. *Akad*. *Nauk SSSR* **150** 238–240. MR0154329

EKSTRÖM, E. and TYSK, J. (2009). Bubbles, convexity and the Black–Scholes equation. *Ann*. *Appl*. *Probab*. **19** 1369–1384. MR2538074

ENGELBERT, H. J. and SCHMIDT, W. (1991). Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations. III. *Math*. *Nachr*. **151** 149–197. MR1121203

EVANS, S. N. and HENING, A. (2016). Markov processes conditioned on their location at large exponential times. Working Paper.

FÖLLMER, H. (1972). The exit measure of a supermartingale. *Z*. *Wahrsch*. *Verw*. *Gebiete* **21** 154–166. MR0309184

GOBET, E. (2000). Weak approximation of killed diffusion using Euler schemes. *Stochastic Process*. *Appl*. **87** 167–197. MR1757112

GOBET, E. and MENOZZI, S. (2004). Exact approximation rate of killed hypoelliptic diffusions using the discrete Euler scheme. *Stochastic Process*. *Appl*. **112** 201–223. MR2073411

HALMOS, P. R. (2014). *Measure Theory*. *Graduate Texts in Mathematics*. Springer, New York.

IKEDA, N. and WATANABE, S. (2014). *Stochastic Differential Equations and Diffusion Processes* **24**. Elsevier, Amsterdam.

ITÔ, K. and MCKEAN, H. P. JR. (1974). *Diffusion Processes and Their Sample Paths*. Springer, Berlin. MR0345224

KARATZAS, I. and RUF, J. (2016). Distribution of the time to explosion for one-dimensional diffusions. *Probab. Theory Related Fields* **164** 1027–1069. MR3477786

KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

KARDARAS, C., KREHER, D. and NIKEGHBALI, A. (2015). Strict local martingales and bubbles. *Ann*. *Appl*. *Probab*. **25** 1827–1867. MR3348996

KOTANI, S. (2006). On a condition that one-dimensional diffusion processes are martingales. In *In Memoriam Paul-André Meyer*: *Séminaire de Probabilités XXXIX*. *Lecture Notes in Math*. **1874** 149–156. Springer, Berlin. MR2276894

LAMBERTON, D. and ZERVOS, M. (2013). On the optimal stopping of a one-dimensional diffusion. *Electron*. *J*. *Probab*. **18** no. 34, 49. MR3035762

LERCHE, H. R. and URUSOV, M. (2007). Optimal stopping via measure transformation: The Beibel–Lerche approach. *Stochastics* **79** 275–291. MR2308076

MARCUS, M. B. and ROSEN, J. (2006). *Markov Processes*, *Gaussian Processes*, *and Local Times*. *Cambridge Studies in Advanced Mathematics* **100**. Cambridge Univ. Press, Cambridge. MR2250510

McKean, H. P. Jr. (1956). Elementary solutions for certain parabolic partial differential equations. *Trans. Amer. Math. Soc.* **82** 519–548. MR0087012

Pal, S. and Protter, P. (2010). Analysis of continuous strict local martingales via h-transforms. *Stochastic Process. Appl.* **120** 1424–1443. MR2653260

Peskir, G. and Shiryaev, A. (2006). *Optimal Stopping and Free-Boundary Problems. Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR2256030

Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357

Rudin, W. (1987). *Real and Complex Analysis*, 3rd ed. McGraw-Hill, New York. MR0924157

Sharpe, M. (1988). *General Theory of Markov Processes. Pure and Applied Mathematics* **133**. Academic Press, Boston, MA. MR0958914

Shiryaev, A. N. (2008). *Optimal Stopping Rules. Stochastic Modelling and Applied Probability* **8**. Springer, Berlin. MR2374974

Stroock, D. W. and Varadhan, S. R. S. (2006). *Multidimensional Diffusion Processes. Classics in Mathematics*. Springer, Berlin. MR2190038

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DYNAMICS OF A PLANAR COULOMB GAS[1]

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We study the long-time behavior of the dynamics of interacting planar Brownian particles, confined by an external field and subject to a singular pair repulsion. The invariant law is an exchangeable Boltzmann–Gibbs measure. For a special inverse temperature, it matches the Coulomb gas known as the complex Ginibre ensemble. The difficulty comes from the interaction which is not convex, in contrast with the case of one-dimensional log-gases associated with the Dyson Brownian motion. Despite the fact that the invariant law is neither product nor log-concave, we show that the system is well-posed for any inverse temperature and that Poincaré inequalities are available. Moreover, the second moment dynamics turns out to be a nice Cox–Ingersoll–Ross process, in which the dependency over the number of particles leads to identify two natural regimes related to the behavior of the noise and the speed of the dynamics.

REFERENCES

[1] ALLEZ, R., BOUCHAUD, J.-P. and GUIONNET, A. (2012). Invariant beta ensembles and the Gauss–Wigner crossover. *Phys. Rev. Lett.* **109** 094102.

[2] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[3] ANÉ, C., BLACHÈRE, S., CHAFAÏ, D., FOUGÈRES, P., GENTIL, I., MALRIEU, F., ROBERTO, C. and SCHEFFER, G. (2000). *Sur les Inégalités de Sobolev Logarithmiques*. *Panoramas et Synthèses* [*Panoramas and Syntheses*] **10**. Société Mathématique de France, Paris. MR1845806

[4] BAKRY, D., BARTHE, F., CATTIAUX, P. and GUILLIN, A. (2008). A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case. *Electron. Commun. Probab.* **13** 60–66. MR2386063

[5] BAKRY, D., GENTIL, I. and LEDOUX, M. (2014). *Analysis and Geometry of Markov Diffusion Operators*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **348**. Springer, Cham. MR3155209

[6] BERMAN, R. J. and ÖNNHEIM, M. (2016). Propagation of chaos for a class of first order models with singular mean field interactions. Preprint arXiv:1610.04327.

[7] BLAIZOT, J.-P., GRELA, J., NOWAK, M. A., TARNOWSKI, W. and WARCHOŁ, P. (2016). Ornstein–Uhlenbeck diffusion of Hermitian and non-Hermitian matrices—unexpected links. *J. Stat. Mech. Theory Exp.* **5** 054037, 22. MR3514219

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[8] BOBKOV, S. G. (1999). Isoperimetric and analytic inequalities for log-concave probability measures. *Ann. Probab.* **27** 1903–1921. MR1742893

[9] BOBKOV, S. G. (2003). Spectral gap and concentration for some spherically symmetric probability measures. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math.* **1807** 37–43. Springer, Berlin. MR2083386

[10] BONAMI, A., BOUCHUT, F., CÉPA, E. and LÉPINGLE, D. (1999). A nonlinear stochastic differential equation involving the Hilbert transform. *J. Funct. Anal.* **165** 390–406. MR1698948

[11] BONNEFONT, M., JOULIN, A. and MA, Y. (2016). Spectral gap for spherically symmetric log-concave probability measures, and beyond. *J. Funct. Anal.* **270** 2456–2482. MR3464047

[12] BORDENAVE, C. and CHAFAÏ, D. (2012). Around the circular law. *Probab. Surv.* **9** 1–89. MR2908617

[13] BOURGADE, P. and DUBACH, G. (2018). The distribution of overlaps between eigenvectors of Ginibre matrices. Preprint arXiv:1801.01219.

[14] CABANAL DUVILLARD, T. and GUIONNET, A. (2001). Large deviations upper bounds for the laws of matrix-valued processes and non-communicative entropies. *Ann. Probab.* **29** 1205–1261. MR1872742

[15] CAGLIOTI, E., LIONS, P.-L., MARCHIORO, C. and PULVIRENTI, M. (1992). A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description. *Comm. Math. Phys.* **143** 501–525. MR1145596

[16] CAGLIOTI, E., LIONS, P.-L., MARCHIORO, C. and PULVIRENTI, M. (1995). A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description. II. *Comm. Math. Phys.* **174** 229–260. MR1362165

[17] CATTIAUX, P. and GUILLIN, A. (2017). Hitting times, functional inequalities, Lyapunov conditions and uniform ergodicity. *J. Funct. Anal.* **272** 2361–2391. MR3603301

[18] CATTIAUX, P., GUILLIN, A., WANG, F.-Y. and WU, L. (2009). Lyapunov conditions for super Poincaré inequalities. *J. Funct. Anal.* **256** 1821–1841. MR2498560

[19] CATTIAUX, P. and PÉDÈCHES, L. (2016). The 2-D stochastic Keller–Segel particle model: Existence and uniqueness. *ALEA Lat. Am. J. Probab. Math. Stat.* **13** 447–463. MR3519253

[20] CÉPA, E. and LÉPINGLE, D. (1997). Diffusing particles with electrostatic repulsion. *Probab. Theory Related Fields* **107** 429–449. MR1440140

[21] CHAFAÏ, D. (2016). http://djalil.chafai.net/blog/2016/12/27/mind-the-gap/. Blog post.

[22] CHAFAÏ, D., BOLLEY, F. and FONTBONA, J. (2018). Dynamics of a planar Coulomb gas II. Preprint in preparation.

[23] CHAFAÏ, D., GOZLAN, N. and ZITT, P.-A. (2014). First-order global asymptotics for confined particles with singular pair repulsion. *Ann. Appl. Probab.* **24** 2371–2413. MR3262506

[24] CHAFAÏ, D., HARDY, A. and MAÏDA, M. (2016). Concentration for Coulomb gases and Coulomb transport inequalities. Preprint arXiv:1610.00980.

[25] COX, J. C., INGERSOLL, J. E. JR. and ROSS, S. A. (1985). A theory of the term structure of interest rates. *Econometrica* **53** 385–407. MR0785475

[26] DUERINCKX, M. (2016). Mean-field limits for some Riesz interaction gradient flows. *SIAM J. Math. Anal.* **48** 2269–2300. MR3514713

[27] ERDŐS, L. and YAU, H.-T. (2017). *A Dynamical Approach to Random Matrix Theory*. *Courant Lecture Notes in Mathematics* **28**. Courant Institute of Mathematical Sciences, New York. MR3699468

[28] FEINGOLD, D. G. and VARGA, R. S. (1962). Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem. *Pacific J. Math.* **12** 1241–1250. MR0151473

[29] FONTBONA, J. (2004). Uniqueness for a weak nonlinear evolution equation and large deviations for diffusing particles with electrostatic repulsion. *Stochastic Process. Appl.* **112** 119–144. MR2062570

[30] FONTBONA, J. and JOURDAIN, B. (2014). On the long time behaviour of stochastic vortices systems. *Markov Process. Related Fields* **20** 675–704. MR3308573

[31] FONTBONA, J. and MARTINEZ, M. (2007). Paths clustering and an existence result for stochastic vortex systems. *J. Stat. Phys.* **128** 699–719. MR2343519

[32] FORRESTER, P. J. (2010). *Log-Gases and Random Matrices. London Mathematical Society Monographs Series* **34**. Princeton Univ. Press, Princeton, NJ. MR2641363

[33] FOURNIER, N., HAURAY, M. and MISCHLER, S. (2014). Propagation of chaos for the 2D viscous vortex model. *J. Eur. Math. Soc.* (*JEMS*) **16** 1423–1466. MR3254330

[34] FOURNIER, N. and JOURDAIN, B. (2017). Stochastic particle approximation of the Keller–Segel equation and two-dimensional generalization of Bessel processes. *Ann. Appl. Probab.* **27** 2807–2861. MR3719947

[35] GINIBRE, J. (1965). Statistical ensembles of complex, quaternion, and real matrices. *J. Math. Phys.* **6** 440–449. MR0173726

[36] HIAI, F. and PETZ, D. (2000). *The Semicircle Law, Free Random Variables and Entropy. Mathematical Surveys and Monographs* **77**. Amer. Math. Soc., Providence, RI. MR1746976

[37] HORN, R. A. and JOHNSON, C. R. (1994). *Topics in Matrix Analysis*. Cambridge Univ. Press, Cambridge. Corrected reprint of the 1991 original. MR1288752

[38] HOUGH, J. B., KRISHNAPUR, M., PERES, Y. and VIRÁG, B. (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes. University Lecture Series* **51**. Amer. Math. Soc., Providence, RI. MR2552864

[39] HWANG, C.-R. (1986). A brief survey on the spectral radius and the spectral distribution of large random matrices with i.i.d. entries. In *Random Matrices and Their Applications* (*Brunswick, Maine*, 1984). *Contemp. Math.* **50** 145–152. Amer. Math. Soc., Providence, RI. MR0841088

[40] IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes. North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. MR0637061

[41] JIN, P., MANDREKAR, V., RÜDIGER, B. and TRABELSI, C. (2013). Positive Harris recurrence of the CIR process and its applications. *Commun. Stoch. Anal.* **7** 409–424. MR3167406

[42] KHORUZHENKO, B. A. and SOMMERS, H.-J. (2011). Non-Hermitian ensembles. In *The Oxford Handbook of Random Matrix Theory* 376–397. Oxford Univ. Press, Oxford. MR2932638

[43] LI, S., LI, X.-D. and XIE, Y.-X. (2013). Generalized Dyson Brownian motion, McKean–Vlasov equation and eigenvalues of random matrices. Preprint arXiv:1303.1240.

[44] MALRIEU, F. (2001). Logarithmic Sobolev inequalities for some nonlinear PDE's. *Stochastic Process. Appl.* **95** 109–132. MR1847094

[45] MEHTA, M. L. (2004). *Random Matrices*, 3rd ed. *Pure and Applied Mathematics* (*Amsterdam*) **142**. Elsevier/Academic Press, Amsterdam. MR2129906

[46] MÉLÉARD, S. (1996). Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models. In *Probabilistic Models for Nonlinear Partial Differential Equations* (*Montecatini Terme*, 1995). *Lecture Notes in Math.* **1627** 42–95. Springer, Berlin. MR1431299

[47] PETZ, D. and HIAI, F. (1998). Logarithmic energy as an entropy functional. In *Advances in Differential Equations and Mathematical Physics* (*Atlanta, GA*, 1997). *Contemp. Math.* **217** 205–221. Amer. Math. Soc., Providence, RI. MR1606719

[48] ROGERS, L. C. G. and SHI, Z. (1993). Interacting Brownian particles and the Wigner law. *Probab. Theory Related Fields* **95** 555–570. MR1217451

[49] ROYER, G. (2007). *An Initiation to Logarithmic Sobolev Inequalities. SMF/AMS Texts and Monographs* **14**. Amer. Math. Soc., Providence, RI. Translated from the 1999 French original by Donald Babbitt. MR2352327

[50] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989*. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185

[51] TANAKA, H. (1983). Some probabilistic problems in the spatially homogeneous Boltzmann equation. In *Theory and Application of Random Fields* (*Bangalore*, 1982). *Lect. Notes Control Inf. Sci*. **49** 258–267. Springer, Berlin. MR0799949

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THE SIZE OF THE BOUNDARY IN FIRST-PASSAGE PERCOLATION

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First-passage percolation is a random growth model defined using i.i.d. edge-weights (t_e) on the nearest-neighbor edges of $\mathbb{Z}^d$. An initial infection occupies the origin and spreads along the edges, taking time t_e to cross the edge e. In this paper, we study the size of the boundary of the infected ("wet") region at time t, $B(t)$. It is known that $B(t)$ grows linearly, so its boundary $\partial B(t)$ has size between ct^{d-1} and Ct^d. Under a weak moment condition on the weights, we show that for most times, $\partial B(t)$ has size of order t^{d-1} (smooth). On the other hand, for heavy-tailed distributions, $B(t)$ contains many small holes, and consequently we show that $\partial B(t)$ has size of order $t^{d-1+\alpha}$ for some $\alpha > 0$ depending on the distribution. In all cases, we show that the exterior boundary of $B(t)$ [edges touching the unbounded component of the complement of $B(t)$] is smooth for most times. Under the unproven assumption of uniformly positive curvature on the limit shape for $B(t)$, we show the inequality $\#\partial B(t) \le (\log t)^C t^{d-1}$ for all large t.

REFERENCES

[1] ALEXANDER, K. S. (1997). Approximation of subadditive functions and convergence rates in limiting-shape results. *Ann. Probab.* **25** 30–55. MR1428498

[2] ANTAL, P. and PISZTORA, A. (1996). On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.* **24** 1036–1048. MR1404543

[3] AUFFINGER, A., DAMRON, M. and HANSON, J. (2017). 50 *Years of First-Passage Percolation*. *University Lecture Series* **68**. Amer. Math. Soc., Providence, RI. MR3729447

[4] BOUCH, G. (2015). The expected perimeter in Eden and related growth processes. *J. Math. Phys.* **56** Article ID 123302. MR3432741

[5] BOUCHERON, S., LUGOSI, G. and MASSART, P. (2013). *Concentration Inequalities. A Nonasymptotic Theory of Independence*. Oxford Univ. Press, Oxford. MR3185193

[6] BURDZY, K. and PAL, S. (2016). Twin peaks. Available at arXiv:1606.08025.

[7] COX, J. T. and DURRETT, R. (1981). Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* **9** 583–603. MR0624685

[8] DAMRON, M., LAM, W.-K. and WANG, X. (2017). Asymptotics for $2D$ critical first passage percolation. *Ann. Probab.* **45** 2941–2970. MR3706736

[9] EDEN, M. (1961). A two-dimensional growth process. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. IV* 223–239. Univ. California Press, Berkeley, CA. MR0136460

[10] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339

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Key words and phrases. First-passage percolation, boundary, Eden model, limit shape.

[11] KESTEN, H. (1986). Aspects of first passage percolation. In *École d'Été de Probabilités de Saint-Flour XIV—1984. Lecture Notes in Math.* **1180** 125–264. Springer, Berlin. MR0876084

[12] LEYVRAZ, F. (1985). The "active perimeter" in cluster growth models: A rigorous bound. *J. Phys. A* **18** L941–L945. MR0814624

[13] NEWMAN, C. M. (1995). A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians*, *Vol*. 1, 2 (*Zürich*, 1994) 1017–1023. Birkhäuser, Basel. MR1404001

[14] RICHARDSON, D. (1973). Random growth in a tessellation. *Math. Proc. Cambridge Philos. Soc.* **74** 515–528. MR0329079

[15] TIMÁR, Á. (2013). Boundary-connectivity via graph theory. *Proc. Amer. Math. Soc.* **141** 475–480. MR2996951

[16] ZABOLITZKY, J. G. and STAUFFER, D. (1986). Simulation of large Eden clusters. *Phys. Rev. A* **34** 1523–1530.

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RANDOM INSCRIBED POLYTOPES HAVE SIMILAR RADIUS FUNCTIONS AS POISSON–DELAUNAY MOSAICS[1]

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Using the geodesic distance on the n-dimensional sphere, we study the expected radius function of the Delaunay mosaic of a random set of points. Specifically, we consider the partition of the mosaic into intervals of the radius function and determine the expected number of intervals whose radii are less than or equal to a given threshold. We find that the expectations are essentially the same as for the Poisson–Delaunay mosaic in n-dimensional Euclidean space. Assuming the points are not contained in a hemisphere, the Delaunay mosaic is isomorphic to the boundary complex of the convex hull in $\mathbb{R}^{n+1}$, so we also get the expected number of faces of a random inscribed polytope. As proved in Antonelli et al. [*Adv. in Appl. Probab.* **9–12** (1977–1980)], an orthant section of the n-sphere is isometric to the standard n-simplex equipped with the Fisher information metric. It follows that the latter space has similar stochastic properties as the n-dimensional Euclidean space. Our results are therefore relevant in information geometry and in population genetics.

REFERENCES

AKIN, E. (1979). *The Geometry of Population Genetics*. *Lecture Notes in Biomathematics* **31**. Springer, Berlin. MR0559137

AMARI, S. and NAGAOKA, H. (2000). *Methods of Information Geometry*. Amer. Math. Soc., Providence, RI.

ANTONELLI, P. L. et al. (1977–80). The geometry of random drift I–VI. *Adv. in Appl. Probab.* **9–12**.

BÁRÁNY, I., FODOR, F. and VÍ GH, V. (2010). Intrinsic volumes of inscribed random polytopes in smooth convex bodies. *Adv. in Appl. Probab.* **42** 605–619. MR2779551

BAUER, U. and EDELSBRUNNER, H. (2017). The Morse theory of Čech and Delaunay complexes. *Trans. Amer. Math. Soc.* **369** 3741–3762. MR3605986

EDELSBRUNNER, H., NIKITENKO, A. and REITZNER, M. (2017). Expected sizes of Poisson–Delaunay mosaics and their discrete Morse functions. *Adv. in Appl. Probab.* **49** 745–767. MR3694316

EDELSBRUNNER, H. and WAGNER, H. (2017). Topological data analysis with Bregman divergences. In 33*rd International Symposium on Computational Geometry*. *LIPIcs*. *Leibniz Int. Proc. Inform.* **77** Art. No. 39, 16. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern. MR3685711

FORMAN, R. (1998). Morse theory for cell complexes. *Adv. Math.* **134** 90–145. MR1612391

HADWIGER, H. (1957). *Vorlesungen Über Inhalt, Oberfläche und Isoperimetrie*. Springer, Berlin. MR0102775

MSC2010 subject classifications. Primary 60D05; secondary 68U05.

Key words and phrases. Voronoi tessellations, Delaunay mosaics, inscribed polytopes, discrete Morse theory, critical simplices, intervals, stochastic geometry, Poisson point process, Blaschke–Petkantschin formula, Fisher information metric.

HEWITT, E. and STROMBERG, K. (1965). *Real and Abstract Analysis*. Springer, Berlin. MR0367121

HUG, D. (2013). Random polytopes. In *Stochastic Geometry*, *Spatial Statistics and Random Fields*. *Lecture Notes in Math*. **2068** 205–238. Springer, Heidelberg. MR3059649

KINGMAN, J. F. C. (1993). *Poisson Processes*. *Oxford Studies in Probability* **3**. The Clarendon Press, New York. MR1207584

KULLBACK, S. and LEIBLER, R. A. (1951). On information and sufficiency. *Ann*. *Math*. *Stat*. **22** 79–86. MR0039968

LERAY, J. (1945). Sur la forme des espaces topologiques et sur les points fixes des représentations. *J*. *Math*. *Pures Appl*. (9) **24** 95–167. MR0015786

LI, S. (2011). Concise formulas for the area and volume of a hyperspherical cap. *Asian J*. *Math*. *Stat*. **4** 66–70. MR2813331

MILES, R. E. (1970). On the homogeneous planar Poisson point process. *Math*. *Biosci*. **6** 85–127. MR0279853

MILES, R. E. (1971). Isotropic random simplices. *Adv*. *in Appl*. *Probab*. **3** 353–382. MR0309164

REITZNER, M. (2010). Random polytopes. In *New Perspectives in Stochastic Geometry* 45–76. Oxford Univ. Press, Oxford. MR2654675

REITZNER, M. and STEMESEDER, J. (2016). Expected number of faces of random polytopes with vertices on the boundary of a smooth convex body. Manuscript, Math. Dept., Univ. Osnabrück, Germany.

RENKA, R. J. (1997). Algorithm 772: STRIPACK: Delaunay triangulation and Voronoi diagram on the surface of a sphere. *ACM Trans*. *Math*. *Software* **23** 416–434. MR1672176

SCHNEIDER, R. (2008). Recent results on random polytopes. *Boll*. *Unione Mat*. *Ital*. (9) **1** 17–39. MR2387995

SCHNEIDER, R. and WEIL, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin. MR2455326

STEMESEDER, J. (2014). Random polytopes with vertices on the sphere. Ph.D. thesis, Math. Dept., Univ. Salzburg, Austria.

WENDEL, J. G. (1962). A problem in geometric probability. *Math*. *Scand*. **11** 109–111. MR0146858

ZÄHLE, M. (1990). A kinematic formula and moment measures of random sets. *Math*. *Nachr*. **149** 325–340. MR1124814

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STOCHASTIC CUCKER–SMALE MODELS: OLD AND NEW

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In this paper we revisit and generalize various stochastic models extending the deterministic Cucker–Smale model for self-organization. We study flocking and swarming properties. We show how these properties strongly depend on the structure and on the variance of the noise.

REFERENCES

[1] AHN, S. M. and HA, S.-Y. (2010). Stochastic flocking dynamics of the Cucker–Smale model with multiplicative white noises. *J. Math. Phys.* **51** Article ID 103301. MR2761313

[2] BOLLEY, F., CAÑIZO, J. A. and CARRILLO, J. A. (2011). Stochastic mean-field limit: Non-Lipschitz forces and swarming. *Math. Models Methods Appl. Sci.* **21** 2179–2210. MR2860672

[3] CUCKER, F. and MORDECKI, E. (2008). Flocking in noisy environments. *J. Math. Pures Appl.* (9) **89** 278–296. MR2401690

[4] CUCKER, F. and SMALE, S. (2007). Emergent behavior in flocks. *IEEE Trans. Automat. Control* **52** 852–862. MR2324245

[5] CUCKER, F. and SMALE, S. (2007). On the mathematics of emergence. *Jpn. J. Math.* **2** 197–227. MR2295620

[6] ERBAN, R., HAŠKOVEC, J. and SUN, Y. (2016). A Cucker–Smale model with noise and delay. *SIAM J. Appl. Math.* **76** 1535–1557. MR3534479

[7] HA, S.-Y., LEE, K. and LEVY, D. (2009). Emergence of time-asymptotic flocking in a stochastic Cucker–Smale system. *Commun. Math. Sci.* **7** 453–469. MR2536447

[8] HA, S.-Y. and LIU, J.-G. (2009). A simple proof of the Cucker–Smale flocking dynamics and mean-field limit. *Commun. Math. Sci.* **7** 297–325. MR2536440

[9] HA, S.-Y. and TADMOR, E. (2008). From particle to kinetic and hydrodynamic descriptions of flocking. *Kinet. Relat. Models* **1** 415–435. MR2425606

[10] MAO, X. (2008). *Stochastic Differential Equations and Applications*, 2nd ed. Horwood Publishing Limited, Chichester. MR2380366

[11] MOTSCH, S. and TADMOR, E. (2011). A new model for self-organized dynamics and its flocking behavior. *J. Stat. Phys.* **144** 923–947. MR2836613

[12] PÉDÈCHES, L. (2018). Asymptotic properties of various stochastic Cucker–Smale dynamics. *Discrete Contin. Dyn. Syst.* **38** 2731–2762. MR3809057

[13] SUN, Y. and LIN, W. (2015). A positive role of multiplicative noise on the emergence of flocking in a stochastic Cucker–Smale system. *Chaos* **25** Article ID 083118. MR3390406

[14] TON, T. V., LINH, N. T. H. and YAGI, A. (2014). Flocking and non-flocking behavior in a stochastic Cucker–Smale system. *Anal. Appl.* (*Singap.*) **12** 63–73. MR3150970

[15] VICSEK, T. and ZAFEIRIS, A. (2012). Collective motion. *Phys. Rep.* **517** 71–140.

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ERRATUM: "PROPAGATION OF CHAOS IN NEURAL FIELDS" [*Ann. Appl. Probab.* **24** (2014) 1298–1328]

BY JONATHAN TOUBOUL

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REFERENCES

[1] BOSSY, M., FAUGERAS, O. and TALAY, D. (2015). Clarification and complement to "Mean-field description and propagation of chaos in networks of Hodgkin–Huxley and FitzHugh–Nagumo neurons." *J. Math. Neurosci.* **5** Art. 19, 23. MR3392551

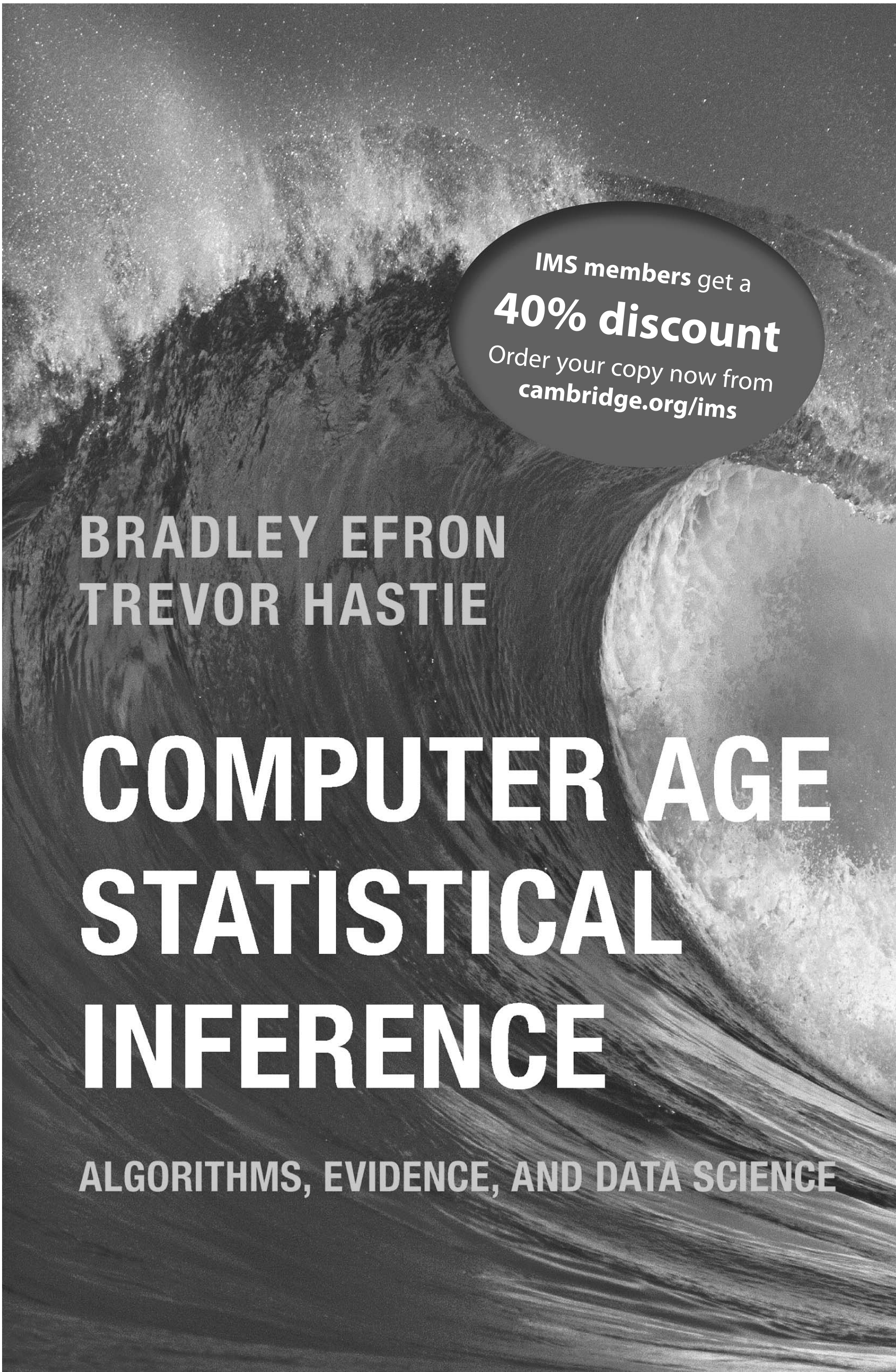
[2] MISCHLER, S., QUIÑINAO, C. and TOUBOUL, J. (2016). On a kinetic Fitzhugh–Nagumo model of neuronal network. *Comm. Math. Phys.* **342** 1001–1042. MR3465438

[3] SCHEUTZOW, M. (1987). Uniqueness and nonuniqueness of solutions of Vlasov–McKean equations. *J. Aust. Math. Soc. A* **43** 246–256. MR0896631

[4] TOUBOUL, J. (2014). Propagation of chaos in neural fields. *Ann. Appl. Probab.* **24** 1298–1328. MR3199987

[5] TOUBOUL, J. (2014). Spatially extended networks with singular multi-scale connectivity patterns. *J. Stat. Phys.* **156** 546–573. MR3217535

MSC2010 subject classifications. 60F99, 60B10, 34C15.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