

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

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THE ANNALS OF APPLIED PROBABILITY

Vol. 28, No. 4, pp. 1977–2634 August 2018

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(Organized September 12, 1935)

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The Annals of Applied Probability [ISSN 1050-5164 (print); ISSN 2168-8737 (online)], Volume 28, Number 4, August 2018. Published bimonthly by the Institute of Mathematical Statistics, 3163 Somerset Drive, Cleveland, Ohio 44122, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Applied Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998, USA.

MIXING TIMES OF RANDOM WALKS ON DYNAMIC CONFIGURATION MODELS

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The mixing time of a random walk, with or without backtracking, on a random graph generated according to the configuration model on n vertices, is known to be of order $\log n$. In this paper, we investigate what happens when the random graph becomes *dynamic*, namely, at each unit of time a fraction α_n of the edges is randomly rewired. Under mild conditions on the degree sequence, guaranteeing that the graph is locally tree-like, we show that for every $\varepsilon \in (0, 1)$ the ε -mixing time of random walk without backtracking grows like $\sqrt{2 \log(1/\varepsilon) / \log(1/(1 - \alpha_n))}$ as $n \rightarrow \infty$, provided that $\lim_{n \rightarrow \infty} \alpha_n (\log n)^2 = \infty$. The latter condition corresponds to a regime of fast enough graph dynamics. Our proof is based on a randomised stopping time argument, in combination with coupling techniques and combinatorial estimates. The stopping time of interest is the first time that the walk moves along an edge that was rewired before, which turns out to be close to a strong stationary time.

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MSC2010 subject classifications. Primary 60C05; secondary 37A25, 05C81.

Key words and phrases. Random graph, random walk, mixing time, coupling, dynamic configuration model.

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CONSISTENCY OF MODULARITY CLUSTERING ON RANDOM GEOMETRIC GRAPHS

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Given a graph, the popular “modularity” clustering method specifies a partition of the vertex set as the solution of a certain optimization problem. In this paper, we discuss scaling limits of this method with respect to random geometric graphs constructed from i.i.d. points $\mathcal{X}_n = \{X_1, X_2, \dots, X_n\}$, distributed according to a probability measure ν supported on a bounded domain $D \subset \mathbb{R}^d$. Among other results, we show, via a Gamma convergence framework, a geometric form of consistency: When the number of clusters, or partitioning sets of \mathcal{X}_n is a priori bounded above, the discrete optimal modularity clusterings converge in a specific sense to a continuum partition of the underlying domain D , characterized as the solution to a “soap bubble” or “Kelvin”-type shape optimization problem.

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MSC2010 subject classifications. Primary 60D05; secondary 62G20, 05C82, 49J55, 49J45, 68R10.

Key words and phrases. Modularity, community detection, consistency, random geometric graph, Gamma convergence, Kelvin’s problem, scaling limit, shape optimization, optimal transport, total variation, perimeter.

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CRITICAL PARAMETER OF RANDOM LOOP MODEL ON TREES

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We give estimates of the critical parameter for random loop models that are related to quantum spin systems. A special case of the model that we consider is the interchange- or random-stirring process. We consider here the model defined on regular trees of large degrees, which are expected to approximate high spatial dimensions. We find a critical parameter that indeed shares similarity with existing numerical results for the cubic lattice. In the case of the interchange process, our results improve on earlier work by Angel and by Hammond, in that we determine the second-order term of the critical parameter.

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MSC2010 subject classifications. 60K35, 82B20, 82B26, 82B31.

Key words and phrases. Random loop model, quantum Heisenberg.

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ORDER STATISTICS OF VECTORS WITH DEPENDENT COORDINATES, AND THE KARHUNEN–LOÈVE BASIS

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Let X be an n -dimensional random centered Gaussian vector with independent but not identically distributed coordinates and let T be an orthogonal transformation of \mathbb{R}^n . We show that the random vector $Y = T(X)$ satisfies

$$\mathbb{E} \sum_{j=1}^k j\text{-min}_{i \leq n} X_i^2 \leq C \mathbb{E} \sum_{j=1}^k j\text{-min}_{i \leq n} Y_i^2$$

for all $k \leq n$, where “ j -min” denotes the j th smallest component of the corresponding vector and $C > 0$ is a universal constant. This resolves (up to a multiplicative constant) an old question of S. Mallat and O. Zeitouni regarding optimality of the Karhunen–Loève basis for the nonlinear signal approximation. As a by-product, we obtain some relations for order statistics of random vectors (not only Gaussian) which are of independent interest.

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MSC2010 subject classifications. 62G30, 60E15, 60G15, 60G35, 94A08.

Key words and phrases. Order statistics, Karhunen–Loève basis, nonlinear approximation, INID case.

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BEATING THE OMEGA CLOCK: AN OPTIMAL STOPPING PROBLEM WITH RANDOM TIME-HORIZON UNDER SPECTRALLY NEGATIVE LÉVY MODELS

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We study the optimal stopping of an American call option in a random time-horizon under exponential spectrally negative Lévy models. The random time-horizon is modeled as the so-called Omega default clock in insurance, which is the first time when the occupation time of the underlying Lévy process below a level y , exceeds an independent exponential random variable with mean $1/q > 0$. We show that the shape of the value function varies qualitatively with different values of q and y . In particular, we show that for certain values of q and y , some quantitatively different but traditional up-crossing strategies are still optimal, while for other values we may have two disconnected continuation regions, resulting in the optimality of two-sided exit strategies. By deriving the joint distribution of the discounting factor and the underlying process under a random discount rate, we give a complete characterization of all optimal exercising thresholds. Finally, we present an example with a compound Poisson process plus a drifted Brownian motion.

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MSC2010 subject classifications. Primary 60G40, 60G51; secondary 60G17.

Key words and phrases. Lévy process, optimal stopping, Omega clock, occupation times, random discount rate, impatience.

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MUTATIONS ON A RANDOM BINARY TREE WITH MEASURED BOUNDARY¹

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Consider a random real tree whose leaf set, or boundary, is endowed with a finite mass measure. Each element of the tree is further given a type, or allele, inherited from the most recent atom of a random point measure (infinitely-many-allele model) on the skeleton of the tree. The partition of the boundary into distinct alleles is the so-called *allelic partition*.

In this paper, we are interested in the infinite trees generated by supercritical, possibly time-inhomogeneous, binary branching processes, and in their boundary, which is the set of particles “coexisting at infinity”. We prove that any such tree can be mapped to a random, compact ultrametric tree called the *coalescent point process*, endowed with a “uniform” measure on its boundary which is the limit as $t \rightarrow \infty$ of the properly rescaled counting measure of the population at time t .

We prove that the clonal (i.e., carrying the same allele as the root) part of the boundary is a regenerative set that we characterize. We then study the allelic partition of the boundary through the measures of its blocks. We also study the dynamics of the clonal subtree, which is a Markovian increasing tree process as mutations are removed.

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MSC2010 subject classifications. Primary 05C05, 60J80; secondary 54E45, 60G51, 60G55, 60G57, 60K15, 92D10.

Key words and phrases. Coalescent point process, branching process, random point measure, allelic partition, regenerative set, tree-valued process.

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***N*-PLAYER GAMES AND MEAN-FIELD GAMES WITH ABSORPTION**

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We introduce a simple class of mean-field games with absorbing boundary over a finite time horizon. In the corresponding N -player games, the evolution of players' states is described by a system of weakly interacting Itô equations with absorption on first exit from a bounded open set. Once a player exits, her/his contribution is removed from the empirical measure of the system. Players thus interact through a renormalized empirical measure. In the definition of solution to the mean-field game, the renormalization appears in form of a conditional law. We justify our definition of solution in the usual way, that is, by showing that a solution of the mean-field game induces approximate Nash equilibria for the N -player games with approximation error tending to zero as N tends to infinity. This convergence is established provided the diffusion coefficient is nondegenerate. The degenerate case is more delicate and gives rise to counter-examples.

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MSC2010 subject classifications. Primary 60K35, 91A06; secondary 60B10, 93E20.

Key words and phrases. Mean-field game, Nash equilibrium, McKean–Vlasov limit, absorbing boundary, weak convergence, martingale problem, optimal control.

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HYPONELLIPTIC STOCHASTIC FITZHUGH–NAGUMO NEURONAL MODEL: MIXING, UP-CROSSING AND ESTIMATION OF THE SPIKE RATE

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The FitzHugh–Nagumo is a well-known neuronal model that describes the generation of spikes at the intracellular level. We study a stochastic version of the model from a probabilistic point of view. The hypoellipticity is proved, as well as the existence and uniqueness of the stationary distribution. The bi-dimensional stochastic process is β -mixing. The stationary density can be estimated with an adaptive nonparametric estimator. Then we focus on the distribution of the length between successive spikes. Spikes are difficult to define directly from the continuous stochastic process. We study the distribution of the number of up-crossings. We link it to the stationary distribution and propose an estimator of its expectation. We finally prove mathematically that the mean length of inter-up-crossings interval is equal to its up-crossings rate. We illustrate the proposed estimators on a simulation study. Different regimes are explored, with no, few or high generation of spikes.

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MSC2010 subject classifications. Primary 60J70; secondary 62M05, 60H10, 60J60, 62P10, 35Q62, 37A50.

Key words and phrases. Hypoelliptic diffusion, FitzHugh–Nagumo model, invariant density, non-parametric estimation, up-crossings, pulse rate, spike rate estimation.

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EXCEPTIONAL TIMES OF THE CRITICAL DYNAMICAL ERDŐS–RÉNYI GRAPH¹

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In this paper, we introduce a network model which evolves in time, and study its largest connected component. We consider a process of graphs $(G_t : t \in [0, 1])$, where initially we start with a critical Erdős–Rényi graph $ER(n, 1/n)$, and then evolve forward in time by resampling each edge independently at rate 1. We show that the size of the largest connected component that appears during the time interval $[0, 1]$ is of order $n^{2/3} \log^{1/3} n$ with high probability. This is in contrast to the largest component in the static critical Erdős–Rényi graph, which is of order $n^{2/3}$.

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MSC2010 subject classifications. 05C80, 82C20, 60F17.

Key words and phrases. Noise sensitivity, Erdős–Rényi, dynamical random graphs, temporal networks, giant component.

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WHICH ERGODIC AVERAGES HAVE FINITE ASYMPTOTIC VARIANCE?¹

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We show that the class of L^2 functions for which ergodic averages of a reversible Markov chain have finite asymptotic variance is determined by the class of L^2 functions for which ergodic averages of its associated jump chain have finite asymptotic variance. This allows us to characterize completely which ergodic averages have finite asymptotic variance when the Markov chain is an independence sampler. From a practical perspective, the most important result identifies a simple sufficient condition for all ergodic averages of L^2 functions of the primary variable in a pseudo-marginal Markov chain to have finite asymptotic variance.

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MSC2010 subject classifications. 60J05, 60J22, 60F05, 65C40.

Key words and phrases. Markov chain Monte Carlo, asymptotic variance, jump chain, independent Metropolis–Hastings, pseudo-marginal method.

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A STOCHASTIC STEFAN-TYPE PROBLEM UNDER FIRST-ORDER BOUNDARY CONDITIONS

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Moving boundary problems allow to model systems with phase transition at an inner boundary. Motivated by problems in economics and finance, we set up a price-time continuous model for the limit order book and consider a stochastic and nonlinear extension of the classical Stefan-problem in one space dimension. Here, the paths of the moving interface might have unbounded variation, which introduces additional challenges in the analysis. Working on the distribution space, the Itô–Wentzell formula for SPDEs allows to transform these moving boundary problems into partial differential equations on fixed domains. Rewriting the equations into the framework of stochastic evolution equations and stochastic maximal L^p -regularity, we get existence, uniqueness and regularity of local solutions. Moreover, we observe that explosion might take place due to the boundary interaction even when the coefficients of the original problem have linear growths.

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MSC2010 subject classifications. Primary 60H15; secondary 91B70, 91G80.

Key words and phrases. Stochastic partial differential equations, Stefan problem, moving boundary problem, limit order book.

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STOCHASTIC APPROXIMATION OF QUASI-STATIONARY DISTRIBUTIONS ON COMPACT SPACES AND APPLICATIONS

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As a continuation of a recent paper, dealing with finite Markov chains, this paper proposes and analyzes a recursive algorithm for the approximation of the quasi-stationary distribution of a general Markov chain living on a compact metric space killed in finite time. The idea is to run the process until extinction and then to bring it back to life at a position randomly chosen according to the (possibly weighted) empirical occupation measure of its past positions. General conditions are given ensuring the convergence of this measure to the quasi-stationary distribution of the chain. We then apply this method to the numerical approximation of the quasi-stationary distribution of a diffusion process killed on the boundary of a compact set. Finally, the sharpness of the assumptions is illustrated through the study of the algorithm in a nonirreducible setting.

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MSC2010 subject classifications. Primary 65C20, 60B12, 60J10; secondary 34F05, 60J20, 60J60.

Key words and phrases. Quasi-stationary distributions, stochastic approximation, reinforced random walks, random perturbations of dynamical systems, extinction rate, Euler scheme.

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GLIVENKO–CANTELLI THEORY, ORNSTEIN–WEISS QUASI-TILINGS, AND UNIFORM ERGODIC THEOREMS FOR DISTRIBUTION-VALUED FIELDS OVER AMENABLE GROUPS

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We consider random fields indexed by finite subsets of an amenable discrete group, taking values in the Banach-space of bounded right-continuous functions. The field is assumed to be equivariant, local, coordinate-wise monotone and almost additive, with finite range dependence. Using the theory of quasi-tilings we prove an uniform ergodic theorem, more precisely, that averages along a Følner sequence converge uniformly to a limiting function. Moreover, we give explicit error estimates for the approximation in the sup norm.

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MSC2010 subject classifications. 60F99, 60B12, 62E20, 60K35.

Key words and phrases. Følner sequence, amenable group, quasi-tilings, Glivenko–Cantelli theory, Uniform convergence, Empirical measures.

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POLYNOMIAL JUMP-DIFFUSIONS ON THE UNIT SIMPLEX

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Polynomial jump-diffusions constitute a class of tractable stochastic models with wide applicability in areas such as mathematical finance and population genetics. We provide a full parameterization of polynomial jump-diffusions on the unit simplex under natural structural hypotheses on the jumps. As a stepping stone, we characterize well-posedness of the martingale problem for polynomial operators on general compact state spaces.

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MSC2010 subject classifications. 60J25, 60H30.

Key words and phrases. Polynomial processes, unit simplex, stochastic models with jumps, Wright–Fisher diffusion, stochastic invariance.

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RAPID MIXING OF GEODESIC WALKS ON MANIFOLDS WITH POSITIVE CURVATURE

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We introduce a Markov chain for sampling from the uniform distribution on a Riemannian manifold \mathcal{M} , which we call the *geodesic walk*. We prove that the mixing time of this walk on any manifold with positive sectional curvature $C_X(u, v)$ bounded both above and below by $0 < m_2 \leq C_X(u, v) \leq M_2 < \infty$ is $\mathcal{O}^*(\frac{M_2}{m_2})$. In particular, this bound on the mixing time does not depend explicitly on the dimension of the manifold. In the special case that \mathcal{M} is the boundary of a convex body, we give an explicit and computationally tractable algorithm for approximating the exact geodesic walk. As a consequence, we obtain an algorithm for sampling uniformly from the surface of a convex body that has running time bounded solely in terms of the curvature of the body.

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MSC2010 subject classifications. Primary 60J05; secondary 68W20, 60J20, 53D25.

Key words and phrases. Markov chain Monte Carlo (MCMC), Hamiltonian Monte Carlo (HMC), manifolds, convex bodies, positive curvature, momentum.

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CONVERGENCE AND QUALITATIVE PROPERTIES OF MODIFIED EXPLICIT SCHEMES FOR BSDES WITH POLYNOMIAL GROWTH

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The theory of Forward–Backward Stochastic Differential Equations (FBSDEs) paves a way to probabilistic numerical methods for nonlinear parabolic PDEs. The majority of the results on the numerical methods for FBSDEs relies on the global Lipschitz assumption, which is not satisfied for a number of important cases such as the Fisher–KPP or the FitzHugh–Nagumo equations. Furthermore, it has been shown in [*Ann. Appl. Probab.* **25** (2015) 2563–2625] that for BSDEs with monotone drivers having polynomial growth in the primary variable y , only the (sufficiently) implicit schemes converge. But these require an additional computational effort compared to explicit schemes.

This article develops a general framework that allows the analysis, in a systematic fashion, of the integrability properties, convergence and qualitative properties (e.g., comparison theorem) for whole families of modified explicit schemes. The framework yields the convergence of some modified explicit scheme with the same rate as implicit schemes and with the computational cost of the standard explicit scheme.

To illustrate our theory, we present several classes of easily implementable modified explicit schemes that can computationally outperform the implicit one and preserve the qualitative properties of the solution to the BSDE. These classes fit into our developed framework and are tested in computational experiments.

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MSC2010 subject classifications. Primary 65C30, 60H35; secondary 60H30.

Key words and phrases. FBSDE, monotone driver, polynomial growth, time discretization, modified explicit schemes, nonexplosion, numerical stability.

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UNIFORM CONTROL OF LOCAL TIMES OF SPECTRALLY POSITIVE STABLE PROCESSES¹

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We establish two results about local times of spectrally positive stable processes. The first is a general approximation result, uniform in space and on compact time intervals, in a model where each jump of the stable process may be marked by a random path. The second gives moment control on the Hölder constant of the local times, uniformly across a compact spatial interval and in certain random time intervals. For the latter, we introduce the notion of a Lévy process restricted to a compact interval, which is a variation of Lambert’s Lévy process confined in a finite interval and of Pistorius’ doubly reflected process. We use the results of this paper to exhibit a class of path-continuous branching processes of Crump–Mode–Jagers-type with continuum genealogical structure. A further motivation for this study lies in the construction of diffusion processes in spaces of interval partitions and \mathbb{R} -trees, which we explore in forthcoming articles. In that context, local times correspond to branch lengths.

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MSC2010 subject classifications. 60G51, 60G52, 60J55, 60J80.

Key words and phrases. Stable process, squared Bessel processes, CMJ process, local time approximation, excursion theory, restricted Lévy process, Hölder continuity.

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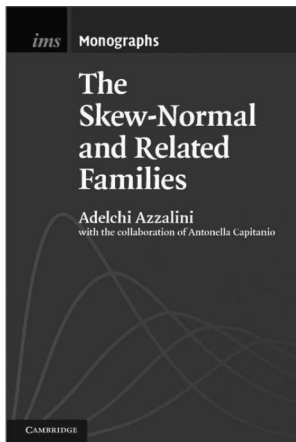
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